Math 608 - Spring 2025
Homework 2
29 January 2025

Read sections 4.3, 4.4, and 4.5. Pay particular attention to Theorem 4.29(c) (we did not do this in class), Lemma 4.32, and Propositions 4.38 and 4.40.

1. If X has the weak topology generated by a family $\mathcal{F}$ of functions, then $\langle x_\alpha \rangle$ converges to $x \in X$ if and only if $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$. (In particular, if $X = \prod_{\beta \in B} X_\beta$, then $x_\alpha \to x$ if and only if $\pi_\beta(x_\alpha) \to \pi_\beta(x)$ for each $\beta \in B$.)

2. Let X be a set and $\mathcal{A}$ the collection of all finite subsets of X, directed by inclusion. Let $f : X \to \mathbb{R}$ be an arbitrary function, and for $A \in \mathcal{A}$, let $z_A = \sum_{x \in A} f(x)$. Show that the net $\langle z_A \rangle$converges in $\mathbb{R}$ if and only if $\{x : f(x) \neq 0\}$ is a countable set $\{x_n\}_{n \in \mathbb{N}}$ and $\sum_{n=1}^{\infty} |f(x_n)| < \infty$, in which case $z_A \to \sum_{n=1}^{\infty} f(x_n)$.

3. Suppose that $(X, \mathcal{T})$ is a compact Hausdorff space and $\mathcal{T}'$ is another topology on X. If $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$ hen $(X, \mathcal{T}')$ is Hausdorff but not compact. If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, then $(X, \mathcal{T}')$ is compact but not Hausdorff.

4. Let X be a locally compact Hausdorff space. Consider $X^* = X \cup \{\infty\}$, where ∞ is a point not in X. Call $O \subset X^*$ open if either $\infty \notin O$ and O is open in X or $\infty \in O$ and $X^* \setminus O$ is compact. Prove that X^* is a compact Hausdorff space; it is called the (Alexandroff) one-point compactification of X.

5. If X and Y are topological spaces, $\phi \in C(X, Y)$ is called *proper* if $\phi^{-1}(K)$ is compact in X for every compact $K \subset Y$. Suppose X and Y are locally compact Hausdorff spaces and X^* and Y^* are their one-point compactifications. Show that if $\phi \in C(X, Y)$, then ϕ is proper if and only if ϕ extends continuously to a map from X^* to Y^* by setting $\phi(\infty_X) = \infty_Y$.

Quiz 2 Throughout this problem, if a result was proven in the text or in class, you can just refer to it. Let X be a locally compact Hausdorff space. Recall that on $\mathbb{C}^X = \{f : X \to \mathbb{C}\}$, the topology of uniform convergence on compact sets is defined by the base

$$U_{g,n,K} = \left\{ f \in \mathbb{C}^X : \|f - g\|_u = \sup_{x \in K} |f(x) - g(x)| < \frac{1}{n} \right\},$$

for $n \in \mathbb{N}$, $g \in \mathbb{C}^X$, and $K \subset X$ compact.

(a) Prove that this is in fact a base for a topology.

(b) Let $(f_\alpha)_{\alpha \in A} \subset \mathbb{C}^X$ be a net. Show that $f_\alpha \to f$ in this topology if and only if for all $K \subset X$ compact, the net of real numbers $\sup_{x \in K} |f_\alpha(x) - f(x)| \to 0$.
For the remaining parts of the problem, we take $X = \mathbb{C}$.

(c) Consider $\mathbb{C}^\mathbb{C} = \{f : \mathbb{C} \to \mathbb{C}\}$ with the metric

$$\rho(f, g) = \sum_{k=1}^{\infty} 2^{-k} \min \left(\sup_{|z| \leq k} |f(z) - g(z)|, 1 \right).$$

(You do not need to prove this is a metric.) Prove that the topology induced by this metric is the topology of uniform convergence on compact sets.

(d) For $X = \mathbb{C}$, in part (b), the convergence of nets can be replaced with the convergence of sequences. (Hint: This almost immediate.)

(e) Why couldn't we have used

$$\rho'(f, g) = \sum_{k=1}^{\infty} 2^{-k} \sup_{|z| \leq k} |f(z) - g(z)|?$$

Additional practice problems Chapter 4, problems 37, 43, 52.