Math 608 - Spring 2025
Homework 12
16 April 2025

1. Show that $L^\infty(\mathbb{R}^n)$ (with Lebesgue measure) is not separable.

2. Suppose μ is a semifinite measure, $q < \infty$,

$$M_q(g) = \sup\left\{\left|\int fg\right| : f \text{ simple, measurable, } \mu(\{f \neq 0\}) < \infty, \|f\|_p = 1\right\} < \infty.$$

Show that $\{x : |g(x)| > \epsilon\}$ has finite measure for all $\epsilon > 0$ and hence $\{g \neq 0\}$ is σ-finite.

3. Suppose $f_n \in L^p$, $1 < p < \infty$, $\sup_n \|f_n\|_p < \infty$, and $f_n \to f$ almost everywhere. Show that $f_n \to f$ weakly in L^p.

4. Let $k(x,y)$ be the function on $[0,1] \times [0,1]$ given by

$$k(x,y) = \begin{cases} \frac{1}{y} & x < y \leq 1 \\ 0 & 0 \leq y \leq x \end{cases}.$$

Let $Tf(x) = \int_0^1 k(x,y)f(y)\,dy$. Show that $T : L^2([0,1]) \to L^2([0,1])$ is bounded. (Hint: Schur's test)

5. Suppose X and Y are σ-finite measure spaces and $K \in L^2(X \times Y)$. For $f \in L^2(Y)$, let $Tf(x) = \int_Y K(x,y)f(y)\,d\nu(y)$. Show that $Tf \in L^2(X)$ for $f \in L^2(Y)$ and that $\|T\|_{L^2 \to L^2} \leq \|K\|_2$.

Quiz 12 Consider the operator M defined by multiplication by x, i.e., $(Mf)(x) = xf(x)$, acting on various different spaces of functions on subsets of the real line.

(a) Let $J \subset \mathbb{R}$ be either a bounded interval $[a,b]$ or a half-line $(-\infty, a]$ or $[a, \infty)$, or else the entire line $\mathbb{R}$. Prove that M is a bounded operator on $L^p(J)$, $1 \leq p < \infty$, if and only if J is a bounded interval.

(b) Let $J = [0,1]$. Show that the range of $M : C(J) \to C(J)$ lies in a proper closed subspace. Is this range closed?