Math 608 - Spring 2025
Homework 10
2 April 2025

1. Show that (with Lebesgue measure to define L^2):

(a) Continuous functions are dense in $L^2([0,1])$.

(b) Polynomials are dense in $L^2([0,1])$.

(c) $L^2([0,1])$ is separable.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $E \in \mathcal{M}$, we can identify $L^2(E, \mu)$ with the subspace of $L^2(X, \mu)$ of functions vanishing outside of E. If E_n is a disjoint sequence in $\mathcal{M}$ with $X = \cup_{n=1}^{\infty} E_n$, show that $L^2(E_n, \mu)$ is a sequence of mutually orthogonal subspaces of $L^2(X, \mu)$ and that every $f \in L^2(X, \mu)$ can be written uniquely as a norm-convergent series $f = \sum_{n=1}^{\infty} f_n$, where $f_n \in L^2(E_n, \mu)$. If each $L^2(E_n, \mu)$ is separable, show that $L^2(X, \mu)$ is as well. Conclude that $L^2(\mathbb{R})$ is separable.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces so that $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$, respectively, and $h_{m,n}(x,y) = f_m(x) g_n(y)$, show that $\{h_{m,n}\}$ is an orthonormal basis for $L^2(\mu \times \nu)$. (Be sure to show completeness!)

4. Show that every orthonormal sequence in an infinite dimensional Hilbert space converges weakly to 0.

5. Let Y be a closed subspace of $L^2([0,1])$ (with Lebesgue measure) that is contained in $C([0,1])$.

(a) Show that there is a constant C so that $\|f\|_u \leq C \|f\|_{L^2}$ for all $f \in Y$. (Hint: Closed graph theorem.)

(b) For each $x \in [0,1]$, show there is a $g_x \in Y$ with $f(x) = \langle f, g_x \rangle$ for all $f \in Y$, and $\|g_x\|_{L^2} \leq C$.

(c) Show that the dimension of Y is at most C^2 (and hence Y is finite dimensional). (Hint: Take an orthonormal sequence f_j in Y; what can you say about $\sum |f_j(x)|^2$?

Quiz 10 In the following problems, we let $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ denote the (normalized) trigonometric functions on the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Recall that $\{e_n\}_{n\in\mathbb{Z}}$ are a complete orthonormal system in $L^2(\mathbb{S}^1)$ and that $L^2(\mathbb{S}^1)$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$ by the map $f \mapsto (\langle f, e_n\rangle)_{n\in\mathbb{Z}}$. To simplify expressions, we use the notation $\hat{f}(n) = \langle f, e_n\rangle$. Define the space $H^s(\mathbb{S}^1)$ by

$$H^s(\mathbb{S}^1) = \{f \in \mathcal{D}'(\mathbb{S}^1) : \sum_{n\in\mathbb{Z}} (1+n^2)^s \left|\hat{f}(n)\right|^2 < \infty.$$

(The space H^s is a Hilbert space when equipped with the inner product $f, g = \sum_{n\in\mathbb{Z}} (1+n^2)^s \hat{f}(n)\overline{\hat{g}(n)}$. The corresponding norm is given by $\|f\|_{H^s}^2 = \sum_{n\in\mathbb{Z}} (1+n^2)^s \left|\hat{(f)}(n)\right|^2$.)

(a) Show that the inclusion map $H^s \hookrightarrow H^{s-2}$ is compact (i.e., any bounded sequence in H^s has a convergent subsequence in H^{s-2}).

(b) Recall that if f is continuously differentiable, then $\langle f', e_n\rangle = in\hat{f}(n)$. Use this to show that the operator L given by $Lf = -f'' + f$ is an isometric isomorphism $H^s \to H^{s-2}$.

(c) Use the above to show that there is a constant C so that for all $f \in H^s$,

$$\|f\|_{H^s} \le C \left(\|f''\|_{H^{s-2}} + \|f\|_{H^{s-2}} \right).$$

Use the first part of the problem to conclude the space $\{f \in H^s : f'' = 0\}$ of solutions to the differential equation $f'' = 0$ is finite-dimensional. (You can identify it, if you like.)

(d) There is a natural pairing $H^s \times H^{-s} \to \mathbb{C}$ given by $(f, u) \mapsto \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{u}(n)}$, which allows us to identify the dual of H^s (which we already know to be isometrically isomorphic to H^s) with H^{-s}. With respect to this pairing, the adjoint of $P = \frac{d^2}{dx^2} : H^s \to H^{s-2}$ is $P^* = \frac{d^2}{dx^2} : H^{2-s} \to H^{-s}$. The estimates above then show that, for all $u \in H^{2-s}$,

$$\|u\|_{H^{2-s}} \le C \left(\|P^* u\|_{H^{-s}} + \|u\|_{H^{-s}} \right),$$

so that $Y = \{u \in H^{2-s} : P^* u = 0\}$ is finite-dimensional. **Problem begins here:** Suppose $f \in H^{s-2}$ is orthogonal to Y (i.e., $\langle f, u\rangle = 0$ for all $u \in Y$. On the subspace

$$X = P^* H^{2-s} = \left\{ v = P^* u \in H^{-s} : u \in H^{2-s} \right\} \subset H^{-s},$$

show that the map $v \mapsto \langle f, u\rangle$ (where $v = P^* u$) is a bounded (conjugate-)linear functional on $X \subset H^{-s}$. Conclude that there is some $g \in H^s$ for which this map is given by $\langle g, v\rangle$. If $s \ge 2$, what is the relationship between g and f? (The same relationship holds generally but in a weaker distributional sense.)