

Read Sections 5.1, 5.2. They are short but dense.

1. Suppose  $X$  and  $Y$  are compact Hausdorff spaces. Show that the algebra generated by functions of the form  $f(x, y) = g(x)h(y)$  for  $g \in C(X)$  and  $h \in C(Y)$  is dense in  $C(X \times Y)$ .
2. Let  $X$  be a metric space. A function  $f \in C(X)$  is called *Hölder continuous with exponent  $\alpha$*  ( $\alpha > 0$ ) if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

is finite. If  $X$  is compact, show that  $\{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$  is compact in  $C(X)$ .

3. Let  $X$  be a compact metric space and  $0 < \alpha \leq 1$ . Show that the Hölder continuous functions on  $X$  with exponent  $\alpha$  form a Banach space with norm

$$\|f\| = \|f\|_u + N_\alpha(f).$$

This space is often denoted  $C^\alpha(X)$  (or  $C^{0,\alpha}(X)$  if  $X$  is a manifold), though Folland uses  $\Lambda_\alpha(X)$ .

4. Suppose  $X$  is a Banach space.

- (a) If  $T \in \mathcal{L}(X, X)$  has  $\|T\| < 1$ , then  $I - T$  is invertible and in fact the series  $\sum_{n=0}^{\infty} T^n$  converges in  $\mathcal{L}(X, X)$  to  $(I - T)^{-1}$ .
  - (b) Show that if  $A \in \mathcal{L}(X, X)$  is invertible and  $\|A - B\| < \|A^{-1}\|^{-1}$ , then  $B$  is invertible. (The set of invertible operators is therefore open in  $\mathcal{L}(X, X)$ .)
5. Suppose that  $X$  is a finite dimensional vector space. Let  $e_1, \dots, e_n$  be a basis for  $X$  and define

$$\left\| \sum_{j=1}^n a_j e_j \right\|_1 = \sum_{j=1}^n |a_j|.$$

- (a) Show that  $\|\cdot\|_1$  is a norm on  $X$ .
- (b) Show that the map  $(a_1, \dots, a_n) \mapsto \sum_{j=1}^n a_j e_j$  is a continuous function from  $K^n$  (with the usual Euclidean topology) to  $X$  with the topology induced by  $\|\cdot\|_1$ .
- (c) Show that the unit sphere in this norm is compact, i.e.,  $\{x \in X : \|x\|_1 = 1\}$  is compact in the topology defined by  $\|\cdot\|_1$ .
- (d) Show that any norm on  $X$  is equivalent to  $\|\cdot\|_1$ .

**Quiz 4** Let  $X$  be a compact Hausdorff space. An *ideal* in  $C(X; \mathbb{R})$  is a subalgebra  $\mathcal{I}$  of  $C(X; \mathbb{R})$  so that if  $f \in \mathcal{I}$  and  $g \in C(X; \mathbb{R})$ ,  $fg \in \mathcal{I}$ .

- (a) If  $\mathcal{I}$  is an ideal in  $C(X; \mathbb{R})$ , let  $V(\mathcal{I}) = \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}\}$ . Show that  $V(\mathcal{I})$  is a closed subset of  $X$ . It is called the hull of  $\mathcal{I}$ .
- (b) If  $E \subset X$ , let  $J(E) = \{f \in C(X; \mathbb{R}) : f(x) = 0 \text{ for all } x \in E\}$ . Show that  $J(E)$  is a closed ideal in  $C(X; \mathbb{R})$ . It is called the kernel of  $E$ .
- (c) Show that if  $E \subset X$ , then  $V(J(E)) = \bar{E}$ .
- (d) If  $\mathcal{I}$  is an ideal in  $C(X; \mathbb{R})$ , then  $J(V(\mathcal{I})) = \bar{\mathcal{I}}$ . (Hint:  $J(V(\mathcal{I}))$  can be identified with a subalgebra of  $C_0(U, \mathbb{R})$ , where  $U = X \setminus V(\mathcal{I})$ .)
- (e) Put the above together to show that the closed subsets of  $X$  are in one-to-one correspondence with the closed ideals of  $C(X; \mathbb{R})$ .

**Additional practice problems** Problems 5.8, 5.9, 5.10, 5.13.