Math 608 - Spring 2025
Homework 8
19 March 2025

Read section 5.5.

1. Suppose V is an inner product space either over $\mathbb{R}$ or over $\mathbb{C}$.

(a) Prove that the inner product can be recovered from the norm by the polarization identity:

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

if the field is $\mathbb{R}$, and

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) - \frac{i}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right)$$

if the field is $\mathbb{C}$.

(b) Prove that a normed vector space $(V, \|\bullet\|)$ is an inner product space (with the induced norm being $\|\bullet\|$) if and only if the norm satisfies the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

for all $x, y \in V$.

(c) Conclude that for $1 \leq p < \infty$, $p \neq 2$, the standard norms on $\ell^p(\mathbb{N})$ and $L^p(\mathbb{R})$ do not arise from inner products. [a]

[a] Recall or learn for first time that $\ell^p(\mathbb{N})$ is the space of complex valued sequences (a_n) with $\sum |a_n|^p < \infty$ and has norm $\|(a_n)\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$. Similarly $L^p(\mathbb{R})$ consists of those measurable functions (up to equality almost everywhere) with $\int_{\mathbb{R}} |f|^p < \infty$ and norm given by $\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}$.

2. Let V be an inner product space, and $\{x_n\}_{n=1}^N$ be an orthonormal set. Prove that

$$\left\| x - \sum_{n=1}^{N} c_n x_n \right\|$$

is minimized by choosing $c_n = \langle x, x_n\rangle$.

3. Let M be any linear subspace of a Hilbert space $\mathcal{H}$. Prove that $M^\perp$ is a closed linear subspace and that $\overline{M} = (M^\perp)^\perp$, with the bar denoting closure.

For the next two problems, we let $C_p^1([0, 2\pi])$ denote the space of 2π-periodic continuously differentiable functions on $\mathbb{R}$. For $n \in \mathbb{Z}$, we also let $e_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$.

4. Suppose $f \in C^1_p([0,2\pi])$ and let $c_n = \langle f, e_n \rangle$ and $b_n = \langle f', e_n \rangle$.

(a) Prove that $\sum |b_n|^2 < \infty$ and conclude that $\sum n^2 |c_n|^2 < \infty$.

(b) Prove that $\sum |c_n| < \infty$.

(c) Prove that $\sum_{n=-M}^{M} c_n e_n(x)$ is uniformly convergent as $M \to \infty$.

5. Let f, c_n, and b_n be as in the previous problem.

(a) Let $S_N(f) = \sum_{n=-N}^{N} c_n e_n(x)$. Prove that

$$(S_N f)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta + x) \frac{\sin\left(\left(N + \frac{1}{2}\right) x\right)}{\sin(x/2)}\, dx.$$

(The 2π-periodic function

$$K_N(x) = \frac{\sin\left(\left(N + \frac{1}{2}\right) x\right)}{\sin(x/2)} = \sum_{n=-N}^{N} e^{inx}$$

is known as the Dirichlet kernel.)

(b) Using that $\frac{1}{2\pi} \int_0^{2\pi} K_N(x)\, dx = 1$, show that for $f \in C^1_p([0,2\pi])$,

$$f(\theta) - S_N f(\theta) = \left\langle g_\theta, \frac{1}{\sqrt{\pi}} \sin\left(\left(N + \frac{1}{2}\right) x\right)\right\rangle,$$

where $g_\theta(x) = \frac{f(\theta+x) - f(\theta)}{2\sqrt{\pi}\sin(x/2)}$, and that $g_\theta \in C^0([0,2\pi])$.

(c) Using that $\frac{1}{\sqrt{\pi}} \sin\left(\left(N + \frac{1}{2}\right) x\right)$, $N = 0, 1, 2, \ldots$, is an orthonormal set in $L^2([0,2\pi])$, show that $S_N f(\theta) \to f(\theta)$, and conclude that the Fourier series of $f \in C^1_p([0,2\pi])$ converges uniformly to f.

Quiz 8 Suppose that X is a Banach space.

(a) The norm-closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is also weakly closed. (Hint: Use Theorem 5.8d from your text.)

(b) If $E \subset X$ is bounded with respect to the norm, so is its weak closure.

(c) If $F \subset X^*$ is bounded with respect to the norm, so is its weak-* closure.

(d) Every weak-* Cauchy sequence in X^* converges. (Use exercise 5 from homework 7.)

(Note that part (a) above should be at least a little surprising, since the weak topology is generally weaker than the norm topology, so you do not typically expect norm-closed sets to be weakly closed.)