Math 608 - Spring 2025
Homework 6
26 February 2025

Read sections 5.3, 5.4, 5.5.

In problems 2–5 below, note that $\ell^\infty(\mathbb{N})$ and $\ell^1(\mathbb{N})$ can be regarded as $L^\infty(\mu)$ and $L^1(\mu)$, respectively, where μ denotes counting measure on $\mathbb{N}$.

1. Prove that $\mathcal{P}$, the set of all polynomials, regarded as a subset of $C^0([0,1])$, is meager (first category) in this space.

2. Let $\ell^\infty(\mathbb{N})$ denote the set of bounded sequences $\mathbb{N} \to \mathbb{C}$ equipped with $\|\cdot\|_\infty$, where

$$\|(a_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |a_n| .$$

(a) Show that $\|\cdot\|_\infty$ is a norm.

(b) Let $c = \{(a_n) \in \ell^\infty(\mathbb{N}) : \lim_{n\to\infty} a_n \text{ exists}\}$. Show that c is a closed subspace of $\ell^\infty(\mathbb{N})$.

(c) For $(a_n) \in c$, let $f((a_n)) = \lim_{n\to\infty} a_n$. Show that f is a bounded linear functional on c and conclude that there is a bounded linear functional on $\ell^\infty(\mathbb{N})$ extending f.

3. Let $\ell^\infty(\mathbb{N})$ be as in the previous problem and let $\ell^1(\mathbb{N})$ denote the space of absolutely summable sequences $a_n : \mathbb{N} \to \mathbb{C}$ equipped with $\|\cdot\|_1$, where

$$\|(a_n)_{n\in\mathbb{N}}\|_1 = \sum_{n\in\mathbb{N}} |a_n| .$$

(You do not have to show it is a norm this time.)

(a) Show that any $(b_n) \in \ell^\infty(\mathbb{N})$ gives a bounded linear functional in $(\ell^1(\mathbb{N}))^*$ by $(a_n) \mapsto \sum a_n b_n$ and that the norm of this linear functional is $\|(b_n)\|_\infty$.

(b) Show that every element of $(\ell^1(\mathbb{N}))^*$ is obtained this way, i.e., $(\ell^1(\mathbb{N}))^* = \ell^\infty(\mathbb{N})$.

(c) Use the last part of the previous problem to show that $(\ell^\infty(\mathbb{N}))^* \neq \ell^1(\mathbb{N})$ (and hence $\ell^1(\mathbb{N})$ is not reflexive).

4. With notation as in the previous two problems:

(a) Show that c and $\ell^1(\mathbb{N})$ are separable.

(b) Show that $\ell^\infty(\mathbb{N})$ is not. (This gives an alternative proof that $\ell^1(\mathbb{N})$ is not reflexive.)

5. Let $Y = \ell^1(\mathbb{N})$ and X denote the set of $(a_n) \in Y$ so that $\sum_{n=1}^\infty n\,|a_n| < \infty$ equipped with the ℓ^1 norm.

(a) Show that X is a proper, dense subspace of Y and hence is not complete.

(b) Let $T : X \to Y$ be given by $T((a_n)) = (na_n)$. Show that T is closed but not bounded.

(c) Let $S = T^{-1}$. Show $S : Y \to X$ is bounded and surjective but not open.

Quiz 6 Suppose that X and Y are normed vector spaces and $T \in \mathcal{L}(X, Y)$.

(a) Define (the *adjoint* or *transpose*) $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$. Show $T^* \in \mathcal{L}(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

(b) Applying the construction twice we get $T^{**} \in \mathcal{L}(X^{**}, Y^{**})$. If X and Y are identified with their natural images $\widehat{X}$ and $\widehat{Y}$ in X^{**} and Y^{**}, respectively, show that $T^{**}|_X = T$.

(c) Show that T^* is injective if and only if the range of T is dense in Y.

(d) Show that if the range of T^* is dense in X^* then T is injective and that the converse is true if X is reflexive.

Challenge. Not to be turned in. Let f be an infinitely differentiable on $I = [0, 1]$. Suppose that for every $x \in I$ there exists an $n(x)$ so that the derivative $f^{(n)}(x) = 0$. Prove that f must be a polynomial.

Additional practice problems Chapter 5: 20, 21, 24, 35.