Math 608 - Spring 2025
Homework 7
5 March 2025

Read sections 5.4, 5.5.

1. Let f_n be a sequence of continuous real-valued functions on $[0,1]$ so that for each $x \in [0,1]$ there is an $n_x \in \mathbb{N}$ so that $f_n(x) \geq 0$ for all $n \geq n_x$. Show there is a non-empty open interval I and an $N \in \mathbb{N}$ so that $f_n(x) \geq 0$ for all $x \in I$ and $n \geq N$.

2. Let $T : X \to Y$ be a surjective linear map between Banach spaces and suppose there is a $\lambda > 0$ so that $\|Tx\| \geq \lambda \|x\|$ for all $x \in X$. Show that T is bounded.

3. Let X be a normed vector space over $K = \mathbb{R}$ or $K = \mathbb{C}$.

(a) If $Y \subset X$ is a closed subspace and $x \in X \setminus Y$, show $Y + Kx$ is closed.

(b) Show that every finite-dimensional subspace of X is closed.

4. Show that every finite-dimensional subspace Y of a normed space X admits a topological complement. In other words, show there is a closed subspace $Z \subset X$ so that $Y \cap Z = \{0\}$ and $Y + Z = X$. (Hint: Choose a basis for Y and use functionals "dual" to each of these basis vectors.)

5. Let X and Y be Banach spaces and let T_n be a sequence in $\mathcal{L}(X,Y)$ so that $\lim_n T_n x$ exists for each $x \in X$. Let $Tx = \lim_n T_n x$. Show $T \in \mathcal{L}(X,Y)$.

Quiz 7 In this problem we let $C^\infty(\mathbb{S}^1) = C^\infty(\mathbb{R}/2\pi\mathbb{Z}) = \{f \in C^\infty(\mathbb{R}) \mid f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R}\}$ denote the space of infinitely differentiable functions on the unit circle. Endow it with the structure of a Fréchet space using the norms

$$\|f\|_{C^k} = \sum_{j=0}^{k} \left\| f^{(j)} \right\|_u .$$

Let $\mathcal{D}'(\mathbb{S}^1)$ denote the topological dual of $C^\infty(\mathbb{S}^1)$. One typically calls elements of $\mathcal{D}'(\mathbb{S}^1)$ distributions. Let $\bar{\imath} : L^1(\mathbb{S}^1) \to \mathcal{D}'(\mathbb{S}^1)$ be the 'inclusion map'

$$\bar{\imath}(\phi)(\psi) = \int_{\mathbb{S}^1} \phi\psi,$$

for $\phi \in L^1(\mathbb{S}^1)$ and $\psi \in C^\infty(\mathbb{S}^1)$.

(a) Show that $\bar{\imath}$ indeed maps into $\mathcal{D}'(\mathbb{S}^1)$, is injective, and is continuous. (We can therefore regard $L^1(\mathbb{S}^1)$ as a subset of $\mathcal{D}'(\mathbb{S}^1)$.)

(b) Let $\imath = \bar{\imath}|_{C^\infty(\mathbb{S}^1)}$. Show that $\imath : C^\infty(\mathbb{S}^1) \to \mathcal{D}'(\mathbb{S}^1)$ is continuous.

(c) Show that $\frac{d}{dx} : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ has a continuous extension to a map $\frac{d}{dx} : \mathcal{D}'(\mathbb{S}^1) \to \mathcal{D}'(\mathbb{S}^1)$, given by

$$\left(\frac{d}{dx}u\right)(\phi) = -u\left(\frac{d\phi}{dx}\right),$$

where $u \in \mathcal{D}'(\mathbb{S}^1)$, $\phi \in C^\infty(\mathbb{S}^1)$. (Thus, every distribution, and in particular every L^1 function, can be differentiated arbitrarily many times in the sense of distributions.)

Additional practice problems 5.37, 5.45.