Math 608 - Spring 2025
Homework 13
23 April 2025

1. Suppose μ is a Radon measure on a locally compact Hausdorff space X.

(a) Let N be the union of all open $U \subset X$ so that $\mu(U) = 0$. Show that N is open and $\mu(N) = 0$. (The complement of N is called the support of μ.)

(b) Show that x is in the support of μ (i.e., in $X \setminus N$) if and only if $\int f \, d\mu > 0$ for every $f \in C_c(X, [0,1])$ with $f(x) > 0$.

2. Suppose X is a locally compact Hausdorff space. Recall that a G_δ set is the countable intersection of open sets.

(a) Show that if $f \in C_c(X, [0, \infty))$, then $f^{-1}([a, \infty))$ is a compact G_δ set for all $a > 0$.

(b) If $K \subset X$ is a compact G_δ set, show there exists $f \in C_c(X, [0, \infty))$ so that $K = f^{-1}(\{1\})$. (Hint: Use Urysohn's lemma for LCH spaces. Also, if you want to combine an infinite family of continuous functions into a continuous function you can sometimes take a weighted average $\sum 2^{-n} f_n$.)

3. Suppose X is a locally compact Hausdorff space. The *Baire sets* are those in the σ-algebra $\mathcal{B}^0_X$ generated by $C_c(X)$. (In other words, $\mathcal{B}^0_X$ is the smallest σ-algebra with respect to which all $f \in C_c(X)$ are measurable functions.) Show that $\mathcal{B}^0_X$ is the σ-algebra generated by the compact G_δ sets.

4. Suppose X is a second countable locally compact Hausdorff space.

(a) Show that every compact subset of X is a G_δ set.

(b) Show that $\mathcal{B}^0_X = \mathcal{B}_X$ (recall that this latter σ-algebra is the Borel σ-algebra, i.e., the one generated by open sets.) (Hint: You may use without proof that every open set in a second countable LCH space is σ-compact.)

5. Let X be an uncountable set with the discrete topology. Show that $\mathcal{B}_X \neq \mathcal{B}^0_X$.

Quiz 13 Suppose that $f_n \in C([0,1])$ is a bounded sequence. Show that $f_n \to 0$ weakly if and only if $f_n(x) \to 0$ for all $x \in [0,1]$. (You may use without proof a result we haven't finished proving yet, namely, that the dual of $C([0,1])$ is the space of signed or complex Radon measures on $[0,1]$.)