Math 608 - Spring 2025
Homework 11
9 April 2025

1. Let $1 \leq p < r \leq \infty$. Show that $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$.

2. Let $1 \leq p < r \leq \infty$. Show that $L^p + L^r$ is a Banach space with norm

$$\|f\| = \inf\left\{\|g\|_p + \|h\|_r \mid g \in L^p, h \in L^r, f = g + h\right\}.$$

3. Suppose $0 < p < q < \infty$. Show that L^p is not contained in L^q if and only if X contains sets of arbitrarily small positive measure. (Hint for the 'if' direction: you can find a disjoint sequence E_n with $0 < \mu(E_n) < 2^{-n}$, then consider $\sum a_n \chi_{E_n}$ for some a_n.)

4. Suppose $0 < p < q < \infty$. Show that L^q is not contained in L^p if and only if X contains sets of arbitrarily large positive measure. (Hint for 'if' direction: you can find a disjoint sequence E_n with $1 \leq \mu(E_n) < \infty$ and consider a similar sum as in the previous problem.)

5. Let $f \in L^p \cap L^\infty$ for some $p < \infty$. You know already that then $f \in L^q$ for all $q > p$. Show that $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Quiz 11 Prove that any weakly convergent sequence in ℓ^1 is also norm convergent.