THE KLEIN–GORDON EQUATION ON ASYMPTOTICALLY MINKOWSKI SPACETIMES: THE FEYNMAN PROPAGATOR

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Abstract. We develop a theory of Feynman propagators for the massive Klein–Gordon equation with asymptotically static perturbations. Building on our previous work on the causal propagators, we employ a framework based on propagation of singularities estimates in Vasy's 3sc-calculus. We combine these estimates to prove global spacetime mapping properties for the Feynman propagator, and to show that it satisfies a microlocal Hadamard condition.

We show that the Feynman propagator can be realized as the inverse of a mapping between appropriate L^2-based Sobolev spaces with additional regularity near the asymptotic sources of the Hamiltonian flow, realized as a family of radial points on a compactified spacetime.

Contents

1. Introduction

Let (M, g) be a globally hyperbolic, asymptotically Minkowski spacetime as considered previously by the authors, [2,3,7]. In this work, we construct the Feynman propagator for the Klein–Gordon operator

$$P_V := \Box_g - m^2 - V\,.$$

where $m > 0$ and V is a smooth potential function with spatial decay.

On flat Minkowski spacetime $(\mathbb{R}^{n+1}, g_{\mathbb{M}})$, the Feynman propagator for the free Klein–Gordon operator, $\Box_{g_{\mathbb{M}}} - m^2 = D_t^2 - \Delta - m^2$, is the Fourier multiplier by the distribution $(\tau^2 - |\zeta|^2 - m^2 + i0)^{-1}$. In this work, we adopt the perspective (taken in the work of the Gell-Redman–Haber–Vasy [7] and Gerard–Wrochna [9]) that an appropriate generalization of the Feynman propagator for P_V is an operator which is both a global spacetime inverse for P_V (e.g. on compactly supported distributions) and which satisfies the well-known wavefront set property, sometimes called the "microlocal Hadamard condition", detailed below.

Our main results are Theorem 5.1 and Theorem 7.3, which prove existence of the Feynman propagator, and Theorem 6.3, which proves a variant of the microlocal Hadamard condition.

We now provide a simplified version these results. On Minkowski space $\mathbb{R}^{n+1}$, let $V = V(t,z)$ be given by

$$V(t,z) = V_0(z) + V_1(t,z)\,, \tag{1}$$

where $V_0 \in S^{-1}(\mathbb{R}^n; \mathbb{R})$ exhibits symbolic decay of order -1 and $V_1 \in \mathcal{C}^\infty(\mathbb{R}^{n+1}; \mathbb{R})$ such that

$$|\partial_t^k \partial_z^\alpha V_1(t,z)| \lesssim \langle t, z\rangle^{-1-k} \langle z\rangle^{-|\alpha|}$$

for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$.

Theorem 1.1. *If $H_{V_0} = \Delta + m^2 + V_0$ has purely absolutely continuous spectrum, then there exists a bounded linear operator $(P_V)^{-1}_{\mathrm{Fey}} : \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+1}) \to \mathcal{C}^{-\infty}(\mathbb{R}^{n+1})$ such that for all $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+1})$,*

$$P_V (P_V)^{-1}_{\mathrm{Fey}} f = (P_V)^{-1}_{\mathrm{Fey}} P_V f = f\,,$$

and

$$\mathrm{WF}_{\mathrm{cl}}((P_V)^{-1}_{\mathrm{Fey}} f) \subset \mathrm{WF}_{\mathrm{cl}}(f) \cup \bigcup_{s\geq 0} \Phi_s(\mathrm{WF}_{\mathrm{cl}}(f) \cap \mathrm{Char}(P_0))\,.$$

where $\mathrm{WF}_{\mathrm{cl}}$ denotes the classical wavefront set of a distribution, Φ_s is the flow generated by the Hamilton vector field of the principal symbol of $P_0 = \Box_g - m^2$ on the cotangent bundle.

Theorem 5.1 is in fact more general and allows for the possibility of **first order perturbations**, as well as for **different limiting potentials** as $t \to \pm\infty$. In the case that the limiting Hamiltonians have **bound states**, we can prove the same conclusions under slightly stronger assumptions on decay of $V - V_0$, see Theorem 7.3 for a precise statement. The appearance of the operator P_0 in the theorem is explained in our previous work [2]. We mention here that the characteristic sets and the flows of P_V and P_0 are identical away from the singular locus of V on a compactified spacetime, a set which consists of two points, the "poles", described in detail below.

There are four standard propagators, or inverses, that arise in the study of wave propagation: the forward and backward causal propagators, which we denote by $G_\pm$, and the Feynman and anti-Feynman propagators. Our work applies equally well to anti-Feynman propagators, which can be constructed exactly as below but with the opposite direction of regularity propagation along the flow.

The causal propagators $G_\pm$ can be characterized easily by the support condition

$$\operatorname{supp} f \subset \{\pm t \geq T\} \implies \operatorname{supp} G_\pm f \subset \{\pm t \geq T\}.$$

The Feynman propagator $(P_V)^{-1}_{\mathrm{Fey}}$, however, does not have such a simple characterization. The physical intuition is that $(P_V)^{-1}_{\mathrm{Fey}}$ propagates positive energy forward in time and negative energy backward in time. This can be made precise either by prescribing asymptotics or using a wavefront set condition. We describe the wavefront set condition below in Section 6.

The asymptotic condition is simple to describe in the case of the free Klein–Gordon equation on Minkowski space. There we can solve $P_0 u = f$ for $f \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$ using the Fourier transform and observe that solutions have asymptotics

$$u = t^{-n/2}\left(a^{\mathrm{f}}_+(z/t)e^{im\sqrt{t^2-|z|^2}} + a^{\mathrm{f}}_-(z/t)e^{-im\sqrt{t^2-|z|^2}}\right)(1+O(1/t)) \tag{2}$$

as $t \to +\infty$ and $|z|/t \leq c < 1$ for some functions a^{f}_+ and a^{f}_-. (Here the superscript f stands for "future", not for the forcing.) Similarly we find $a^{\mathrm{p}}_\pm$ for $t \to -\infty$. *The Feynman propagator produces the solution operator with* $a^{\mathrm{f}}_- \equiv 0$ *and* $a^{\mathrm{p}}_+ \equiv 0$.

The relationship between the Hamiltonian flow and the asymptotic form for Feynman solutions can be understood through consideration of the phase functions

$$\phi_\pm(t,z) = \pm m\sqrt{t^2 - |z|^2}.$$

The graphs of these functions in the cotangent bundle, i.e., the submanifold

$$\{(t, z, \partial_t\phi_\pm, \partial_z\phi_\pm) : (t,z) \in \mathbb{R}^{n+1}\}$$

defines the surface

$$\{\tau^2 - (|\zeta|^2 + m^2) = 0,\ \tau z + t\zeta = 0, \pm\tau > 0\}. \tag{3}$$

We consider a compactification $\overline{\mathbb{R}^{n+1}}$ of $\mathbb{R}^{n+1}$; the boundary of this compactification is (locally) given by $x = \pm 1/t = 0$ and is parametrized by the variable $y = \pm z/t$. The set (3) can be restricted to the boundary; this restriction has four components, corresponding to the choices of $\pm t > 0$ and $\pm\tau > 0$, and can be identified precisely as the limit points of the trajectories of the Hamiltonian flow of the Hamiltonian function $\tau^2 - (|\zeta|^2 + m^2)$, i.e., the symbol of P_0.

The coefficients $a^{\mathrm{f}}_-, a^{\mathrm{p}}_+$ correspond to the sources of the Hamiltonian flow, $\mathcal{R}_{\mathrm{src}}$. (In terms of the sign choices above, $\mathcal{R}_{\mathrm{src}}$ corresponds to $t < 0, \tau > 0$ and $t > 0, \tau < 0$.)

The coefficients $a_+^{\mathrm{f}}, a_-^{\mathrm{p}}$ microlocally correspond to the sinks $\mathcal{R}_{\mathrm{snk}}$ ($t > 0, \tau > 0$ and $t < 0, \tau < 0$); nontrivial asymptotics at the former are excluded by the mapping properties of the Feynman propagator, while at the latter they are allowed.

To construct the Feynman propagator, we follow the method developed by Vasy [26], in which a propagator is realized as the inverse of a Fredholm operator. That is, we define families of Hilbert spaces $\mathcal{X}, \mathcal{Y}$ of tempered distributions on $\mathbb{R}^{n+1}$ such that $P_V : \mathcal{X} \longrightarrow \mathcal{Y}$ is a Fredholm operator, and then subsequently show that this operator is: (1) invertible under certain assumptions on the potential V, and (2) its inverse satisfies the defining properties of the Feynman propagator.

In the absence of a potential V, or more generally in the setting in which $V = V(t, z)$ is a decaying scattering potential on $\mathbb{R}^{n+1}$ (i.e., V enjoys symbolic decay jointly in space and time), Gerard–Wrochna [9] constructed the Feynman propagator via a related method. In the same setting, Vasy [28] proved estimates which lead directly to an alternative construction of a Feynman propagator, which we reviewed in our prior work [2, Sect. 2].

In each of these constructions, the spaces $\mathcal{X}, \mathcal{Y}$ referred to above are subspaces of scattering Sobolev spaces $H_{\mathrm{sc}}^{s;\ell}(\mathbb{R}^{n+1}) = \langle t, x\rangle^{-\ell} H_{\mathrm{sc}}^{s}(\mathbb{R}^{n+1})$. In Vasy's treatment [28] (and in our treatment of causal propagators [2]) one takes $\ell = \boldsymbol{\ell}$ to be a variable order, i.e., a function $\boldsymbol{\ell} = \boldsymbol{\ell}(t, z, \tau, \zeta)$. Which propagator is selected depends on the properties of $\boldsymbol{\ell}$, namely whether it is above or below the threshold value of $-1/2$ at the four components of the radial set. For the Feynman propagator, one uses $\boldsymbol{\ell} > -1/2$ at $\mathcal{R}_{\mathrm{src}}$ and $\boldsymbol{\ell} < -1/2$ at $\mathcal{R}_{\mathrm{snk}}$.

To analyze P_V for asymptotically static potentials V, we use an adaptation of Vasy's 3sc-pseudodifferential calculus introduced in [24, 25]. The modifications needed to treat the Klein–Gordon operator were presented in our prior paper [2], where we proved propagation estimates at all parts of phase space for the operators P_V. In that paper, we gave a construction of the causal propagators, in which the range of, say, the forward causal propagator G_+ is contained in a variable order weighted space $H_{\mathrm{sc}}^{s;\boldsymbol{\ell}_+}(\mathbb{R}^{n+1})$, in which $\boldsymbol{\ell}_+ = \boldsymbol{\ell}_+(t, z)$ is a function on space time which satisfies $\boldsymbol{\ell}_+ > -1/2$ at ι^- and $\boldsymbol{\ell}_+ < -1/2$ at ι^+, where ι^+, ι^- are future/past timelike infinity, respectively. The significance of these conditions is that solutions $G_+ f$ are allowed below threshold asymptotics only to the future.

With this method, the main distinction between the treatment of the free Klein–Gordon operator P_0 and the perturbed operators P_V is that the potential V is not a smooth function on the standard radial compactification $X = \overline{\mathbb{R}^{n+1}}$. It fails to be smooth precisely at the "north/south poles", NP/SP, depicted in Figure 1 below. To analyze the behavior of P_V near the poles, one uses the (operator-valued) indicial operators $\hat{N}_{\mathrm{ff},\pm}(P_V)$, which are essentially the Fourier transform (in time) of the limiting operator.

The global nature of the indicial operator presents a technical obstacle in that variable decay orders $\boldsymbol{\ell}$ are incompatible with the indicial family. Since both $\mathcal{R}_{\mathrm{snk}}$ and $\mathcal{R}_{\mathrm{src}}$ have nontrivial projections at both poles, any construction of the Feynman propagator with Vasy's method must confront this issue. We do so by *avoiding the use of variable order spaces altogether,* which is a key technical novelty of our construction. Instead, we define spaces which enforce above threshold behavior at $\mathcal{R}_{\mathrm{src}}$ through the use of a microlocal cutoff. That is, we use microlocal cutoffs Q_{src} supported at $\mathcal{R}_{\mathrm{src}}$ to define the $\mathcal{X}, \mathcal{Y}$ spaces and then subsequently show that the Feynman propagator thus constructed does not depend on our choices.

As in our analysis of the causal propagators, the possible existence of bound states for the limiting spatial Hamiltonians $H_{V_\pm}$ plays an important role in determining the potential existence of a kernel or cokernel in the Fredholm map constructed above. When $H_{V_\pm}$ are positive, we are able to construct the Feynman propagator as an operator on $\mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ (or more generally on distributions $\mathcal{S}'$ satisfying an appropriate wavefront set condition.) This is done in Section 7.

1.1. **Related work.** Our work generalizes the results from Gérard–Wrochna [9, 10] and the implied construction in [28]. These works are the closest in outline to the present work and yield Feynman propagators for Klein–Gordon operators whose potential perturbations are "scattering" in spacetime; in particular their potentials decay in spacetime.

The results of Derezinski–Siemssen [4, 5] are perhaps the most similar to the present one in that they construct the Feynman propagator in a non-trivial time-dependent setting using Kato's "method of evolution equations" to construct propagators. Related work on the existence of Feynman propagators via essential self-adjointness of the Klein–Gordon operator by Nakamura–Taira [20, 21] and Vasy [29]. Moreover, the Feynman propagator appears in index theory on Lorentzian manifolds [1].

As our results use many-body microlocal methods, the only assumptions made in the finite time region are dynamical, namely the assumption that the flow is non-trapping. Therefore, the global spectral theoretic assumptions in Derezinksi–Siemssen (see point 3 in the introduction [5]) are not needed. Interestingly, assumptions on the point spectrum do arise in the $t \to \pm\infty$ limit. Specifically, we obtain a Feynman-type Fredholm problem so long as there is no point spectrum for the limiting Hamiltonians at 0. We then obtain a Feynman *propagator*, i.e., inverse to the Fredholm mapping, if the limiting Hamiltonians are positive.

Although we do not study many-body Hamiltonians directly here, since our work uses the 3sc-calculus of Vasy, the analysis herein is related to previous work on such operators. We do not include an exhaustive list of related work in that field here, but we note that the use of microlocal cutoffs in many -body scattering has a long history.

We draw particular attention to the pioneering work of Gérard–Isozaki–Skibsted [8], which uses functions of the Hamiltonian to construct commutators. This strategy was later adopted by Vasy, and informs our propagator construction, both here and in our previous work.

Constructing Fredholm problems for operators on non-compact manifolds requires a precise analysis at infinity. However, microlocal analysis and in particular the calculus of Fourier integral operators has been used by Duistermaat–Hörmander [6] to prove the existence of distinguished parametrices, which are inverses modulo smoothing operators, for hyperbolic operators. In particular, they constructed a Feynman parametrix. Radzikowski [22] showed that the wavefront set condition that distinguishes the Feynman parametrix is equivalent to the Hadamard condition of quantum field theory. We also refer to Islam–Strohmaier [17] for a more modern treatment of distinguished parametrices which also allows for vector-valued operators.

For other recent work using tools from the many-body analysis of Vasy to study hyperbolic PDE, see; Hintz and Hintz–Vasy [11–15]. See also the work of Ma on many-body Hamiltonians [18] as well as recent work of Sussman on Klein–Gordon [23] using microlocal methods.

The present paper differs from the works above in at least three significant ways. To our knowledge, this paper is the first to construct the Feynman propagator for asymptotically static potential without explicitly appealing to a global time-splitting or spectral hypotheses on each time-slice. Our construction also has the advantage that a microlocal Hadamard condition is easily proved (indeed, it is nearly automatic). Finally, in contrast with other related constructions, our construction does not use variable order Sobolev spaces. Vasy [30, 31] employs an approach using second microlocalization to analyze $\Delta - \lambda^2$ in the $\lambda \to 0^+$ limit. The estimates proven in that work also avoid the use of variable order spaces. That setting is structurally similar to that of the Klein–Gordon operator plus a potential, provided the potential is "spacetime scattering", in particular meaning the potential functions decay in time.

Outline of the paper. The paper is organized as follows. In Section 2 we consider the case that $V \equiv 0$, which puts us in the setting of scattering operators and while it is possible in this case to use the same approach as for the causal propagator, we construct the Feynman propagator using localizers to the radial set giving a simplified version of the proofs of the main theorems. In Section 3 we recall the main properties of the 3sc-calculus introduced by Vasy [24, 25] and the extensions from [2] needed to treat the Klein–Gordon operator. In Section 4 we define the class of localization operators that are used to define the Sobolev spaces for the Feynman propagator and we state a slightly modified radial set estimate. We construct the Feynman propagator in Section 5 in the case that there are no bound states and prove the Hadamard property in Section 6. Finally, in Section 7, we discuss the modifications of the argument that are needed to treat the case that the limiting Hamiltonians have bound states.

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2. The Feynman propagator for the free Klein–Gordon equation

In this section, we consider the case $V \equiv 0$. This is simpler than the case of general asymptotically static V, in that it can be treated using only the scattering calculus of Melrose [19]. In contrast with related treatments [28] of this case, including our own work [2] on causal propagators, here we do not use variable order weight functions for reasons outlined in the introduction. Treating the case $V \equiv 0$ separately allows us to focus on the new features of the construction, which uses microlocalizers to define the Feynman spaces as opposed to variable order weights.

We write

$$P_0 := \Box_g - m^2,$$

in particular we think of the background asymptotically Minkowski metric g as fixed; the subscript 0 refers only to the potential V being identically zero.

Our construction of the Feynman propagator relies on a detailed understanding of the Hamiltonian flow of P_0 on the characteristic set, which we describe in Section 2.1. We in particular identify the family $\mathcal{R}_{\mathrm{src}}$ of sources of the Hamiltonian flow. In Sections 2.2 we construct microlocal cutoffs Q_{src} to $\mathcal{R}_{\mathrm{src}}$, and in Section 2.3 use to impose an above threshold decay condition on distributions in the Feynman domain, thereby making a Fredholm problem for P_V. Then in Section 2.4 we discuss the Feynman propagator as it arises as an inverse to these Fredholm problems.

2.1. **Geometry of the Hamiltonian flow.**

To analyze P_0, we use the scattering formalism of Melrose [19]. In particular, we work on the radial compactification

$$X = \overline{\mathbb{R}^{n+1}}, \tag{4}$$

a compact manifold whose boundary is defined by the vanishing of a function $\rho_{\mathrm{base}} \in \mathcal{C}^\infty(X)$, i.e., $\rho_{\mathrm{base}}^{-1}(0) = \partial X$ and $d\rho_{\mathrm{base}}|_{\partial X} \neq 0$. We take

$$\rho_{\mathrm{base}} = \langle t, z\rangle^{-1} = \left(1 + t^2 + |z|^2\right)^{-1/2} \geq 1. \tag{5}$$

With coordinates t, z on $\mathbb{R}^{n+1}$, writing

$$g_{\mathbb{M}} = dt^2 - dz^2 \tag{6}$$

we assume that our background metric g is a non-trapping Lorentzian metric and that the difference of the components of g and $g_{\mathbb{M}}$ satisfies, for all $j, k \in \{1, \dots, n+1\}$,

$$g_{jk} - (g_{\mathbb{M}})_{jk} \in S^{-2}(\mathbb{R}^{n+1}) = \rho_{\mathrm{base}}^2 C^\infty(X). \tag{7}$$

The d'Alembertian is $\Box_g = -(1/\sqrt{|g|})\partial_\mu g^{\mu\nu}\sqrt{|g|}\partial_\nu$, so for the Minkowski metric it is just the free wave operator:

$$\Box_{g_{\mathbb{M}}} = -\partial_t^2 + \sum_{j=1}^n \partial_{z_j}^2 = D_t^2 - \Delta.$$

The radial compactification used here is different from the Penrose compactification of Minkowski space; in the radial compactification, future and past causal infinity are the limit loci of forward and backward timelike rays, respectively:

$$\begin{aligned}\iota^+ &:= \{(t, z) \colon t \geq |z|\} \cap \partial X\,,\\ \iota^- &:= \{(t, z) \colon -t \geq |z|\} \cap \partial X\,.\end{aligned}$$

Near ι^+, we typically use coordinates $x = 1/t, y = z/t$, which are valid in any region in which $0 \leq x, |y| \leq C$ where $C > 0$. In these coordinates,

$$\iota^+ = \{x = 0, |y| \leq 1\}.$$

The full symbol of P_0 is

$$p(t, z, \tau, \zeta) = g^{-1}_{(t,z)}\left((\tau, \zeta), (\tau, \zeta)\right) + m^2\,.$$

where $g^{-1}_{(t,z)}$ is the inverse of the metric $g_{(t,z)}$. This symbol p is an example of a scattering symbol of order $(2, 0)$, as we describe now.

The scattering cotangent bundle ${}^{\mathrm{sc}}T^*X = X \times \mathbb{R}^{n+1}$ is simply the compactification of the spacetime factor of the standard phase space $T^*\mathbb{R}^{n+1}$, the latter written with coordinates (t, z, τ, ζ). Its fiber compactification

$${}^{\mathrm{sc}}\overline{T}^*X = \overline{\mathbb{R}^{n+1}_{t,z}} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}, \tag{8}$$

on which the momentum/energy variables – the "fibers" – are also radially compactified, is a manifold with corners, where the fiber boundary is defined by the vanishing of $\rho_{\mathrm{fib}} \in \mathcal{C}^\infty(\overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ where

$$\rho_{\mathrm{fib}} = \langle \tau, \zeta\rangle^{-1}. \tag{9}$$

The classical scattering symbols can then be characterized simply by

$$ {}^{\mathrm{sc}}S^{m,r}(\mathbb{R}^{n+1}) = \rho_{\mathrm{fib}}^{-m}\rho_{\mathrm{base}}^{-r}\mathcal{C}^\infty({}^{\mathrm{sc}}\overline{T}^*X). \tag{10} $$

The quantization of these symbols yields the space of scattering pseudodifferential operators:

$$ {}^{\mathrm{sc}}\Psi^{m,r} = \mathrm{Op}_L({}^{\mathrm{sc}}S^{m,r}(X)), \tag{11} $$

and the differential operators in this class are denoted $\mathrm{Diff}_{\mathrm{sc}}^{m,r}(X)$. Equivalently, $L \in \mathrm{Diff}_{\mathrm{sc}}^{m,r}(X)$ if an only if

$$ L = \sum_{j+|\alpha|\le m} a_{j,\alpha}(t,z)D_t^j D_z^\alpha, \quad a_{t,\alpha} \in S^r(\mathbb{R}^{n+1}). \tag{12} $$

Therefore $P_0 \in \mathrm{Diff}_{\mathrm{sc}}^{2,0}$, and we can apply general scattering calculus results to P_0. In particular, we can define the scattering principal symbol by the restriction

$$ j_{\mathrm{sc},2,0}(P_0) \coloneqq \rho_{\mathrm{fib}}^2 \cdot p|_{\partial^{\mathrm{sc}}\overline{T}^*X}, $$

which we write in two components corresponding to the two boundary hypersurfaces of ${}^{\mathrm{sc}}\overline{T}^*X$ defined by $\rho_{\mathrm{fib}} = 0$ (fiber infinity) and $\rho_{\mathrm{base}} = 0$ (spacetime or “base” infinity),

$$ \partial^{\mathrm{sc}}\overline{T}^*X = {}^{\mathrm{sc}}\overline{T}^*_{\partial X}X \cup {}^{\mathrm{sc}}S^*X, \tag{13} $$

where ${}^{\mathrm{sc}}S^*X \cong X \times \partial\overline{\mathbb{R}^{n+1}}$ denote the sphere bundle of ${}^{\mathrm{sc}}T^*X$. Following Melrose [19, Proposition 3] we write

$$ j_{\mathrm{sc},2,0}(P_0) = \left(\sigma_{\mathrm{sc},2,0}(P_0), \hat{N}_{\mathrm{sc},2,0}(P_0)\right). \tag{14} $$

In the interior of $\mathbb{R}^{n+1}$, $\sigma_{\mathrm{sc},2,0}(P_0)$ is identical to the standard principal symbol of P_0, while in the interior of the fiber, i.e., for finite (τ,ζ), $\hat{N}_{\mathrm{sc},2,0}(P_0)$ is the restriction of the total symbol to the space time boundary ∂X.

Both components of $j_{\mathrm{sc},2,0}(P_0)$ are functions, and the characteristic set is merely the vanishing locus of the symbol restricted to the boundary, i.e., the union of the vanishing loci of the components. In the case of P_0 this is easy to write down:

$$ \mathrm{Char}(P_0) = \left\{(q,\tau,\zeta) \colon q \in X, g_q(\tau,\zeta) = m^2\right\}, $$

where, when $q \in \partial X$ it is simply the condition $\tau^2 = |\zeta|^2 + m^2$, and over the interior of X it is interpreted as the equation $g_q(\tau,\zeta) = 0$ holding on points in the fiber boundary $\partial\overline{\mathbb{R}^{n+1}_{\tau,\xi}}$.

The main object of study in these results is the Hamiltonian flow on the characteristic set. In the scattering setting, the Hamilton vector field is also rescaled so that a well defined flow is induced on the characteristic set in $\partial^{\mathrm{sc}}\overline{T}^*X$. Namely, we define the scattering Hamilton vector field

$$ {}^{\mathrm{sc}}H_p \coloneqq (\rho_{\mathrm{fib}}/\rho_{\mathrm{base}}) \cdot H_p \tag{15} $$

where we use the standard definition of the Hamilton vector field

$$H_p := \frac{\partial p}{\partial \tau}\partial_t + \frac{\partial p}{\partial \zeta}\partial_z - \frac{\partial p}{\partial t}\partial_\tau - \frac{\partial p}{\partial z}\partial_\zeta\,. \tag{16}$$

We now discuss coordinates which are particularly convenient for the computation of ${}^{\mathrm{sc}}H_p$ and also for quantities of interest in later sections. First we note the general fact that boundary defining functions such as ρ_{base} and ρ_{fib} are not unique. Locally over a boundary component $\bullet \in \{\mathrm{base}, \mathrm{fib}\}$, any function $\tilde{\rho}$ satisfying $1/C < \tilde{\rho}/\rho_\bullet < C$ for $C > 0$ is also a valid boundary defining function in that region. For us the most convenient choices to replace ρ_{base} and ρ_{fib} are

$$x = (\operatorname{sgn} t)/t, \text{ and } \rho = (\operatorname{sgn} \tau)/\tau, \tag{17}$$

respectively. In regions in which $x, \rho < C$ we have coordinates

$$x, \quad y = z/t, \quad \rho, \quad \mu = \zeta/\tau, \tag{18}$$

and we obtain the expression

$$(1/2)^{\mathrm{sc}}H_p = -(\operatorname{sgn} t)(\operatorname{sgn} \tau)\left(x\partial_x + (\mu + y)\cdot \partial_y\right), \tag{19}$$

where, as the reader pleases, one can think of this as a redefinition of ${}^{\mathrm{sc}}H_p$ where $\rho_{\mathrm{base}}, \rho_{\mathrm{fib}}$ are replaced by x, ρ, or one can think of the prefactor of $(\rho_{\mathrm{fib}}/\tau)(x/\rho_{\mathrm{base}})$ as suppressed. (For $a > 0$ a smooth, positive function on ${}^{\mathrm{sc}}\overline{T}^*X$, the flow of $a\,{}^{\mathrm{sc}}H_p$ on the characteristic set of P_0 is a smooth, non-degenerate reparametrization of the flow of ${}^{\mathrm{sc}}H_p$ and is therefore irrelevant in all statements below regarding the Hamiltonian flow.)

The radial set is the vanishing locus of the scattering Hamilton vector field in the boundary inside the characteristic set:

$$\mathcal{R} := \mathrm{Char}(P_0) \cap {}^{\mathrm{sc}}H_p^{-1}(0) \subset \partial\,{}^{\mathrm{sc}}\overline{T}^*X\,.$$

The radial set of P_0 is computed in [2, Sect. 2.5], where it is shown that it lies over causal infinity, i.e., if $\pi\colon {}^{\mathrm{sc}}\overline{T}^*X \longrightarrow X$ is the projection to the base, then

$$\pi(\mathcal{R}) = \iota^+ \cup \iota^-,$$

and we write

$$\mathcal{R} = \mathcal{R}^f \sqcup \mathcal{R}^p\,.$$

for the parts of $\mathcal{R}$ lying over future/past causal infinity, meaning

$$\mathcal{R}^f := \mathcal{R} \cap {}^{\mathrm{sc}}\overline{T}^*_{\iota^+}X, \quad \mathcal{R}^p := \mathcal{R} \cap {}^{\mathrm{sc}}\overline{T}^*_{\iota^-}X\,.$$

Both $\mathcal{R}^f$ and $\mathcal{R}^p$ consist of two connected components,

$$\mathcal{R}^f = \mathcal{R}^f_+ \sqcup \mathcal{R}^f_-\,, \quad \mathcal{R}^p = \mathcal{R}^p_+ \sqcup \mathcal{R}^p_-\,,$$

where

$$\mathcal{R}^\bullet_\pm := \mathcal{R}^\bullet \cap \{\pm\tau \geq m\}\,.$$

For example, it is easy to show that, with $w = (\zeta/\tau) + y$, then in the coordinates x, w, ρ, μ, in the region $t > 0, \tau > 0$,

$$\mathcal{R}^f_+ = \{x = 0,\ w = 0\},$$

where in these coordinates

$$(1/2)^{\mathrm{sc}}H_p = -x\partial_x - w \cdot \partial_w,$$

showing that $\mathcal{R}^f_+$ is a sink of the flow. The other three components are obtained similarly. Finally, we denote the sources and sinks by

$$\begin{aligned}\mathcal{R}_{\mathrm{src}} &:= \mathcal{R}^f_- \cup \mathcal{R}^p_+\,,\\ \mathcal{R}_{\mathrm{snk}} &:= \mathcal{R}^f_+ \cup \mathcal{R}^p_-\,.\end{aligned}$$

The main theorem regarding the global structure of the Hamiltonian flow on $\mathrm{Char}(P_0)$, which is central to the development of the theory below is the following.

Theorem 2.1 (cf. [2, Sect. 2.5])**.** *The sets $\mathcal{R}_{\mathrm{src}}$ and $\mathcal{R}_{\mathrm{snk}}$ are, respectively, global sources and sinks for the Hamiltonian flow on* $\mathrm{Char}(P_0)$*.*

2.2. **Microlocal cutoffs.** The definitions of the spaces we use to construct the Feynman propagator rely on microlocalization to the radial set, and we review the basic features of this briefly.

Recall from [19, Section 7], that for $\mathrm{Op}_L(a) = A \in {}^{\mathrm{sc}}\Psi^{m,r}$, the characteristic set is in general the vanishing locus of $j_{\mathrm{sc},m,r}(A)$ in $\partial^{\mathrm{sc}}\overline{T}^*X$, and the elliptic set is its complement, $\mathrm{Ell}(A) = \partial^{\mathrm{sc}}\overline{T}^*X \setminus \mathrm{Char}(A)$. The operator wavefront set $\mathrm{WF}'(A)$ is the essential support $\text{ess-supp}(a)$ of the symbol.

For any compact subset $K \subset \partial^{\mathrm{sc}}\overline{T}^*X$ and any open neighborhood $U \subset \partial^{\mathrm{sc}}\overline{T}^*X$ of K, a microlocal cutoff to K supported in U is a $Q \in {}^{\mathrm{sc}}\Psi^{0,0}$ such that

$$K \subset \mathrm{Ell}(Q) \text{ and } \mathrm{WF}'(Q) \subset U.$$

We often assume in addition to this that, for some open neighborhood V of K with $\overline{V} \subset U$, that

$$\mathrm{WF}'(I - Q) \cap \overline{V} = \varnothing,$$

which is to say that Q is microlocally equal to the identity near K.

Microlocal elliptic regularity (cf. [28, Corollary 5.5]) states that if $A \in {}^{\mathrm{sc}}\Psi^{m,r}$ and $B, G \in {}^{\mathrm{sc}}\Psi^{0,0}$ satisfy that $\mathrm{WF}'(G) \subset \mathrm{Ell}(A)$ and $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$, then for any $M, N \in \mathbb{R}$, there is $C > 0$ such that, for any $s, \ell \in \mathbb{R}$,

$$\|Bu\|_{s,\ell} \leq C\left(\|GAu\|_{s-m,\ell-r} + \|u\|_{-N,-M}\right). \tag{20}$$

Recall the scattering wavefront set of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^{n+1})$, defined by its complement:

$$\mathrm{WF}^{m,r}(u)^c = \{q \in \partial^{\mathrm{sc}}\overline{T}^* X \colon \exists A \in {}^{\mathrm{sc}}\Psi^{m,r},\ q \in \mathrm{Ell}(A),\ Au \in L^2\}.$$

Below we are interested in distributions u which lie globally in some Sobolev space $H^{s,\ell_0}_{\mathrm{sc}}$ with $\ell_0 < -1/2$ but lie in a better space, namely one with above threshold decay, near $\mathcal{R}_{\mathrm{src}}$. Specifically, we work with u which also satisfy, for some $\ell_+ > -1/2$, that $\mathrm{WF}^{s,\ell_+}(u) \cap \mathcal{R}_{\mathrm{src}} = \varnothing$. However, this condition is difficult to work with directly to obtain global Fredholm estimates. We therefore fix microlocal cutoffs to $\mathcal{R}_{\mathrm{src}}$ and work with spaces with manifest dependence on those cutoffs; we then show that the solutions obtained by that method do not depend on choices.

Microlocal cutoffs Q_{src} to $\mathcal{R}_{\mathrm{src}}$ are thus in particular operators in ${}^{\mathrm{sc}}\Psi^{0,0}$ such that $\mathcal{R}_{\mathrm{src}} \subset \mathrm{Ell}(Q_{\mathrm{src}})$. To construct them is straightforward given the fact that $\mathcal{R}_{\mathrm{src}}$ is a disjoint union of smooth submanifolds which intersect the corner transversely. Indeed, following the coordinate description above, the future radial source $\mathcal{R}^f_-$ is the vanishing locus of the smooth coordinate functions x, w in the region where $0 \le -1/\tau < C$ and $|\zeta/\tau| < C$. Thus a cutoff to this set is a quantization of a symbol

$$q_1 = \chi(x)\chi(|w|), \tag{21}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is a smooth bump function with $\chi(s) = 1$ for $|s| < \delta/2$ and $\chi(s) \equiv 0$ for $|s| \ge \delta$. If q_2 is the analogous symbol supported at the other component of the sources, $\mathcal{R}^p_+$, then we can use

$$Q_{\mathrm{src}} = \mathrm{Op}_L(q_1) + \mathrm{Op}_L(q_2). \tag{22}$$

Thus we can take cutoffs Q_{src} supported in arbitrarily small neighborhoods of $\mathcal{R}_{\mathrm{src}}$ by taking $\delta > 0$ small in this definition.

2.3. **Sobolev spaces for the Feynman problem.** Fix a microlocalizer to the source, Q_{src}. For $s, \ell_-, \ell_+ \in \mathbb{R}$ with $\ell_- < -1/2, \ell_+ > -1/2$, we estimate the quantity

$$\|u\|_{s,\ell_0} + \|Q_{\mathrm{src}} u\|_{s,\ell_+}$$

in terms of appropriate norms of $P_0 u$ and weaker norms of u. Note that the finiteness of the displayed quantity implies that $u \in H^{s,\ell_0}$ globally, and in addition that $\mathrm{WF}^{s,\ell_+}(u) \cap \mathcal{R}_{\mathrm{src}} = \varnothing$. In particular, as we discuss below, such a distribution u is above the threshold weight at the radial sources.

Digressing briefly, it is convenient to introduce notation for a Hilbert space with a norm equivalent to the sum of these norms of u just given. Indeed, for $\ell_0 < \ell_+$ and $A \in {}^{\mathrm{sc}}\Psi^{0,0}$ (and, in later sections, $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$), we set

$$H^{s,\ell_0,\ell_+}_A := \{u \in H^{s,\ell_0}_{\mathrm{sc}} \colon Au \in H^{s,\ell_+}_{\mathrm{sc}}\}\,. \tag{23}$$

This is complete with respect to the norm

$$\|u\|^2_{A,s,\ell_0,\ell_+} := \|u\|^2_{s,\ell_0} + \|Au\|^2_{s,\ell_+} .$$

We observe that

$$H^{s,\ell_+}_{\mathrm{sc}} \subset H^{s,\ell_0,\ell_+}_{A} \subset H^{s,\ell_0}_{\mathrm{sc}} .$$

To state the global Fredholm estimate, now assume that we are given an additional microlocalizer to the sources $Q'_{\mathrm{src}} \in {}^{\mathrm{sc}}\Psi^{0,0}$ such that

$$\mathcal{R}_{\mathrm{src}} \subset \mathrm{Ell}(Q_{\mathrm{src}}), \quad \mathrm{WF}'(Q_{\mathrm{src}}) \subset \mathrm{Ell}(Q'_{\mathrm{src}}), \quad \mathrm{WF}'(Q'_{\mathrm{src}}) \cap \mathcal{R}_{\mathrm{snk}} = \varnothing. \tag{24}$$

We must require, moreover, that $\mathrm{Ell}(Q'_{\mathrm{src}})$ contains all flow segments whose endpoints lie in $\mathrm{WF}'(Q_{\mathrm{src}})$. That is, if $q \in \mathrm{WF}'(Q_{\mathrm{src}}) \cap \mathrm{Char}(P_0)$, then we must have $\Phi_s(q) \in \mathrm{Ell}(Q'_{\mathrm{src}})$ for all $s < 0$. This holds automatically, for example, if $\mathcal{R}_{\mathrm{src}} \subset \mathrm{Ell}(Q_{\mathrm{src}})$ and $\mathrm{WF}'(Q_{\mathrm{src}})$ is convex under the flow on $\mathrm{Char}(P_0)$, as we will arrange in our construction. (This avoids the possibility of a flow line exiting both $\mathrm{WF}'(Q_{\mathrm{src}})$ and $\mathrm{Ell}(Q'_{\mathrm{src}})$, in which case propagation of singularities would not apply.)

With these assumptions on $Q_{\mathrm{src}}, Q'_{\mathrm{src}}$, we have a global estimate, which says that, for some $C > 0$,

$$\|u\|_{Q_{\mathrm{src}},s,\ell_0,\ell_+} \le C \left(\|P_0 u\|_{Q'_{\mathrm{src}},s-1,\ell_0+1,\ell_++1} + \|u\|_{-N,-M} \right). \tag{25}$$

We briefly discuss this estimate heuristically here, but we elide its formal proof as we prove a similar estimate below in the case $V \not\equiv 0$, see Lemma 5.2.

As with other estimates similar to (25), this estimate is obtained by combining three types of estimates which hold at different parts of phase space: 1) elliptic estimates, which are used away from the characteristic set, 2) real principal-type propagation estimates, which are used on the characteristic set away from the radial sets, and 3) radial points estimates, which hold near the radial set and themselves come in two varieties, one for above threshold decay and the other for below [2, Sect. 2.7 and Sect. 7].

The above threshold radial points estimates [2, Proposition 2.11] imply that, for any ℓ'_+ with $-1/2 < \ell'_+ < \ell_+$ and any $N \in \mathbb{R}$,

$$\|Q_{\mathrm{src}} u\|_{s,\ell_+} \lesssim \|Q'_{\mathrm{src}} P_0 u\|_{s-1,\ell_++1} + \|Q'_{\mathrm{src}} u\|_{-N,\ell'_+} + \|u\|_{-N,-M} \tag{26}$$

This can be read as saying that u is controlled in $H^{s,\ell_+}_{\mathrm{sc}}$ in some neighborhood U of $\mathcal{R}_{\mathrm{src}}$ by $P_0 u$ in $H^{s-1,\ell_++1}_{\mathrm{sc}}$ on a neighborhood U' of $\mathcal{R}_{\mathrm{src}}$ with $\overline{U} \subset U'$, provided u lies in an above threshold space near $\mathcal{R}_{\mathrm{src}}$. Obtaining the global estimate (25) from (26) and the other propagation estimates is at this point a standard argument; see, among many others, the arguments in [2, Sect. 8]. We also review this in greater detail in Section 5.

Following Vasy's development, natural Sobolev spaces corresponding to the global Fredholm estimates are those in which both the left and the right hand sides of (25) are finite. We thus define:

$$\mathcal{X}^{s,\ell_0,\ell_+} := \{u \in H^{s,\ell_0,\ell_+}_{Q_{\mathrm{src}}} \colon P_V u \in H^{s-1,\ell_0+1,\ell_++1}_{Q'_{\mathrm{src}}}\}, \quad \mathcal{Y}^{s,\ell_0,\ell_+} := H^{s,\ell_0,\ell_+}_{Q'_{\mathrm{src}}}.$$

We prefer the notation $\mathcal{X}^{s,\ell_0,\ell_+}$ to the more cumbersome $\mathcal{X}^{s,\ell_0,\ell_+}_{Q_{\mathrm{src}},Q'_{\mathrm{src}}}$ despite the obvious dependence of these spaces on the choices $Q_{\mathrm{src}}, Q'_{\mathrm{src}}$. We prove the following:

Theorem 2.2. *For $s, \ell_0, \ell_+ \in \mathbb{R}$, $\ell_0 < -1/2, \ell_+ > -1/2$, and $Q_{\mathrm{src}}, Q'_{\mathrm{src}} \in {}^{\mathrm{sc}}\Psi^{0,0}$ satisfying (24). Then the map*

$$P_0 \colon \mathcal{X}^{s,\ell_0,\ell_+} \longrightarrow \mathcal{Y}^{s-1,\ell_0+1,\ell_++1} \tag{27}$$

is Fredholm.

Proof. This proof is covered in detail in Section 5 below in the more general case of nonzero potential V. In brief summary, the fact that the operator is bounded follows immediately from the definitions of the spaces. Moreover, the fact that the mapping has closed range and finite dimensional kernel follows from the estimate (25) by a standard argument using the compactness of $H^{s,\ell_0,\ell_+}_{Q_{\mathrm{src}}}$ in $H^{-N,-M}_{\mathrm{sc}}$ for sufficiently large M, N. The cokernel is then identified with the kernel of P_0^* with domain $(\mathcal{Y}^{s,\ell_0,\ell_+})'$, whose elements have above threshold decay at $\mathcal{R}_{\mathrm{snk}}$. Similar estimates apply to those distributions, giving the finite dimensionality by the same compactness argument. □

2.4. **The Feynman propagator and the wavefront set condition.** The fact that the mapping (27) is in fact invertible (seen below in Section 5) provides the basis of our definition of the Feynman propagator. However, to uniquely define the Feynman propagator, we must ensure that the solutions u to $P_0 u = f$ obtained by applying the inverse mapping in (27) do not depend on any choices made in the construction. Specifically, a given f lies in many spaces $\mathcal{Y}^{s,\ell_0,\ell_+} = H^{s,\ell_0,\ell_+}_{Q'_{\mathrm{src}}}$ for different values of s, ℓ_0, ℓ_+ and different choices of cutoff Q'_{src}, and we must show that the inverse mapping gives the same distribution independent of these choices and the choice of Q_{src} in the definition of $\mathcal{X}^{s,\ell_0,\ell_+}$. This, and more, is summarized in the following theorem.

Theorem 2.3. *Under the assumptions of Theorem 2.2, the mapping (27) is invertible.*

The inverse $(P_0)^{-1}_{\mathrm{Fey}}$ is defined independently of the parameters s, ℓ_0, ℓ_+ and the cutoffs $Q_{\mathrm{src}}, Q'_{\mathrm{src}}$ in the following sense. Let $u_1, u_2 \in \mathcal{S}'$ satisfy

$$u_i \in H^{s_i,\ell_{0,i},\ell_{+,i}}_{Q_{\mathrm{src},i}} \text{ for some } s_i, \ell_{0,i}, \ell_{+,i} \in \mathbb{R}, \ell_{+,i} > -1/2,$$

with $\mathcal{R}_{\mathrm{src}} \subset \mathrm{Ell}(Q_{\mathrm{src},i})$. Then

$$P_0 u_1 = P_0 u_2 \implies u_1 = u_2.$$

In particular, the mapping

$$(P_0)^{-1}_{\mathrm{Fey}} \colon \mathcal{C}_c^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X) \tag{28}$$

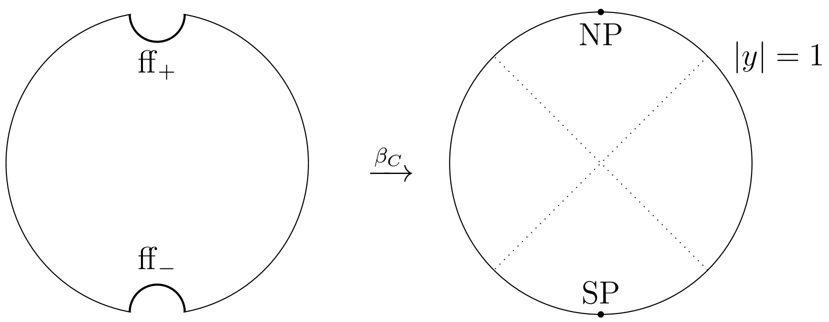

FIGURE 1. The blow-down map $\beta_C : [X; C] \to X$.

is well defined and satisfies the following Hadamard property; for $f \in \mathcal{C}_c^{-\infty}$,

$$\mathrm{WF}((P_0)^{-1}_{\mathrm{Fey}} f) \subset \mathrm{WF}(f) \cup \bigcup_{s \geq 0} \Phi_s \left(\mathrm{WF}(f) \cap \mathrm{Char}(P_0)\right) \cup \mathcal{R}_{\mathrm{snk}} . \tag{29}$$

where Φ_s *is the Hamiltonian flow on* $\partial^{\mathrm{sc}}\overline{T}^* X$.

3. 3sc-calculus for the Klein–Gordon operator

3.1. **Basics of the** 3sc-**calculus.** In this section, we recall the basics of the 3sc-calculus first introduced by Vasy [24] with adaptions to treat the Klein–Gordon equation in [2]. On the spacetime compactification $X = \overline{\mathbb{R}^{n+1}}$ discussed in Section 2.1, we set $C = \{\mathrm{NP}, \mathrm{SP}\}$, where

$$\mathrm{NP} = \partial X \cap \overline{\{z = 0\}} \cap \overline{\{t > 0\}} , \quad \mathrm{SP} = \partial X \cap \overline{\{z = 0\}} \cap \overline{\{t < 0\}} .$$

Thus $\mathrm{NP} \in \iota^+$ and $\mathrm{SP} \in \iota^-$, and due to their placement we refer to them as the north and south poles, respectively.

To use Vasy's three-body framework, we consider the blown-up space $[X; C]$ with the canonical blow-down map

$$\beta_C : [X; C] \to X.$$

The space $[X; C]$ is a manifold with corners with three boundary hypersurfaces,

$$\mathrm{ff}_+ := \beta_C^*(\mathrm{NP}) , \quad \mathrm{ff}_- := \beta_C^*(\mathrm{SP}) , \quad \mathrm{mf} := \beta_C^*(\partial X) ,$$

depicted in Figure 1. The resolution $[X; C]$ has the important property that, while the potential V defined in (1) is not smooth on X, it is smooth on $[X; C]$.

Our analysis below is based on the realization of P_V as a 3sc-operator. To review, the space of differential operators in the 3sc-calculus is given by

$$\mathrm{Diff}^m_{3\mathrm{sc}}(X) := \mathrm{Diff}^m_{\mathrm{sc}}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty([X; C]) .$$

Thus, a differential operator L in $\mathrm{Diff}^m_{3\mathrm{sc}}(X)$ is given by

$$L = \sum_{|\alpha|+k\le m} a_{k,\alpha} D_t^k D_z^\alpha \,,$$

where $a_{k,\alpha} \in \mathcal{C}^\infty([X;C])$. It is easy to verify that $P_V \in \mathrm{Diff}^2_{3\mathrm{sc}}(X)$ if $V \in S^{-1}(\mathbb{R}^n_z)$. More generally, we define weighted differential operators as

$$\mathrm{Diff}^{m,r}_{3\mathrm{sc}}(X) := \langle t,z\rangle^r \,\mathrm{Diff}^m_{3\mathrm{sc}}(X)\,.$$

The compactified 3sc-cotangent bundle is defined to be the pullback bundle

$${}^{3\mathrm{sc}}\overline{T}^*[X;C] := \beta_C^* \,{}^{\mathrm{sc}}\overline{T}^*X = [X;C] \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}}\,.$$

This is a manifold with corners with *four* boundary hypersurfaces,

$${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}_\pm}[X;C]\,, \quad {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]\,, \quad {}^{3\mathrm{sc}}S^*[X;C]\,,$$

with the latter being the fiber boundary $[X;C] \times \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}}$, which can be identified with the sphere bundle of ${}^{3\mathrm{sc}}T^*[X;C]$. The corresponding boundary defining functions are denoted by

$$\rho_{\mathrm{ff}_\pm}\,, \quad \rho_{\mathrm{mf}}\,, \quad \rho_{\mathrm{fib}}\,.$$

Moreover, we define the functions

$$\rho_{\mathrm{ff}} := \rho_{\mathrm{ff}_+}\rho_{\mathrm{ff}_-}\,, \quad \rho_\infty := \rho_{\mathrm{ff}}\rho_{\mathrm{mf}}\,.$$

Here $\rho_\infty = \beta_C^*\rho_{\mathrm{base}}$ defines the boundary of $[X;C]$.

The space of classical 3sc-symbols is

$${}^{3\mathrm{sc}}S^{m,r}(X;C) := \rho_{\mathrm{fib}}^{-m}\rho_\infty^{-r}\mathcal{C}^\infty({}^{3\mathrm{sc}}\overline{T}^*[X;C])\,.$$

and the 3sc-pseudodifferential operators are quantizations of such symbols,

$${}^{3\mathrm{sc}}\Psi^{m,r} := \mathrm{Op}_L({}^{3\mathrm{sc}}S^{m,r})\,.$$

Let $A \in {}^{3\mathrm{sc}}\Psi^{m,r}$ and $s,\ell \in \mathbb{R}$, then ([24, Cor. 8.2])

$$A : H^{m+s,r+\ell}_{\mathrm{sc}}(X) \to H^{s,\ell}_{\mathrm{sc}}(X)$$

is a bounded mapping. Below, we recall additional facts about 3sc-pseudodifferential operators necessary for the statements and proofs of our estimates. Our first goal, however, is to recall the principal symbol construction, which differs from that of the scattering calculus as it includes the indicial operator.

If $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{0,0}$, then we define the symbols over mf and fiber infinity as

$$\hat{N}_{\mathrm{mf}}(A) = a|_{{}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]} \in \mathcal{C}^\infty({}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C])$$

and

$$\sigma_{3\mathrm{sc}}(A) = a|_{{}^{3\mathrm{sc}}S^*[X;C]} \in \mathcal{C}^\infty({}^{3\mathrm{sc}}S^*[X;C])\,.$$

In particular, away from $\mathrm{ff}_\pm$, the symbol of a 3sc operator is equal to its symbol in the standard scattering sense.

The indicial operator $\hat{N}_{\mathrm{ff}}$ is a family of operators parametrized by the vector bundle

$$W^\perp := \operatorname{span}_{\mathbb{R}}\left(\frac{dx}{x^2}\right) \subset {}^{\mathrm{sc}}T^*_C X$$

and we have the orthogonal projection

$$\pi^\perp : {}^{\mathrm{sc}}T^*_C X \to W^\perp .$$

The space $W^\perp$ is simply two copies of the line $\mathbb{R}_\tau$ corresponding to the forms τdt over NP and SP and we write $W^\perp_\pm$ for the restriction of $W^\perp$ to NP and SP, respectively. We also need the compactification of $W^\perp$,

$$\overline{W^\perp} \cong \{\pm\infty\} \times \overline{\mathbb{R}} .$$

We describe below that the indicial operator is a semiclassical scattering operator with semiclassical parameter $1/|\tau|$ as $\tau \to \pm\infty$, and its semiclassical principal symbol is equal to its total symbol's value at $\tau = \pm\infty$.

Given $A \in {}^{3\mathrm{sc}}\Psi^{m,0}$, we now recall the indicial operator $\hat{N}_{\mathrm{ff}}(A)$ (cf. [24, Chap. 6], [2, Sect. 4.2]). As the constructions at $\mathrm{ff}_\pm$ are identical, we work at ff_+ and write $\mathrm{ff} = \mathrm{ff}_+$. We reintroduce the $\pm$ notation when it is useful below. Each $A \in {}^{3\mathrm{sc}}\Psi^{m,0}$ defines an operator $A_{\mathrm{ff}} \in {}^{\mathrm{sc}}\Psi^{m,0}(\mathrm{ff})$ by

$$A_{\mathrm{ff}}(f) = (Au)_{\mathrm{ff}} ,$$

where $u \in \mathcal{C}^\infty(X)$ is any extension of f in the sense that $u|_{\mathrm{ff}} = f$. The indicial operator on ff is then defined by

$$\hat{N}_{\mathrm{ff}}(A)(\tau_0) := \left(e^{i\tau_0/x} A e^{-i\tau_0/x}\right)_{\mathrm{ff}} .$$

As an example, we recall that if $P_V = D_t^2 - (\Delta + m^2 + V)$ and $V_0 \in S^{-1}(\mathbb{R}^n_z)$ with $V(t,z) - V_0(z) \in {}^{3\mathrm{sc}}\Psi^{0,-1}$, then

$$\hat{N}_{\mathrm{ff}}(P_V)(\tau) = \tau^2 - H_{V_0} ,$$

so $\hat{N}_{\mathrm{ff}}(P_V)$ is the partial Fourier transform of P_{V_0} in the t variable.

The principal symbol of an operator $A \in {}^{3\mathrm{sc}}\Psi^{m,r}$ is described in detail in our previous paper [2, Sect. 4]. It consists of two "local" components, which are rescaled restrictions of the symbol of A to the boundary components ${}^{3\mathrm{sc}}S^*[X;C]$ and ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]$, respectively. The former is the restriction to fiber infinity, and its restriction to the interior $X^\circ = \mathbb{R}^{n+1}$ is the standard principal symbol. The rescaled restriction to ${}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]$ corresponds with the spacetime infinity scattering principal symbol, i.e., it is the sc, as opposed to

3sc, symbol, which is well defined on mf°. We use $\rho_{\mathrm{fib}} = \langle \tau, \zeta \rangle^{-1}$ and $\rho_{\mathrm{base}} = x = 1/t$:

$$\begin{aligned} \hat{N}_{\mathrm{mf},m,r}(A) &:= \langle \tau, \zeta \rangle^{-m} x^r a|_{^{3\mathrm{sc}}\overline{T}^*_{\mathrm{mf}}[X;C]} , \\ \sigma_{3\mathrm{sc},m,r}(A) &:= \langle \tau, \zeta \rangle^{-m} x^r a|_{^{3\mathrm{sc}}S^*[X;C]} , \\ \hat{N}_{\mathrm{ff},\pm,r}(A) &:= \hat{N}_{\mathrm{ff},\pm}(x^r A) . \end{aligned}$$

The 3sc-principal symbol consists of these four components, and we denote the principal symbol mapping by

$$\begin{aligned} j_{3\mathrm{sc},m,r} : {}^{3\mathrm{sc}}\Psi^{m,r} &\to \mathcal{C}^\infty({}^{3\mathrm{sc}}T^*_{\mathrm{mf}}[X;C]) \times \mathcal{C}^\infty({}^{3\mathrm{sc}}S^*[X;C]) \times \mathcal{C}^\infty(\mathbb{R}_\tau; {}^{\mathrm{sc}}\Psi^{m,0}(\mathrm{ff}_\pm)) \\ A &\mapsto \left(\sigma_{3\mathrm{sc},m,r}(A), \hat{N}_{\mathrm{mf},m,r}(A), \hat{N}_{\mathrm{ff},\pm,r}(A) \right) . \end{aligned}$$

The indicial operator is equal to the quantization of the restriction of the symbol of A to the slice τ. Specifically, if $a \in {}^{3\mathrm{sc}}S^{m,r}$ satisfies, at $\mathrm{ff} = \mathrm{ff}_+$,

$$a_{\mathrm{ff}} := (x^r a)|_{^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C]},$$

then, [2, Lemma 4.4]

$$\hat{N}_{\mathrm{ff},r}(A)(\tau) = \mathrm{Op}_{L,z}(a_{\mathrm{ff}}(z, \tau, \zeta)). \tag{30}$$

Thus, the principal symbol satisfies the obvious matching conditions, namely that the restriction of each component of the symbol the boundary of its domain matches the restriction of the other components of the symbol there. Matching is the only condition required for quantization. This is summarized in the following proposition.

Proposition 3.1 ([2, Proposition 4.6 and Proposition 4.8])**.** *The kernel of the principal symbol mapping is* ${}^{3\mathrm{sc}}\Psi^{m-1,r-1}$ *and the image is the set of those* $(q_1, q_2, \{Q_\tau\})$ *such that if* q_0 *denotes the left reduction of* Q_τ*, we have* $\langle \tau, \zeta \rangle^{-m} q_0 \in \mathcal{C}^\infty(\mathrm{ff} \times \overline{\mathbb{R}^{n+1}_{\tau,\zeta}})$ *and the matching conditions*

$$\begin{aligned} q_1|_{^{3\mathrm{sc}}S^*_{\mathrm{mf}}[X;C]} = q_2|_{^{3\mathrm{sc}}S^*_{\mathrm{mf}}[X;C]} , \quad \langle \tau, \zeta \rangle^{-m} q_0|_{^{3\mathrm{sc}}S^*_{\mathrm{ff}}[X;C]} = q_1|_{^{3\mathrm{sc}}S^*_{\mathrm{ff}}[X;C]} , \\ \langle \tau, \zeta \rangle^{-m} q_0|_{^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}\,\cap\,\mathrm{mf}}[X;C]} = q_1|_{^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}\,\cap\,\mathrm{mf}}[X;C]} \end{aligned}$$

hold.

Moreover, $j_{3\mathrm{sc}}$ *is multiplicative in the sense that if* $A \in {}^{3\mathrm{sc}}\Psi^{m_1,r_1}$ *and* $B \in {}^{3\mathrm{sc}}\Psi^{m_2,r_2}$*, then* $AB \in {}^{3\mathrm{sc}}\Psi^{m_1+m_2,r_1+r_2}$ *and*

$$j_{3\mathrm{sc},m_1+m_2,r_1+r_2}(AB) = j_{3\mathrm{sc},m_1,r_1}(A) j_{3\mathrm{sc},m_2,r_2}(B) .$$

3.2. **Elliptic and wavefront sets.** The appropriate notions of operator wavefront set, elliptic set, and characteristic set, are influenced by the spatially global nature of the indicial operator. In particular, given $A \in {}^{3\mathrm{sc}}\Psi^{m,r}$ and $\tau \in W^\perp$, we have: (1) $\tau_0 \notin {}^{3\mathrm{sc}}\mathrm{WF}(A)$ if, for some $\epsilon > 0$, the total symbol of A vanishes to all orders at the entire lens of slices $\mathrm{ff} \times \{(\tau, \zeta) : |\tau - \tau_0| < \epsilon\} \subset {}^{3\mathrm{sc}}\overline{T}^*_{\mathrm{ff}}[X;C]$, and (2) $\tau_0 \in {}^{3\mathrm{sc}}\mathrm{Ell}(A)$ if $\hat{N}_{\mathrm{ff},r}(A)(\tau_0)$ is an invertible operator.

Moreover, as $\tau \to \pm\infty$, $\hat{N}_{\mathrm{ff},r}(A)(\tau_0)$ is semiclassical in $h = 1/|\tau|$, and its principal symbol at $\tau \to \pm\infty$ is the restriction of the total symbol to the "hemisphere"

$$UH_\pm = {}^{3\mathrm{sc}}S^*_{\mathrm{ff}}[X;C] \cap \mathrm{cl}(\{\pm\tau > 0\}).$$

In particular, as $\tau \to +\infty$,

$$\sigma_{\mathrm{scl},h=1/\tau}(\hat{N}_{\mathrm{ff},r}(A)(1/h)) = \langle \tau, \zeta\rangle^{-m} a_{\mathrm{ff}}|_{UH_+}. \tag{31}$$

At a basic level, the principal symbol is a function on a compactification of the compressed cotangent bundle ${}^{\mathrm{sc}}\dot{T}^*X = ({}^{\mathrm{sc}}T^*X \setminus {}^{\mathrm{sc}}T^*_C X) \cup W^\perp$, specifically on

$${}^{\mathrm{sc}}\dot{\overline{T}}^* X = ({}^{\mathrm{sc}}\overline{T}^*X \setminus {}^{\mathrm{sc}}\overline{T}^*_C X) \cup \overline{W^\perp}.$$

The domain of principal symbol, $\dot{C}_{3\mathrm{sc}}[X;C]$, is a subset of the boundary faces of ${}^{\mathrm{sc}}\dot{\overline{T}}^* X$:

$$\dot{C}_{3\mathrm{sc}}[X;C] := {}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X \cup \overline{W^\perp}.$$

To describe the operator wavefront set, we define a mapping $\gamma_{3\mathrm{sc}}$, which associates to each point in the domain of the principal symbol the appropriate subset of phase space which determines the symbolic behavior:

$$\gamma_{3\mathrm{sc}} : \dot{C}_{3\mathrm{sc}}[X;C] \longrightarrow \mathcal{P}(\partial\, {}^{3\mathrm{sc}}\overline{T}^*[X;C]) \tag{32}$$

as

$$\begin{aligned} \gamma_{3\mathrm{sc}}(p) &= \{p\} && \text{for } p \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X\,, \\ \gamma_{3\mathrm{sc}}(\tau) &= \beta_C^{-1}(\pi^\perp)^{-1}\{\tau\} && \text{for } \tau \in W^\perp\,, \\ \gamma_{3\mathrm{sc}}(\pm\infty) &= UH_\pm && \text{for } \pm\infty \in \partial\overline{W^\perp}\,. \end{aligned}$$

In particular, $\gamma_{3\mathrm{sc}}$ maps $\tau_0 \in W^\perp_+$ to the entire slice $\mathrm{ff}_+ \times \overline{\{(\tau_0,\zeta) \colon \zeta \in \mathbb{R}^n\}}$. By abuse of notation, for a set $S \subset \dot{C}_{3\mathrm{sc}}[X;C]$ we write

$$\gamma_{3\mathrm{sc}}(S) := \bigcup_{p\in S} \gamma_{3\mathrm{sc}}(p)\,. \tag{33}$$

We used above that ${}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X$ is naturally identified with ${}^{3\mathrm{sc}}S^*_{[X;C]\setminus\mathrm{ff}}[X;C] \cup {}^{3\mathrm{sc}}\overline{T}^*_{\partial[X;C]\setminus\mathrm{ff}}[X;C]$ via the blow down map.

For a symbol $a \in {}^{3\mathrm{sc}}S^{m,r}$, we define the essential support of a, ${}^{3\mathrm{sc}}\text{ess-supp}(a) \subset \partial\,{}^{3\mathrm{sc}}\overline{T}^*[X;C]$, by declaring $p \in {}^{3\mathrm{sc}}\text{ess-supp}(a)^c$ if and only if there exists $U \subset {}^{3\mathrm{sc}}\overline{T}^*[X;C]$ open and $\chi \in \mathcal{C}^\infty_c({}^{3\mathrm{sc}}\overline{T}^*[X;C])$ such that $p \in U$, $\chi|_U \equiv 1$ and $\chi a \in {}^{3\mathrm{sc}}S^{-\infty,-\infty}$.

Definition 3.2. *Let $A = \mathrm{Op}_L(a) \in {}^{3\mathrm{sc}}\Psi^{m,r}(X)$. The operator wavefront set*

$${}^{3\mathrm{sc}}\mathrm{WF}'(A) \subset \dot{C}_{3\mathrm{sc}}[X;C]$$

is defined as follows: a point $p \in \dot{C}_{3\mathrm{sc}}[X;C]$ is not in the wavefront set,

$$p \in {}^{3\mathrm{sc}}\mathrm{WF}'(A)^c \text{ if and only if } \gamma_{3\mathrm{sc}}(p) \cap {}^{3\mathrm{sc}}\text{ess-supp}(a) = \varnothing\,.$$

Moreover, we define

$$\begin{aligned}
\mathrm{WF}'_{\mathrm{fib}}(A) &:= {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{\mathrm{sc}}S^*_{X\setminus C}X\,,\\
\mathrm{WF}'_{\mathrm{mf}}(A) &:= {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X\,,\\
\mathrm{WF}'_{\mathrm{ff}}(A) &:= {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap \overline{W^\perp}\,.
\end{aligned}$$

We can write the complements of each of the components as

$$\begin{aligned}
\mathrm{WF}'_{\mathrm{fib}}(A)^c = \{&\alpha \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \colon \exists U \subset {}^{\mathrm{sc}}S^*_{X\setminus C}X \text{ open such that } \alpha \in U\\
&\text{and } a(A) \text{ vanishes to infinite order on } \overline{U}\}\,,\\
\mathrm{WF}'_{\mathrm{mf}}(A)^c = \{&\alpha \in {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \colon \exists U \subset {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \text{ open such that } \alpha \in U\\
&\text{and } a(A) \text{ vanishes to infinite order on } \overline{U}\}\,,\\
\mathrm{WF}'_{\mathrm{ff}}(A)^c = \{&\tau \in W^\perp \colon \exists\ \epsilon > 0 \text{ such that } a(A) \text{ vanishes to}\\
&\text{infinite order on } \beta_C^{-1}(\pi^\perp)^{-1}[\tau - \epsilon, \tau + \epsilon]\}\\
&\cup \{\pm\infty : \exists \text{ open } U \subset \partial^{3\mathrm{sc}}\overline{T}^*[X;C] \text{ such that } UH_\pm \subset U\\
&\text{and } a(A) \text{ vanishes to infinite order on } \overline{U}\}.
\end{aligned}$$

Lemma 3.3. *Let* $A \in {}^{3\mathrm{sc}}\Psi^{m_1,r_1}$*,* $B \in {}^{3\mathrm{sc}}\Psi^{m_2,r_2}$*, then*

$$\begin{aligned}
{}^{3\mathrm{sc}}\mathrm{WF}'(AB) &\subset {}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{3\mathrm{sc}}\mathrm{WF}'(B)\,,\\
{}^{3\mathrm{sc}}\mathrm{WF}'(A) &= \varnothing \Rightarrow A \in {}^{3\mathrm{sc}}\Psi^{-\infty,-\infty}\,.
\end{aligned}$$

Proof. The first property follows from microlocality of the composition and the second claim easily follows from the fact that $\gamma_{3\mathrm{sc}}$ is surjective meaning that $\gamma_{3\mathrm{sc}}(\dot{C}_{3\mathrm{sc}}[X;C]) = \partial^{3\mathrm{sc}}\overline{T}^*[X;C]$. $\square$

Now we define the elliptic sets. Over mf and fiber infinity, the definition of ellipticity is exactly as in the standard scattering case, i.e., non-vanishing (or, for operators acting on sections of vector bundles, invertibility) of the principal symbol. Over ff in $W^\perp$, the correct notion of ellipticity is invertibility between appropriate scattering Sobolev spaces.

To define the elliptic set, we note that the two components of the symbol $\sigma_{3\mathrm{sc},m,r}(A)$ and $\hat{N}_{\mathrm{mf},m,r}(A)$ define, by restriction, functions on ${}^{\mathrm{sc}}S^*_{X\setminus C}X$ and ${}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X$, respectively.

Definition 3.4. *Let* $A \in {}^{3\mathrm{sc}}\Psi^{m,r}$*. The* 3sc*-elliptic set* ${}^{3\mathrm{sc}}\mathrm{Ell}(A)$ *is*

$$^{3\mathrm{sc}}\mathrm{Ell}(A) = \mathrm{Ell}_{\mathrm{fib}}(A) \cup \mathrm{Ell}_{\mathrm{mf}}(A) \cup \mathrm{Ell}_{\mathrm{ff}}(A) \subset \dot{C}_{3\mathrm{sc}}[X;C]\,,$$

with

$$\mathrm{Ell}_{\mathrm{fib}}(A) = \{\alpha \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \colon \sigma_{3\mathrm{sc},m,r}(A)(\alpha) \neq 0\}\,,$$
$$\mathrm{Ell}_{\mathrm{mf}}(A) = \{\alpha \in {}^{\mathrm{sc}}\overline{T}^*_{\partial X\setminus C}X \colon \hat{N}_{\mathrm{mf},m,r}(A)(\alpha) \neq 0\}\,,$$

while

$$\mathrm{Ell}_{\mathrm{ff}}(A) = \{\tau \in W^\perp \colon \hat{N}_{\mathrm{ff},r}(A)(\tau) \text{ is scattering elliptic and invertible}\}$$
$$\cup \{\pm\infty \in \partial\overline{W^\perp} : \sigma_{3\mathrm{sc},m,r}(A) \text{ is nowhere vanishing on } UH_\pm\}.$$

Moreover, we set

$$ {}^{3\mathrm{sc}}\mathrm{Char}(A) \coloneqq \dot{C}_{3\mathrm{sc}}[X;C] \setminus {}^{3\mathrm{sc}}\mathrm{Ell}(A)\,, \tag{34}$$

the 3sc*-characteristic set of* A.

As expected, one obtains elliptic estimates on the elliptic set.

Proposition 3.5 (3sc-elliptic regularity, cf. [2, Prop. 4.20], [24, Lemma 9.3])**.** *Let* $u \in \mathcal{S}'$ *and* $A \in {}^{3\mathrm{sc}}\Psi^{m,r}$*,* $B, Q' \in {}^{3\mathrm{sc}}\Psi^{0,0}$*. Assume that* ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(A) \cap {}^{3\mathrm{sc}}\mathrm{Ell}(Q')$*. For* $s, \ell \in \mathbb{R}$*, if* $Q'Au \in H^{s-m,\ell-r}_{\mathrm{sc}}$*, then* $Bu \in H^{s,\ell}_{\mathrm{sc}}$ *and for any* $M, N \in \mathbb{R}$ *there is* $C > 0$ *such that*

$$\|Bu\|_{s,\ell} \leq C\left(\|Q'Au\|_{s-m,\ell-r} + \|u\|_{-N,-M}\right).$$

In order to state a wavefront set condition for the Feynman propagator, we have to define the 3sc-wavefront set. Our definition is very similar to the one in [24], with the modification that we use $\dot{C}_{3\mathrm{sc}}[X;C]$. We note that there is a related wavefront set defined in [25], which is slightly weaker.

Definition 3.6. *Let* $u \in \dot{\mathcal{C}}^{-\infty}(X)$*. A point* $\zeta \in \dot{C}_{3\mathrm{sc}}[X;C]$ *is not in the* 3sc*-wavefront set,* $\zeta \notin {}^{3\mathrm{sc}}\mathrm{WF}(u)$*, if there exists* $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$ *with* $\zeta \in {}^{3\mathrm{sc}}\mathrm{Ell}(A)$ *such that* $Au \in \dot{\mathcal{C}}^\infty(X)$*.*

For $\zeta \in {}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X$, the 3sc-wavefront set ${}^{3\mathrm{sc}}\mathrm{WF}(u)$ coincides with the normal (scattering-) wavefront set $\mathrm{WF}(u)$. Over the poles, we have that if $\tau_0 \in W^\perp$ with $\mathrm{WF}(u) \cap \overline{(\pi^\perp)^{-1}(\tau_0)} = \varnothing$, then $\tau_0 \notin \mathrm{WF}_{\mathrm{ff}}(u)$, where $\mathrm{WF}_{\mathrm{ff}}(u) \coloneqq {}^{3\mathrm{sc}}\mathrm{WF}(u) \cap \overline{W^\perp}$.

4. 3sc-localizers

As in the scattering case in Section 2, we define a domain $\mathcal{X}$ for the operator P_V, which will become the range of the Feynman propagator, and this domain is again defined using microlocalizers to the radial sources analogous with those in equation (22). In this section we describe these microlocalizers and record several necessary modifications of estimates in our previous work [2].

4.1. **Localization to the characteristic set.** Following Vasy [24], we use the functional calculus as in our previous work [2, Section 5] to adapt the microlocalizers to the radial sets over the poles.

We first describe the functions of the operators used in the construction. Let $V_0 \in {}^{\mathrm{sc}}\Psi^{1,-1}(\mathbb{R}^n)$, $V_0^* = V_0$. For $E > 0$ sufficiently large and $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, the operator

$$G_{\psi,V_0} := \psi\left((D_t^2 + H_{V_0} + E)^{-1} P_{V_0}\right)$$

is well-defined by the functional calculus of the (static) operator H_{V_0}, $G_{\psi,V_0} \in {}^{3\mathrm{sc}}\Psi^{0,0}$, and satisfies (cf. [2, Sect. 5.1])

- $\hat{N}_{\mathrm{ff}}(G_{\psi,V_0})(\tau) = \psi\left((\tau^2 + H_{V_0} + E)^{-1}(\tau^2 - H_{V_0})\right)$,
- $\hat{N}_{\mathrm{ff}}(G_{\psi,V_0}) \in \Psi^{-\infty,0,0}_{\mathrm{scl,sc}}$.

In [2], we only used G_{ψ,V_0} near the poles, now we need a global operator G_ψ that coincides with G_{ψ,V_+} near NP and G_{ψ,V_-} near SP.

Definition 4.1. *Let $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ and $r \in (0,\infty)$. We say that V is asymptotically static of order r at C, if there exist $V_\pm \in S^{-1}_{\mathrm{cl}}(\mathbb{R}^n_z)$ and $\chi_\pm \in \mathcal{C}_c^\infty(X)$ such that*

(1) $\chi_+(\mathrm{NP}) = 1$ and $\chi_-(\mathrm{SP}) = 1$,
(2) $\chi_\pm \cdot (V - V_\pm) \in {}^{3\mathrm{sc}}\Psi^{1,-r}$.

We note that if V is asymptotically static of order $r > 0$, then the static parts $V_\pm$ are uniquely determined.

For the definition of G_ψ, we choose $\chi_\pm$ as in the previous definition and with the additional property that $\chi_+ = 0$ in a neighborhood of SP and $\chi_- = 0$ in a neighborhood of NP. We set

$$G_\psi := \chi_+ G_{\psi,V_+} + \chi_- G_{\psi,V_-} + (1 - \chi_+ - \chi_-)G_{\psi,0}\,.$$

We have that $G_\psi \in {}^{3\mathrm{sc}}\Psi^{0,0}$ and since by [2, Eq. (5.8) and Eq. (5.9)] the mf and fib-symbols of G_{ψ,V_0} are independent of V_0, we have that G_ψ is defined independently of the choice of $\chi_\pm$ up to ${}^{3\mathrm{sc}}\Psi^{-1,-1}$.

We can estimate Bu by $BG_\psi u$ and $G' P_V u$ if G' is elliptic on the wavefront set of B:

Proposition 4.2 (cf. [2, Proposition 5.14]). *Let $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ be asymptotically static of order r. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\varphi|_{(-\varepsilon,\varepsilon)} \equiv 1$ for some $\varepsilon > 0$, $B, G', B' \in {}^{3\mathrm{sc}}\Psi^{0,0}$ such that*

$${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G') \cap {}^{3\mathrm{sc}}\mathrm{Ell}(B')\,.$$

For all $M, N \in \mathbb{N}$ and $s, \ell \in \mathbb{R}$ there exists $C > 0$ such that for all $u \in H_{\mathrm{sc}}^{-N,-M}$,

$$\|Bu\|_{s,\ell} \le C\left(\|BG_\varphi u\|_{s,\ell} + \|G' P_V u\|_{s-2,\ell} + \|B'u\|_{s-1,\ell-r} + \|u\|_{-N,-M}\right)\,,$$

provided that the right hand side is finite.

Proof. Recall [2, Proposition 5.14] that we have $E_{\psi,V_0} \in {}^{3\mathrm{sc}}\Psi^{-2,0}$ which satisfies $(\mathrm{Id} - G_{\psi,V_0}) = E_{\psi,V_0} P_{V_0}$. We set

$$\tilde{E} = \chi_+ E_{\psi,V_+} + \chi_- E_{\psi,V_-}$$

and

$$R = \chi_+ E_{\psi,V_+}(P_{V_+} - P_V) + \chi_- E_{\psi,V_-}(P_{V_-} - P_V) \in {}^{3\mathrm{sc}}\Psi^{-1,-r}$$

Therefore, we have that

$$\mathrm{Id} = G_\psi + \tilde{E} P_V + R + (1 - \chi_+ - \chi_-)(\mathrm{Id} - G_{\psi,0})$$

Since ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G')$, we obtain

$$\|B\tilde{E}P_V u\|_{s,\ell} \lesssim \|G' P_V u\|_{s-2,\ell} + \|u\|_{-N,-M},$$

while ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B')$ implies

$$\|BRu\|_{s,\ell} \lesssim \|B'u\|_{s-1,\ell-r} + \|u\|_{-N,-M},$$

and, since P_V is elliptic on the microsupport of $(1 - \chi_+ - \chi_-)(\mathrm{Id} - G_{\psi,0}) \in {}^{\mathrm{sc}}\Psi^{0,0}$,

$$\|B(1 - \chi_+ - \chi_-)(\mathrm{Id} - G_{\psi,0})u\|_{s,\ell} \lesssim \|G' P_V u\|_{s-2,\ell} + \|u\|_{-N,-M}.$$

Combining these estimates gives the claimed inequality. □

Moreover, we have an elliptic estimate for $BG_\varphi u$ by $QG_\psi u$ given that Q is elliptic on the wavefront set of BG_φ and the support of φ is contained in the set $\{\psi = 1\}$.

Proposition 4.3. *Let $B, Q \in {}^{3\mathrm{sc}}\Psi^{m,r}$ and $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\varphi\psi = \varphi$ and $\varphi(0) = 1$. If ${}^{3\mathrm{sc}}\mathrm{WF}'(BG_\varphi) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(Q)$, then for any $N, M \in \mathbb{N}$ there exists $C > 0$ such that for $u \in H_{\mathrm{sc}}^{-N,-M}$,*

$$\|BG_\varphi u\| \leq C\left(\|QG_\psi u\| + \|u\|_{-N,-M}\right).$$

Proof. Let $\tilde{\chi}_\pm \in \mathcal{C}_c^\infty(X)$ with $\operatorname{supp} \tilde{\chi}_\pm \subset \{\chi_\pm \equiv 1\}$, $\tilde{\chi}_+ \equiv 1$ in a small neighborhood of NP, and $\tilde{\chi}_- \equiv 1$ in a small neighborhood of SP. Write $B = B\tilde{\chi}_+ + B\tilde{\chi}_- + B(1 - \tilde{\chi}_+ - \tilde{\chi}_-)$. We have that

$$B\tilde{\chi}_+ G_\varphi = B\tilde{\chi}_+ G_{\varphi,V_+} = B\tilde{\chi}_+ G_{\varphi,V_+} G_{\psi,V_+}.$$

On the support of $B\tilde{\chi}_+ G_{\varphi,V_+}$, G_{ψ,V_+} is equal to to G_ψ modulo ${}^{\mathrm{sc}}\Psi^{-\infty,-\infty}$ and together with elliptic regularity, we have Then, we have that

$$\|B\tilde{\chi}_+ G_\varphi u\| \lesssim \|QG_\psi u\| + \|u\|_{-N,-M}.$$

Similarly, we can estimate $B\tilde{\chi}_- G_\varphi u$. for the interior term, we notice that $B(1 - \tilde{\chi}_+ - \tilde{\chi}_-)G_\varphi$ is microsupported in the elliptic set of QG_ψ and therefore we can apply elliptic regularity directly. □

In order to localize to slices $\{\tau = \tau_0\}$, we want to quantize symbols whose restrictions to C are purely functions of τ. For this, we denote the fiber equator by

$$\mathrm{fibeq} := \partial\overline{\mathbb{R}^{n+1}_{\tau,\zeta}} \cap \{\tau = \tau_0\}\,.$$

Note that this set is independent of the choice of τ_0.

We recall from [2, Proposition 5.15] that if $q \in \mathcal{C}^\infty([{}^{\mathrm{sc}}\overline{T}^*X; \mathrm{fibeq}])$, using $\rho = 1/\tau$ as a rescaling function,

$$\mathrm{Op}_L(x^{-\ell}\rho^{-s}q)G_\psi \in {}^{3\mathrm{sc}}\Psi^{s,\ell}$$

for any $\psi \in \mathcal{C}^\infty_c$ and $s, \ell \in \mathbb{R}$.

We have the following variant of [24, Lemma 13.5]:

Lemma 4.4. *Assume that $q|_{\mathrm{NP}} = f$, where $f \in \mathcal{C}^\infty(\overline{W^\perp})$, i.e., q_{NP} depends only on τ. The operator wavefront set of $Q := \mathrm{Op}_L(x^{-\ell}\rho^{-s}q)G_\psi$ satisfies*

$$\begin{aligned}
\mathrm{WF}'_{\mathrm{ff}}(Q) &\subset \text{ess-supp}(f) \cap \mathrm{WF}'_{\mathrm{ff}}(G_\psi)\,,\\
\mathrm{WF}'_{\mathrm{mf}}(Q) &\subset \text{ess-supp}_{\mathrm{mf}}\left(q\psi\left(\frac{\tau^2 - (|\zeta|^2 + m^2)}{\tau^2 + |\zeta|^2 + m^2 + E}\right)\right)\,,\\
\mathrm{WF}'_{\mathrm{fib}}(Q) &\subset \text{ess-supp}_{\mathrm{fib}}\left(q\psi\left(\frac{\tau^2 - |\zeta|^2}{\tau^2 + |\zeta|^2}\right)\right)\,.
\end{aligned}$$

4.2. **Localization to the radial set.** The radial set is naturally defined on ${}^{\mathrm{sc}}\overline{T}^*X$, but the natural phase space for 3sc-operators is $\dot{C}_{3\mathrm{sc}}[X;C]$. Over the poles, we therefore define the 3sc-radial set as subsets of $W^\perp$, namely

$$\begin{aligned}
\mathcal{R}_{\mathrm{ff,\,src}} &:= (\mathrm{NP} \times \{-m\}) \cup (\mathrm{SP} \times \{+m\}) \subset W^\perp\,,\\
\mathcal{R}_{\mathrm{ff,\,snk}} &:= (\mathrm{NP} \times \{+m\}) \cup (\mathrm{SP} \times \{-m\}) \subset W^\perp\,.
\end{aligned}$$

The entire 3sc-radial set is given by

$${}^{3\mathrm{sc}}\mathcal{R} := \left(\mathcal{R} \cap \left({}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X\right)\right) \cup (C \times \{\pm m\})$$

and

$$\begin{aligned}
{}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} &:= \left(\mathcal{R}_{\mathrm{src}} \cap \left({}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X\right)\right) \cup \mathcal{R}_{\mathrm{ff,src}}\,,\\
{}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}} &:= \left(\mathcal{R}_{\mathrm{snk}} \cap \left({}^{\mathrm{sc}}S^*_{X\setminus C}X \cup {}^{\mathrm{sc}}T^*_{\partial X\setminus C}X\right)\right) \cup \mathcal{R}_{\mathrm{ff,snk}}\,.
\end{aligned}$$

We have that ${}^{3\mathrm{sc}}\mathcal{R} = {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} \sqcup {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}}$ and ${}^{3\mathrm{sc}}\mathcal{R}_\bullet \subset \gamma_{3\mathrm{sc}}(\mathcal{R}_\bullet)$ for $\bullet \in \{\mathrm{src}, \mathrm{snk}\}$.

We note that localizing to the characteristic set over the poles is nuanced: if $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$ is elliptic at $\mathcal{R}_{\mathrm{ff,src}}$, then there exists a neighborhood $U \subset \partial X$ of C such that A is elliptic at ${}^{\mathrm{sc}}T^*_{U\setminus C}X \cap \mathcal{R}_{\mathrm{src}}$. We therefore must further localize and employ a localizer of the form QG_ψ for appropriate ψ and $Q = \mathrm{Op}_L(q)$, with the symbol q restricting over NP to a function of τ.

Definition 4.5. *Let $\delta > 0$. We call $Q \in {}^{3\mathrm{sc}}\Psi^{0,0}$ a δ-localizer to ${}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}}$ if*

(1) Q is microlocally the identity near the sources,

$$ {}^{3\mathrm{sc}}\mathrm{WF}'(\mathrm{Id} - Q) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} = \varnothing \,, $$

(2) for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (-\delta, \delta)$ we have

$$ {}^{3\mathrm{sc}}\mathrm{WF}'(QG_\varphi) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}} = \varnothing \,. $$

We denote the set of all δ-localizers to ${}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}}$ by ${}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$. We also define ${}^{3\mathrm{sc}}\Psi_{\mathrm{snk},\delta}$, where the roles of $\mathcal{R}_{\mathrm{src}}$ and $\mathcal{R}_{\mathrm{snk}}$ are interchanged.

Lemma 4.6. *The set ${}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$ is non-empty for all $\delta > 0$.*

Proof. Let $\varepsilon > 0$ small and choose $f \in \mathcal{C}_c^\infty(\overline{W^\perp})$ such that $f(+\infty, \tau) = f(-\infty, -\tau) = 0$ and $f(+\infty, -\tau) = f(-\infty, \tau) = 1$ for $\tau \in (m - \varepsilon, m + \varepsilon)$. We choose a function $q \in \mathcal{C}^\infty([{}^{\mathrm{sc}}\overline{T}^* X; \mathrm{fibeq}])$ such that $q|_C(\pm\infty, \tau, \zeta) = f(\pm\infty, \tau)$ and $q = 1$ in a neighborhood of $\mathcal{R}_{\mathrm{src}}$ and $q = 0$ in a neighborhood of $\mathcal{R}_{\mathrm{snk}}$.

Fix $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\psi(s) \equiv 1$ for $s \in (-\delta, \delta)$ and $\psi(s) \equiv 0$ for $|s| > 2\delta$. By [2, Proposition 5.15], we have that

$$ Q \coloneqq \mathrm{Op}_L(q) G_\psi + (\mathrm{Id} - G_\psi) $$

is a 3sc-operator of order $(0, 0)$.

To show that ${}^{3\mathrm{sc}}\mathrm{WF}'(\mathrm{Id} - Q) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} = \varnothing$, we use that

$$ \mathrm{Id} - Q = \mathrm{Op}_L(1 - q) G_\psi \,. $$

Since $q = 1$ in a neighborhood of $\mathcal{R}_{\mathrm{src}}$ and $f = 1$ near $\tau = -m$, we have that ${}^{3\mathrm{sc}}\mathrm{WF}'(\mathrm{Id} - Q) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} = \varnothing$ by Lemma 4.4.

For $\varphi \in \mathcal{C}_c^\infty$ with $\operatorname{supp}\varphi \subset (-\delta, \delta)$, we calculate

$$ QG_\varphi = \mathrm{Op}_L(q) G_\varphi $$

and consequently,

$$ \mathrm{WF}'_{\mathrm{ff}}(QG_\varphi) \subset \text{ess-supp}(f) \cap \mathrm{WF}'_{\mathrm{ff}}(G_\varphi) \,. $$

Since $f(+\infty, m) = 0$ it then follows that $m \notin \mathrm{WF}'_{\mathrm{ff}}(QG_\varphi)$.

The same arguments apply to SP and therefore $Q \in {}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$. ☐

4.3. **Propagation estimates.**

Definition 4.7. *Let $U, V, W \subset \dot{C}_{3\mathrm{sc}}[X; C]$. We say that U is (backward) controlled by V through W if for all $\alpha \in \mathrm{Char}(P_0)$ that are incoming to U, in sense that*

$$ \pi_{X,\tau}(\alpha) \in \pi_{X,\tau}(\gamma_{3\mathrm{sc}}(U) \cap \mathrm{Char}(P_0)) \,, $$

there exists $s_\alpha < 0$ *such that*

$$\exp(s_\alpha{}^{\mathrm{sc}}H_p)(\alpha) \in V$$

and for all $s \in [s_\alpha, 0]$,

$$\exp(s^{\mathrm{sc}}H_p)(\alpha) \in W\,.$$

We recall from [2] the various propagation estimates. We assume throughout that $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ is asymptotically static of order $r \geq 1$ and

$$V - V^* \in {}^{3\mathrm{sc}}\Psi^{0,-2}\,.$$

Moreover, we assume that $H_{V_\pm}$ have purely absolutely continuous spectrum in $[m^2, \infty)$. This condition is guaranteed on (m^2, ∞) by well-known results in scattering theory; it is really only an assumption at the bottom of the continuous spectrum m^2.

Proposition 4.8 (Regular propagation estimate [2, Proposition 6.2])**.** *Let* $\delta > 0$ *sufficiently small,* $\varphi, \psi_1, \psi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset (-\delta, \delta)$ *and* $\psi_j|_{(-\delta,\delta)} \equiv 1$, *and* $B, E, G, G', B' \in {}^{3\mathrm{sc}}\Psi^{0,0}$ *such that*

(1) $\mathrm{WF}'_{\mathrm{ff}}(E) = \varnothing$,
(2) ${}^{\mathrm{sc}}H_p(\alpha) \neq 0$ *for all* $\alpha \in \gamma_{3\mathrm{sc}}({}^{3\mathrm{sc}}\mathrm{Ell}(G))$,
(3) ${}^{3\mathrm{sc}}\mathrm{WF}'(BG_\varphi)$ *is controlled by* ${}^{3\mathrm{sc}}\mathrm{Ell}(E)$ *through* ${}^{3\mathrm{sc}}\mathrm{Ell}(G)$,
(4) ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G') \cap {}^{3\mathrm{sc}}\mathrm{Ell}(B')$.

and $\psi_1, \psi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ *with* $\psi_j|_{(-\delta,\delta)} \equiv 1$.

For all $M, N, s, \ell \in \mathbb{R}$ *and* $u \in H_{\mathrm{sc}}^{-N,-M}$ *with* $EG_{\psi_1}u \in H_{\mathrm{sc}}^{s,\ell}$, $GG_{\psi_2}P_Vu \in H_{\mathrm{sc}}^{s-1,\ell+1}$, $G'P_Vu \in H_{\mathrm{sc}}^{s-2,\ell}$, *and* $B'u \in H_{\mathrm{sc}}^{s-1,\ell-r}$, *it follows that* $Bu \in H_{\mathrm{sc}}^{s,\ell}$ *and*

$$\begin{aligned}\|Bu\|_{s,\ell} \leq C\Big(&\|EG_{\psi_1}u\|_{s,\ell} + \|GG_{\psi_2}P_Vu\|_{s-1,\ell+1} + \|G'P_Vu\|_{s-2,\ell} + \|B'u\|_{s-1,\ell-r}\\ &+ \|u\|_{-N,-M}\Big)\,.\end{aligned}$$

Near the radial sets ${}^{3\mathrm{sc}}\mathcal{R}$, we have two different estimates, depending on whether $\ell > 1/2$ or $\ell < -1/2$:

Proposition 4.9 (Above threshold radial point estimate [2, Proposition 7.1])**.** *Let* $\delta > 0$ *sufficiently small,* $\varphi, \psi_1, \psi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}\varphi \subset (-\delta, \delta)$ *and* $\psi_j|_{(-\delta,\delta)} \equiv 1$, $\tau_0 \in \{\pm m\} \subset W^\perp$, *and* $B, B_1, G, G', B' \in {}^{3\mathrm{sc}}\Psi^{0,0}$ *such that*

(1) ${}^{3\mathrm{sc}}\mathrm{WF}'(BG_\varphi)$ *is contained in a sufficiently small neighborhood of* $\tau_0 \in \dot{C}_{3\mathrm{sc}}[X; C]$,
(2) $\tau_0 \in \mathrm{Ell}_{\mathrm{ff}}(B) \cap \mathrm{Ell}_{\mathrm{ff}}(B_1)$,
(3) ${}^{3\mathrm{sc}}\mathrm{WF}'(BG_\varphi) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B_1) \cap {}^{3\mathrm{sc}}\mathrm{Ell}(G)$,
(4) ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B') \cap {}^{3\mathrm{sc}}\mathrm{Ell}(G')$.

For all $M, N, s, s', \ell, \ell' \in \mathbb{R}$ with $\ell > \ell' > -1/2$, $s > s'$, and $u \in H_{\mathrm{sc}}^{-N,-M}$ with $B_1 G_\phi u \in H_{\mathrm{sc}}^{s',\ell'}$, $G G_\psi P_V u \in H_{\mathrm{sc}}^{s-1,\ell+1}$, $G' P_V u \in H_{\mathrm{sc}}^{s-2,\ell}$, and $B' u \in H_{\mathrm{sc}}^{s-1,\ell-r}$, it follows that $Bu \in H_{\mathrm{sc}}^{s,\ell}$ and

$$\|Bu\|_{s,\ell} \le C\Big(\|B_1 G_{\psi_1} u\|_{s',\ell'} + \|G G_{\psi_2} P_V u\|_{s-1,\ell+1} + \|G' P_V u\|_{s-2,\ell} + \|B' u\|_{s-1,\ell-r} + \|u\|_{-N,-M}\Big).$$

Proposition 4.10 (Below threshold radial point estimate [2, Proposition 7.2])**.** *Let $\delta > 0$ sufficiently small, $\varphi, \psi_1, \psi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (-\delta,\delta)$ and $\psi_j|_{(-\delta,\delta)} \equiv 1$, and $B, E, G, G', B' \in {}^{3\mathrm{sc}}\Psi^{0,0}$ such that*

(1) ${}^{3\mathrm{sc}}\mathrm{WF}'_{\mathrm{ff}}(E) = \varnothing$,
(2) ${}^{3\mathrm{sc}}\mathrm{WF}'(BG_\varphi) \cup {}^{3\mathrm{sc}}\mathrm{Ell}(E) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(G)$,
(3) ${}^{3\mathrm{sc}}\mathrm{Ell}(G) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} = \varnothing$,
(4) ${}^{3\mathrm{sc}}\mathrm{WF}'(B) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B') \cap {}^{3\mathrm{sc}}\mathrm{Ell}(G')$,
(5) ${}^{3\mathrm{sc}}\mathrm{WF}'(BG_\varphi) \setminus {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}}$ is backward controlled by ${}^{3\mathrm{sc}}\mathrm{Ell}(E)$ through ${}^{3\mathrm{sc}}\mathrm{Ell}(G)$.

For all $M, N, s, \ell \in \mathbb{R}$ with $\ell < -1/2$ and $u \in H_{\mathrm{sc}}^{-N,-M}$ with $E G_{\psi_1} u \in H_{\mathrm{sc}}^{s,\ell}$, $G G_{\psi_2} P_V u \in H_{\mathrm{sc}}^{s-1,\ell+1}$, $G' P_V u \in H_{\mathrm{sc}}^{s-2,\ell}$, and $B' u \in H_{\mathrm{sc}}^{s-1,\ell-r}$, it follows that $Bu \in H_{\mathrm{sc}}^{s,\ell}$ and

$$\|Bu\|_{s,\ell} \le C\Big(\|E G_{\psi_1} u\|_{s,\ell} + \|G G_{\psi_2} P_V u\|_{s-1,\ell+1} + \|G' P_V u\|_{s-2,\ell} + \|B' u\|_{s-1,\ell-r} + \|u\|_{-N,-M}\Big).$$

The same statement holds if the roles of ${}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}}$ and ${}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}}$ are interchanged and forward control is used instead of backward control.

5. Construction of the Feynman propagator

We now construct the Feynman propagator. We follow the general structure of the construction in Section 2, which is to say that we begin with the construction of a Fredholm problem for P_V. The spaces involved in this construction are similar to those in Section 2 in the sense that they have an overall regularity and below threshold weight with the assumption of above threshold weight imposed at the radial sources using a microlocalizer, the primary difference here being that we must now use the 3sc-microlocalizers discussed above.

We choose $\delta > 0$, $Q_{\mathrm{src}} \in {}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$, $Q'_{\mathrm{src}} \in {}^{3\mathrm{sc}}\Psi_{\mathrm{src},2\delta}$ and cutoff functions $\phi, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$ be bump functions supported near 0 with $\operatorname{supp}\phi \subset (-\delta,\delta)$, $\operatorname{supp}\psi \subset (-2\delta, 2\delta)$, and $\psi(s) \equiv 1$ on $(-\delta,\delta)$. (${}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$ is from Definition 4.5.) We assume

$${}^{3\mathrm{sc}}\mathrm{WF}'(Q_{\mathrm{src}} G_\phi) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(Q'_{\mathrm{src}}).$$

Moreover, we require, as in the scattering case in Section 2.2, that the segments of broken bicharacteristic rays with endpoints in ${}^{3\mathrm{sc}}\mathrm{WF}'(Q_{\mathrm{src}} G_\phi)$ lie in ${}^{3\mathrm{sc}}\mathrm{Ell}(Q'_{\mathrm{src}})$.

We additionally choose a $B_{\mathrm{src}} \in {}^{3\mathrm{sc}}\Psi^{0,0}$ with $\mathcal{R}_{\mathrm{ff,src}} \subset \mathrm{Ell}_{\mathrm{ff}}(B_{\mathrm{src}})$. For $s, \ell_0, \ell_+ \in \mathbb{R}$, we define the Feynman–Sobolev spaces as

$$\begin{aligned}\mathcal{Y}^{s,\ell_0,\ell_+} &:= \left\{v \in H_{\mathrm{sc}}^{s-1,\ell_0+1} \colon Q'_{\mathrm{src}} G_\psi v \in H_{\mathrm{sc}}^{s-1,\ell_++1}, B_{\mathrm{src}} v \in H_{\mathrm{sc}}^{s-2,\ell_+}\right\}, \\ \mathcal{X}^{s,\ell_0,\ell_+} &:= \left\{u \in H_{\mathrm{sc}}^{s,\ell_0} \colon Q_{\mathrm{src}} G_\phi u \in H_{\mathrm{sc}}^{s,\ell_+}, P_V u \in \mathcal{Y}^{s,\ell_0,\ell_+}\right\}\end{aligned}$$

with norms

$$\begin{aligned}\|v\|^2_{\mathcal{Y}^{s,\ell_0,\ell_+}} &:= \|v\|^2_{s-1,\ell_0+1} + \|Q'_{\mathrm{src}} G_\psi v\|^2_{s-1,\ell_++1} + \|B_{\mathrm{src}} v\|^2_{s-2,\ell_+}, \\ \|u\|^2_{\mathcal{X}^{s,\ell_0,\ell_+}} &:= \|u\|^2_{s,\ell_0} + \|Q_{\mathrm{src}} G_\phi\|^2_{s,\ell_+} + \|P_V u\|^2_{\mathcal{Y}^{s,\ell_0,\ell_+}}.\end{aligned}$$

In particular, this means that u is supposed to have above threshold regularity at $\mathcal{R}_{\mathrm{src}}$ and below threshold regularity at $\mathcal{R}_{\mathrm{snk}}$. We have to include the $B_{\mathrm{src}} v$ term in the $\mathcal{Y}^{s,\ell_0,\ell_+}$ space because the above threshold radial set estimate near the poles requires control of $B_{\mathrm{src}} P_V u$ with a 3sc-elliptic operator and the operator $Q_{\mathrm{src}} G_\psi$ is not 3sc-elliptic there. If we interchange the roles of $\mathcal{R}_{\mathrm{src}}$ and $\mathcal{R}_{\mathrm{snk}}$, then the resulting spaces give the Fredholm problem associated to the anti-Feynman propagator.

As in [2] we have two results, which differ by the assumption on bound states and the decay of the non-static part of the potential. We treat the case where there are no bound states for the limiting Hamiltonians in this section and the case with bound states in Section 7.

Theorem 5.1. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+ < 1/2$, and $r \geq \max\{1, \ell_+ - \ell_0\}$. Let $V \in \rho_{\mathrm{mf}} {}^{3\mathrm{sc}}\Psi^{1,0}$ be an asymptotically static perturbation of order r and the limiting Hamiltonians $H_{V_\pm} = \Delta + m^2 + V_\pm$ have purely absolutely continuous spectrum near $[m^2, \infty)$, have no bound states, and the leading part of V is self-adjoint in the sense that*

$$V - V^* \in {}^{3\mathrm{sc}}\Psi^{0,-2},$$

then the map

$$P_V : \mathcal{X}^{s,\ell_0,\ell_+} \to \mathcal{Y}^{s,\ell_0,\ell_+} \tag{35}$$

is Fredholm.

If we assume that $V \in \rho_{\mathrm{mf}} \operatorname{Diff}^1_{3\mathrm{sc}}$ and $V = V^*$, then P_V is invertible.

To prove that P_V is a Fredholm operator, we have to show that $P_V : \mathcal{X}^{s,\ell_0,\ell_+} \to \mathcal{Y}^{s,\ell_0,\ell_+}$ has finite dimensional kernel and cokernel.

Lemma 5.2. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+$, and $r \geq \max\{1, \ell_+ - \ell_0\}$. Assume that $V \in \rho_{\mathrm{mf}} {}^{3\mathrm{sc}}\Psi^{1,0}$ satisfies the assumptions of Theorem 5.1. There exists $C > 0$ such that for all $u \in H_{\mathrm{sc}}^{s-1,\ell_0-1} \cap \mathcal{X}^{s,\ell_0,\ell_+}$,*

$$\|u\|_{\mathcal{X}^{s,\ell_0,\ell_+}} \leq C \left(\|P_V u\|_{\mathcal{Y}^{s,\ell_0,\ell_+}} + \|u\|_{s-1,\ell_0-1}\right).$$

In particular, $\ker_{\mathcal{X}^{s,\ell_0,\ell_+}} P_V$ *is finite-dimensional and has closed range.*

Proof. The claimed global estimate follows, using a interpolation argument as in [2, Eq. (2.39)] applied to $\|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_0}$, from the two estimates

$$\begin{aligned}\|u\|_{s,\ell_0} &\lesssim \|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_0} + \|P_V u\|_{s-1,\ell_0+1} + \|u\|_{s-1,\ell_0-1} \\ \|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_+} &\lesssim \|B_{\mathrm{src}} P_V u\|_{s-2,\ell_+} + \|Q'_{\mathrm{src}} G_\psi P_V u\|_{s-1,\ell_++1} \\ &\quad + \|u\|_{s,\ell_0}\,.\end{aligned}$$

The first inequality is the result of using a microlocal partition of unity together with elliptic estimates, propagation estimates and below threshold estimates. Near the poles, we use Proposition 3.5, Proposition 4.8, and Proposition 4.10 and away from the poles, we can use the estimates from the scattering calculus as in Section 2.

More precisely, we take an open cover O_1, O_2, O_3, O_4 of the compressed cotangent bundle ${}^{\mathrm{sc}}\dot{\overline{T}}X$ similar as in [2, p. 83] such that

(1) ${}^{3\mathrm{sc}}\mathrm{WF}'(Q_{\mathrm{src}} G_\phi) \subset O_1$,
(2) $\mathcal{R}_{\mathrm{snk}} \subset O_2 \subset {}^{3\mathrm{sc}}\mathrm{WF}'(\mathrm{Id} - Q_{\mathrm{src}})$,
(3) ${}^{3\mathrm{sc}}\mathrm{Char}(P_V) \subset O_1 \cup O_2 \cup O_3$,
(4) $O_3 \cap {}^{3\mathrm{sc}}\mathrm{Char}(P_V)$ is controlled along ${}^{\mathrm{sc}}H_p$ by O_1,
(5) $(O_2 \cap {}^{3\mathrm{sc}}\mathrm{Char}(P_V)) \setminus \mathcal{R}_{\mathrm{snk}}$ is controlled along ${}^{\mathrm{sc}}H_p$ by O_3, and
(6) $O_4 \subset {}^{3\mathrm{sc}}\mathrm{Ell}(P_V)$.

The proof that that such a cover exists is similar to the case in [2].

We also choose a collection $Q_2, Q_3, Q_4 \in {}^{3\mathrm{sc}}\Psi^{0,0}$ with

$${}^{3\mathrm{sc}}\mathrm{WF}'(Q_i G_\phi) \subset O_i$$

and

$$\dot{C}_{3\mathrm{sc}}[X; C] \subset {}^{3\mathrm{sc}}\mathrm{Ell}(Q_{\mathrm{src}}) \cup \bigcup_{i=2}^{4} {}^{3\mathrm{sc}}\mathrm{Ell}(Q_i)\,.$$

We have the estimate

$$\|u\|_{s,\ell_0} \lesssim \|Q_{\mathrm{src}} u\|_{s,\ell_0} + \|Q_2 u\|_{s,\ell_0} + \|Q_3 u\|_{s,\ell_0} + \|Q_4 u\|_{s,\ell_0} + \|u\|_{s-1,\ell_0-1}\,.$$

By Proposition 4.2, we have

$$\|Q_{\mathrm{src}} u\|_{s,\ell_0} \lesssim \|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_0} + \|P_V u\|_{s-1,\ell_0+1} + \|u\|_{s-1,\ell_0-1}\,.$$

Using the propagation estimates, Proposition 4.8 and [2, Proposition 2.6], we can estimate $Q_3 u$ by the localizer near the sources, $Q_{\mathrm{src}} u$, and $P_V u$. More precisely, we have

$$\|Q_3 u\|_{s,\ell_0} \lesssim \|Q_{\mathrm{src}} u\|_{s,\ell_0} + \|P_V u\|_{s-1,\ell_0+1} + \|u\|_{s-1,\ell_0-1}\,.$$

The below threshold estimates, Proposition 4.10 and [2, Proposition 2.13], imply

$$\|Q_2 u\|_{s,\ell_0} \lesssim \|Q_3 u\|_{s,\ell_0} + \|P_V u\|_{s-1,\ell_0+1} + \|u\|_{s-1,\ell_0-1}\,.$$

Lastly, we use the elliptic estimate [2, Proposition 2.2] to bound

$$\|Q_4 u\|_{s,\ell_0} \lesssim \|P_V u\|_{s-2,\ell_0} + \|u\|_{-N,-M} \leq \|P_V u\|_{s-1,\ell_0+1} + \|u\|_{s-1,\ell_0-1}\,.$$

Putting the estimates together, we obtain

$$\|u\|_{s,\ell_0} \lesssim \|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_0} + \|P_V u\|_{s-1,\ell_0+1} + \|u\|_{s-1,\ell_0-1}\,,$$

which is the first inequality.

The second inequality is a "global" above threshold estimate. As a consequence of Proposition 4.9 with $s' = s-1$, $-N = s-1$, and $-M = \ell_0$ and using that $\ell_+ - r \leq \ell_0$, we obtain

$$\begin{aligned} \|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_+} \lesssim \|B_1 G_\psi u\|_{s-1,\ell'} &+ \|Q'_{\mathrm{src}} G_\psi P_V u\|_{s-1,\ell_++1} + \|B_{\mathrm{src}} P_V u\|_{s-2,\ell_+} \\ &+ \|u\|_{s-1,\ell_0}\,, \end{aligned} \tag{36}$$

where $\ell' \in (-1/2, \ell_+)$ and for some $B_1 \in {}^{3\mathrm{sc}}\Psi^{0,0}$ that satisfies ${}^{3\mathrm{sc}}\mathrm{WF}'(Q_{\mathrm{src}} G_\phi) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B_1)$ and ${}^{3\mathrm{sc}}\mathrm{WF}'(B_1 G_\psi) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(B_{\mathrm{src}}) \cup {}^{3\mathrm{sc}}\mathrm{Ell}(Q'_{\mathrm{src}} G_\psi)$.

To remove the $B_1 G_\psi$ term, we claim that

$$\begin{aligned} \|B_1 G_\psi u\|_{s-1,\ell'} \lesssim \|Q_{\mathrm{src}} G_\phi u\|_{s-1,\ell'} &+ \|B_{\mathrm{src}} P_V u\|_{s-3,\ell'} + \|Q'_{\mathrm{src}} G_\psi P_V u\|_{s-2,\ell'+1} \\ &+ \|u\|_{s-2,\ell_0}\,. \end{aligned} \tag{37}$$

Combining (36) and (37), we arrive at the estimate

$$\begin{aligned} \|Q_{\mathrm{src}} G_\phi u\|_{s,\ell_+} \lesssim \|B_{\mathrm{src}} P_V u\|_{s-2,\ell_+} &+ \|Q'_{\mathrm{src}} G_\psi P_V u\|_{s-1,\ell_++1} \\ &+ \|Q_{\mathrm{src}} G_\phi u\|_{s-1,\ell'} + \|u\|_{s,\ell_0}\,. \end{aligned}$$

Using the interpolation inequality [2, Eq. (2.39)], we can absorb $\|Q_{\mathrm{src}} G_\phi u\|_{s-1,\ell'}$ into the left hand side.

It remains to prove (37). From Proposition 4.2 we obtain

$$\|B_1 G_\psi u\|_{s-1,\ell'} \lesssim \|B_1 G_\varphi u\|_{s-1,\ell'} + \|B_{\mathrm{src}} P_V u\|_{s-3,\ell'} + \|u\|_{s-1,\ell_0}\,,$$

where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\varphi\phi = \varphi$ and $\varphi \equiv 1$ in a small neighborhood of 0. We choose $\tilde{Q} \in {}^{3\mathrm{sc}}\Psi^{0,0}$ such that ${}^{3\mathrm{sc}}\mathrm{WF}'(\mathrm{Id} - \tilde{Q}) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{src}} = \varnothing$ and ${}^{3\mathrm{sc}}\mathrm{WF}'(\tilde{Q} G_\varphi) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(Q_{\mathrm{src}})$. Using Proposition 4.3, we further estimate

$$\begin{aligned} \|B_1 G_\varphi u\|_{s-1,\ell'} \lesssim \|B_1(\mathrm{Id} - \tilde{Q}) G_\varphi u\|_{s-1,\ell'} &+ \|Q_{\mathrm{src}} G_\phi u\|_{s-1,\ell'} \\ &+ \|u\|_{s-1,\ell_0}\,. \end{aligned}$$

Since $B_1(\mathrm{Id} - \tilde{Q})G_\varphi$ is localized away from the radial sources and controlled by $Q_{\mathrm{src}}G_\phi$, we obtain using Proposition 4.8,

$$\|B_1(\mathrm{Id} - \tilde{Q})G_\varphi u\|_{s-1,\ell'} \lesssim \|Q_{\mathrm{src}}G_\phi u\|_{s-1,\ell'} + \|Q'_{\mathrm{src}}G_\psi P_V u\|_{s-2,\ell'+1} \\ + \|B_{\mathrm{src}}P_V\|_{s-3,\ell'} + \|u\|_{s-2,\ell_0} .$$

Putting all of these estimates together, we arrive at (37).

The conclusion that $\ker_{\mathcal{X}^{s,\ell_0,\ell_+}} P_V$ is finite-dimensional and that P_V has closed range follow from a standard argument (cf. [27]) because $H_{\mathrm{sc}}^{s-1,\ell_0-1} \to \mathcal{X}^{s,\ell_0,\ell_+}$ is compact. ☐

To complete the proof of Theorem 5.1, we have to show that the cokernel of $P_V : \mathcal{X}^{s,\ell_0,\ell_+} \to \mathcal{Y}^{s,\ell_0,\ell_+}$ is finite-dimensional as well. The cokernel is given by

$$\operatorname{coker}(P_V) = \left\{ v \in (\mathcal{Y}^{s,\ell_0,\ell_+})' \colon \langle P_V u, v\rangle = 0 \text{ for all } u \in \mathcal{X}^{s,\ell_0,\ell_+} \right\},$$

which is equal to $\ker_{(\mathcal{Y}^{s,\ell_0,\ell_+})'} P_V^*$ since P_V has closed range.

Lemma 5.3. *There exists $\tilde{Q} \in {}^{3\mathrm{sc}}\Psi_{\mathrm{snk},\delta}$ such that for all $\phi' \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}\phi' \subset (-\delta,\delta)$ and $u \in (\mathcal{Y}^{s,\ell_0,\ell_+})'$, we have $\tilde{Q}G_{\phi'}u \in H_{\mathrm{sc}}^{1-s,\ell'_+}$, where $\ell'_+ = \min\{-\ell_0 - 1, -\ell_+\}$.*

Proof. Since $Q'_{\mathrm{src}} \in {}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$, we can construct the following operator $\tilde{Q}$:

We choose $f \in \mathcal{C}_c^\infty(W^\perp)$ such that

$$f|_{\mathrm{WF}'_{\mathrm{ff}}(Q'_{\mathrm{src}})^c} = 0 , \\ f|_{W_+^\perp}(\tau) = f|_{W_-^\perp}(-\tau) = 1$$

for $\tau \in (m-\varepsilon, m+\varepsilon)$ and $\varepsilon > 0$ sufficiently small. Moreover, we choose $\tilde{q} \in \mathcal{C}^\infty([{}^{\mathrm{sc}}\overline{T}^* X; \mathrm{fibeq}])$ such that $\tilde{q} = 1$ near $\mathcal{R}_{\mathrm{snk}}$, $\tilde{q} = 0$ on ${}^{3\mathrm{sc}}\mathrm{WF}'(Q_{\mathrm{src}}) \setminus \overline{W^\perp}$ and $\tilde{q}|_C(\tau,\zeta) = f(\tau)$.

We set

$$\tilde{Q} \coloneqq \mathrm{Op}_L(\tilde{q})G_\varphi + (\mathrm{Id} - G_\varphi) ,$$

which is an element in ${}^{3\mathrm{sc}}\Psi_{\mathrm{snk},\delta}$ by construction.

To show that $\tilde{Q}G_{\phi'}$ maps $(\mathcal{Y}^{s,\ell_0,\ell_+})'$ to $H_{\mathrm{sc}}^{1-s,\ell'_+}$, we show that the adjoint $G_{\phi'}\tilde{Q}^*$ maps $H_{\mathrm{sc}}^{s-1,-\ell'_+}$ to $\mathcal{Y}^{s,\ell_0,\ell_+}$. Let $v \in H_{\mathrm{sc}}^{s-1,-\ell'_+}$, we have to show that

$$G_{\phi'}\tilde{Q}^* v \in H_{\mathrm{sc}}^{s-1,\ell_0+1} , \\ Q'_{\mathrm{src}}G_\psi G_{\phi'}\tilde{Q}^* v \in H_{\mathrm{sc}}^{s-1,\ell_++1} , \\ B_{\mathrm{src}}G_{\phi'}\tilde{Q}^* v \in H_{\mathrm{sc}}^{s-2,\ell_+} .$$

Since $-\ell'_+ \geq \ell_0 + 1$ and $-\ell'_+ \geq \ell_+$, the first and the third property are trivially satisfied. Finally, we have that

$${}^{3\mathrm{sc}}\mathrm{WF}'(\tilde{Q}G_{\phi'}) \cap {}^{3\mathrm{sc}}\mathrm{WF}'(Q'_{\mathrm{src}}G_\psi) = \varnothing$$

by construction of $\tilde{Q}$ and therefore $Q'_{\mathrm{src}} G_\psi G_{\phi'} \tilde{Q}^*$ is regularizing. □

Lemma 5.4. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+ < 1/2$, and $r \geq \max\{1, \ell_+ - \ell_0\}$. If $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}$ is an admissible asymptotically static perturbation of order r, then the kernel of P_V^*, $\ker_{(\mathcal{Y}^{s,\ell_0,\ell_+})'} P_V^*$, is finite-dimensional.*

Proof. Set $\ell'_+ := \min\{-\ell_0 - 1, -\ell_+\}$ and $\ell'_0 = -\ell_+ - 1$. By assumption, $\ell'_+ > -1/2$ and $\ell'_0 < -1/2$ and we note that $r \geq \ell'_+ - \ell'_0$. Therefore, $s' = 1 - s, \ell'_0, \ell'_+$, and r satisfy the assumptions of Lemma 5.2.

We choose $\tilde{Q} \in {}^{3\mathrm{sc}}\Psi_{\mathrm{snk},\delta}$ as in the previous lemma, $\tilde{Q}' \in {}^{3\mathrm{sc}}\Psi_{\mathrm{snk},2\delta}$ with ${}^{3\mathrm{sc}}\mathrm{WF}'(\tilde{Q}) \subset {}^{3\mathrm{sc}}\mathrm{Ell}(\tilde{Q}')$ and as in Lemma 5.2. Moreover, we choose a $\tilde{B}_{\mathrm{snk}} \in {}^{3\mathrm{sc}}\Psi^{0,0}$ with $\mathcal{R}_{\mathrm{ff,snk}} \subset {}^{3\mathrm{sc}}\mathrm{Ell}(\tilde{B}_{\mathrm{snk}})$. We set

$$\tilde{\mathcal{Y}} := \left\{ v \in H_{\mathrm{sc}}^{-s,\ell'_0+1} \colon \tilde{Q}' G_\psi v \in H_{\mathrm{sc}}^{-s,\ell'_++1}, \tilde{B}_{\mathrm{snk}} v \in H_{\mathrm{sc}}^{-s-1,-\ell'_+} \right\},$$
$$\tilde{\mathcal{X}} := \left\{ u \in H_{\mathrm{sc}}^{1-s,\ell'_0} \colon \tilde{Q} G_\phi u \in H_{\mathrm{sc}}^{1-s,\ell'_+}, P_V^* u \in \tilde{\mathcal{Y}} \right\}$$

We claim that $\ker_{(\mathcal{Y}^{s,\ell_0,\ell_+})'} P_V^* \subset \tilde{\mathcal{X}}$. We have to show that for $u \in (\mathcal{Y}^{s,\ell_0,\ell_+})'$ with $P_V^* u = 0$, we have that

$$u \in H_{\mathrm{sc}}^{1-s,\ell'_0},$$
$$\tilde{Q} G_\phi u \in H_{\mathrm{sc}}^{1-s,\ell'_+}.$$

The first inclusion follows by duality from $H_{\mathrm{sc}}^{s-1,-\ell'_0} = H_{\mathrm{sc}}^{s-1,\ell_++1} \subset \mathcal{Y}^{s,\ell_0,\ell_+}$ and the second follows from the previous lemma.

The claim now follows from Lemma 5.2 with $P_V^* : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$ and interchanging the roles of $\mathcal{R}_{\mathrm{src}}$ and $\mathcal{R}_{\mathrm{snk}}$. □

If $P_V u \in \dot{\mathcal{C}}^\infty(X)$, then we have the following regularity result, which is an immediate consequence of the estimates in Section 4.3.

Lemma 5.5. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+$, and $r \geq \max\{1, \ell_+ - \ell_0\}$. Let $V \in \rho_{\mathrm{mf}}{}^{3\mathrm{sc}}\Psi^{1,0}(X)$ be an admissible asymptotically static perturbation of order r and assume that $H_{V_\pm}$ have no bound states. If $u \in \mathcal{X}^{s,\ell_0,\ell_+}$ and $P_V u \in \dot{\mathcal{C}}^\infty(X)$, then for any $\delta > 0$,*

$$u \in H_{\mathrm{sc}}^{\infty,-1/2-\delta}, \text{ and } Au \in H_{\mathrm{sc}}^{\infty,-1/2-\delta+r}$$

provided that $A \in {}^{3\mathrm{sc}}\Psi^{0,0}$ satisfies ${}^{3\mathrm{sc}}\mathrm{WF}'(A) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}} = \varnothing$.

The previous lemma states that if $P_V u \in \dot{\mathcal{C}}^\infty$, then u is above threshold except at the radial sinks. If we assume that $P_V u = 0$, then u is above threshold everywhere.

Proposition 5.6. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+$, and $r \geq \max\{1, \ell_+ - \ell_0\}$. Assume that V is an admissible asymptotically static perturbation of order r, and in addition that $V \in \rho_{\mathrm{mf}} \operatorname{Diff}^1_{3\mathrm{sc}}$ and $V = V^*$. Then, for all $\delta > 0$,*

$$\ker_{\mathcal{X}^{s,\ell_0,\ell_+}} P_V \subset H^{\infty,-1/2-\delta+r}_{\mathrm{sc}}.$$

Proof. We employ essentially the same argument as the one provided by Vasy [24, Proposition 17.8] (see also [29, Proposition 7]). For $\alpha \in (-1/2, 0)$ and $\varepsilon > 0$, we introduce the family of cut-off functions $\chi_\varepsilon(x)$ given by

$$\chi_\varepsilon(x) = \varepsilon^{-2\alpha-1} \int_0^{x/\varepsilon} \varphi(s)^2 s^{-2\alpha-2}\, ds\,,$$

where $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ is non-negative, $\varphi \equiv 0$ on $(-\infty, 1]$, and $\varphi \equiv 1$ on $[2, \infty)$. For fixed $\varepsilon > 0$, χ_ε is compactly supported in the interior of M and hence an element of ${}^{\mathrm{sc}}\Psi^{0,-\infty}(X)$. As a family in $\varepsilon \in (0, 1)$, however, χ_ε is not uniformly bounded in any symbol space. On the other hand, its commutator with $x^2\partial_x$ is uniformly bounded in ${}^{\mathrm{sc}}\Psi^{0,2\alpha}$, as

$$[x^2\partial_x, \chi_\varepsilon] = x^2\partial_x\chi_\varepsilon(x) = x^{-2\alpha}\varphi\,(x/\varepsilon)^2\,.$$

For $u \in \ker_{\mathcal{X}^{s,\ell_0,\ell_+}} P_V$, we consider the pairing

$$0 = i\left(\langle \chi_\varepsilon u, P_V u\rangle - \langle P_V u, \chi_\varepsilon u\rangle\right) = i\langle [P_V, \chi_\varepsilon]u, u\rangle\,,$$

where the second equality follows from $P_V^* = P_V$ and the fact that χ_ε is compactly supported.

Since $V \in \rho_{\mathrm{mf}} \operatorname{Diff}^1_{3\mathrm{sc}}$, we have that $[V, \chi_\varepsilon] = \varphi(x/\varepsilon) f(x, z)$ for some $f \in {}^{3\mathrm{sc}}S^0$. Therefore, we can write

$$i[P_V, \chi_\varepsilon] = 2x^{-2\alpha}\varphi(x/\varepsilon)^2(x^2 D_x) + F_\varepsilon\,,$$

where $F_\varepsilon \in {}^{3\mathrm{sc}}\Psi^{0,2\alpha-1}$ is uniformly bounded.

We now work in $t \gg 0$, as the argument in $t \ll 0$ is similar with a sign change. Choose $b \in \mathcal{C}^\infty(\overline{W^\perp})$ such that $b \equiv 1$ for $\tau > m/2$ and $b \equiv 0$ for $\tau < m/4$ and set

$$\begin{aligned} B &:= \operatorname{Op}_L(\tau^{1/2} b(\tau)) G_\psi\,, \\ E &:= (x^2 D_x)(\operatorname{Id} - G_\psi^2) + \operatorname{Op}_L(\tau(1 - b(\tau)^2)) G_\psi^2\,, \end{aligned}$$

so that

$$x^2 D_x = B^* B + E + R$$

with $R \in {}^{3\mathrm{sc}}\Psi^{0,-1}$, hence we obtain that

$$i[P_V, \chi_\varepsilon] = 2x^{-2\alpha}\varphi(x/\varepsilon)^2\left(B^*B + E\right) + F'_\varepsilon\,,$$

where $F'_\varepsilon \in {}^{3\mathrm{sc}}\Psi^{0,2\alpha-1}$ is uniformly bounded.

We write $E = E_1 + E_2$, where

$$E_1 := (x^2 D_x)(\mathrm{Id} - G_\psi^2)\,,$$
$$E_2 := \mathrm{Op}_L(\tau(1-b(\tau)^2))G_\psi^2\,.$$

By Proposition 4.2, we have that $\|E_1 u\|_{s,-1/2-\delta+r} \lesssim \|u\|_{s-1,-1/2-\delta}$ and therefore $E_1 u \in H_{\mathrm{sc}}^{\infty,-1/2-\delta+r}$. The operator E_2 has 3sc-wavefront set away from the sinks of the Hamiltonian flow, ${}^{3\mathrm{sc}}\mathrm{WF}'(E_2) \cap {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}} = \varnothing$, therefore $E_2 u \in H_{\mathrm{sc}}^{\infty,-1/2-\delta+r}$ by Lemma 5.5.

Since commutators with $\varphi(x/\varepsilon)$ decrease the order and are uniformly bounded, we obtain that

$$\|x^{-\alpha}\varphi(x/\varepsilon)Bu\|^2 \lesssim |\langle x^{-\alpha}\varphi(x/\varepsilon)Eu, x^{-\alpha}\varphi(x/\varepsilon)u\rangle| + |\langle F_\varepsilon'' u, u\rangle|$$

with $F_\varepsilon'' \in {}^{3\mathrm{sc}}\Psi^{0,2\alpha-1}$ being uniformly bounded. Taking $\delta = -\alpha \in (0,1/2)$, we observe that the right hand side is finite and hence $x^{-\alpha}\varphi(x/\varepsilon)Bu$ is uniformly bounded. This implies that Bu is bounded in $H_{\mathrm{sc}}^{0,\alpha}$ and by Proposition 4.2, we conclude that $u \in H_{\mathrm{sc}}^{0,\alpha}$ near $\gamma_{3\mathrm{sc}}({}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}})$, in particular, u is above threshold on ${}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}}$. This implies that u is in $H_{\mathrm{sc}}^{s,-1/2-\delta+r}$ for $t \gg 0$ by the above threshold radial set estimates. $\square$

With this, we can prove that under the same assumptions as in the case of the causal propagators, $P_V : \mathcal{X}^{s,\ell_0,\ell_+} \to \mathcal{Y}^{s,\ell_0,\ell_+}$ is invertible.

Proof of Theorem 5.1. By Lemma 5.2 and Lemma 5.4, we have that P_V is a Fredholm operator. Let $u \in \ker P_V$. By Proposition 5.6, u has above threshold regularity at the entire radial set. Hence, u is an element in a causal space for P_V (for any choice of $s, \boldsymbol{\ell}$). By [2, Theorem 8.2], $P_V : \mathcal{X}^{s,\boldsymbol{\ell}} \to \mathcal{Y}^{s,\boldsymbol{\ell}}$ is invertible, hence $u \equiv 0$.

Because Lemma 5.4 shows that the cokernel is contained in one of the $\mathcal{X}$ spaces and P_V^* satisfies the same estimates, the same argument proves that the cokernel is trivial as well. $\square$

Proposition 5.7. *The inverse $(P_V)_{\mathrm{Fey}}^{-1}$ is independent of the parameters s, ℓ_0, ℓ_+ and the cut-offs in the following sense: let $j \in \{1,2\}$ and $\delta_j > 0$, $Q_{\mathrm{src},j} \in {}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta_j}$, $Q'_{\mathrm{src},j}$, $B_{\mathrm{src},j} \in {}^{3\mathrm{sc}}\Psi^{0,0}$, $\phi_j, \psi_j \in \mathcal{C}_c^\infty(\mathbb{R})$, and $s_j, \ell_{0,j}, \ell_{+,j} \in \mathbb{R}$ satisfying the assumptions of Theorem 5.1. Assume that $u_j \in H_{\mathrm{sc}}^{s_j,\ell_{0,j}}$ and $Q_{\mathrm{src},j}G_{\phi_j}u \in H_{\mathrm{sc}}^{s_j,\ell_{+,j}}$ with $P_V u_j \in H_{\mathrm{sc}}^{s_j-1,\ell_{0,j}+1}$ and $Q'_{\mathrm{src},j}G_{\psi_j}P_V u_j \in H_{\mathrm{sc}}^{s_j-1,\ell_{+,j}+1}$, $B_{\mathrm{src},j}P_V u_j \in H_{\mathrm{sc}}^{s_j-2,\ell_{+,j}}$. If $P_V u_1 = P_V u_2$, then $u_1 = u_2$.*

In particular, $(P_V)_{\mathrm{Fey}}^{-1}$ defines a continuous linear operator

$$(P_V)_{\mathrm{Fey}}^{-1} : \dot{\mathcal{C}}^{-\infty}(X) \to \mathcal{C}^{-\infty}(X)\,.$$

Proof. Set $u := u_1 - u_2$. We have that $P_V u = 0$ and we can find parameters $s, \ell_0, \ell_+ \in \mathbb{R}$ and cut-offs such that u is in a $\mathcal{X}$-space. Since $P_V : \mathcal{X} \to \mathcal{Y}$ is invertible, it follows that $u \equiv 0$. □

6. Regularity of solutions with compactly supported forcing

We now relate the propagators constructed here to the distinguished parametrices of Duistermaat–Hörmander [6]. We follow the exposition of Gérard–Wrochna [9, 10].[1]

A bounded linear operator $A : \dot{\mathcal{C}}^\infty(X) \to \mathcal{C}^{-\infty}(X)$ can be identified with its Schwartz kernel $K \in \mathcal{C}^{-\infty}(X \times X)$. We set

$$\widetilde{\mathrm{WF}}(A) := \{(p, \xi, p', \xi') \colon (p, p', \xi, -\xi') \in \mathrm{WF}_{\mathrm{cl}}(K)\}\,.^{[2]}$$

The microlocal Hadamard condition as introduced by Radzikowski [22] is the following:

Definition 6.1. *A bounded linear operator $E : \mathcal{C}_c^\infty \to \mathcal{C}^{-\infty}$ is called a Feynman parametrix if $\mathrm{Id} - EP_V$ and $\mathrm{Id} - P_V E$ have smooth Schwartz kernels and*

$$\widetilde{\mathrm{WF}}(E) = \mathrm{diag}_{T^*X\setminus 0} \cup \bigcup_{s\geq 0} \tilde{\Phi}_s(\mathrm{diag}_{\mathrm{Char}(P_0)})\,.$$

Here, $\tilde{\Phi}_s$ denotes the bicharacteristic flow acting on the left component of $T^*X \times T^*X$.

Proposition 6.2. *Let E be a Feynman parametrix. Then E uniquely extends to a map $\mathcal{C}_c^{-\infty} \to \mathcal{C}^{-\infty}$ and for $f \in \mathcal{C}_c^{-\infty}$ we have that*

$$\mathrm{WF}_{\mathrm{cl}}(Ef) \subset \mathrm{WF}_{\mathrm{cl}}(f) \cup \bigcup_{s\geq 0} \Phi_s\left(\mathrm{WF}_{\mathrm{cl}}(f) \cap \mathrm{Char}(P_0)\right)\,.$$

Proof. This follows almost directly from Hörmander [16, Theorem 8.2.13]. Denote by K_E the Schwartz kernel of E. By the condition on $\widetilde{\mathrm{WF}}(E)$, we have that

$$\mathrm{WF}'(K_E)_X := \{(p', \xi') \colon (p, p', 0, -\xi') \in \mathrm{WF}_{\mathrm{cl}}(K_E) \text{ for some } p \in X\} = \varnothing,$$

hence Ef is uniquely defined for any $f \in \mathcal{C}_c^{-\infty}(X)$. Moreover, we have that $\mathrm{WF}(K_E)_X := \{(p, \xi) \colon (p, p', \xi, 0) \in \mathrm{WF}_{\mathrm{cl}}(K_E) \text{ for some } p' \in X\} = \varnothing$, so

$$\begin{aligned}\mathrm{WF}_{\mathrm{cl}}(Ef) &\subset \widetilde{\mathrm{WF}}(E) \circ \mathrm{WF}_{\mathrm{cl}}(f)\\ &= \mathrm{WF}_{\mathrm{cl}}(f) \cup \{(\exp(s^{\mathrm{sc}}H_p)q, q) \colon q \in \mathrm{WF}_{\mathrm{cl}}(f) \cap \mathrm{Char}(P_0)\}\,.\end{aligned}$$

□

This version of the microlocal Hadamard condition is more in line with the support condition for the causal propagators. The inverse $(P_V)^{-1}_{\mathrm{Fey}}$ as constructed in Proposition 5.7 has the same property:

---

[1] Note that the sign conventions are different, hence the flow direction is reversed.

[2] This set is usually denoted by WF', but this causes confusion with the operator wavefront set.

Theorem 6.3. *Let $f \in \dot{\mathcal{C}}^{-\infty}(X)$, then*

$$\mathrm{WF}_{\mathrm{cl}}((P_V)^{-1}_{\mathrm{Fey}} f) \subset \mathrm{WF}_{\mathrm{cl}}(f) \cup \bigcup_{s \geq 0} \Phi_s \left(\mathrm{WF}_{\mathrm{cl}}(f) \cap \mathrm{Char}(P_0)\right) .$$

This theorem follows directly from the propagation estimates in the scattering calculus.

Proof of Theorem 1.1. Theorem 1.1 is a direct consequence of Theorem 5.1, Proposition 5.7, and Theorem 6.3. □

7. The case of bound states

We now discuss the case that one or both of the limiting spatial Hamiltonians $H_{V_\pm}$ admits bound states. This follows the treatment of bound states in our work [2, Sect. 8.2] on the causal propagators. We now assume that there are only finitely many bound states of the limiting Hamiltonians, $H_{V_\pm}$ and they are contained in the interval $(-\infty, m^2)$. Moreover, we assume throughout that

$$0 \notin \sigma(H_{V_\pm}), \tag{38}$$

i.e., 0 is not an eigenvalue of $H_{V_\pm}$, as the corresponding solutions with linear growth in time complicate the analysis substantially.

Setting $\lambda(\tau) = \sqrt{m^2 - \tau^2}$, denote the eigenspace of $\Delta_z + V_\pm$ at frequency λ by

$$E_\pm(\lambda(\tau_0)) = \left\{ w \in L^2(\mathbb{R}^n) \colon \hat{N}_{\mathrm{ff},\pm}(P_V)(\tau_0) w = 0 \right\}.$$

Since the operators $\hat{N}_{\mathrm{ff},\pm}(P_V)(\tau_0)$ are scattering elliptic for $|\tau_0| < m$, such bound states w are Schwartz:

$$E_\pm(\lambda(\tau_0)) \subset \mathcal{S}(\mathbb{R}^n).$$

We denote the set of values τ at which there are non-trivial bound states by

$$\mathcal{B}_{V,\pm} = \{\tau \in (-m, m) \colon E_\pm(\lambda(\tau)) \neq \{0\}\}. \tag{39}$$

Thus, under our assumptions, $\mathcal{B}_{V,\pm}$ is finite and $0 \notin \mathcal{B}_{V,\pm}$. An element $\tau_0 \in \mathcal{B}_{V,+}$ and a choice of $w \in E_+(\lambda(\tau))$ gives solutions

$$u_{w,\tau_0}(t,z) := e^{i\tau_0 t} w(z), \quad P_{V_+}(u_{w,\tau_0}) = 0, \tag{40}$$

and the same for V_-; these solutions lie exactly at threshold, meaning $e^{\pm i\tau_0 t} w(z) \in H_{\mathrm{sc}}^{-\infty,-1/2-\epsilon}$ for all $\epsilon > 0$ and no better. They have wavefront set at $\tau_0 \in W_+^\perp$.

From the discussion in the introduction, it is clear which asymptotic solutions $e^{i\tau_0 t} w(z)$ should be allowed in the domain and which should be excluded (see (2)) to obtain a Feynman type problem. Near NP we allow asymptotics as in (40) with $\tau_0 > 0$ and

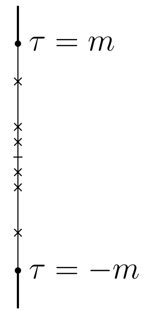

FIGURE 2. The set $\mathcal{B}_{V,+} \subset W_+^\perp$

exclude those with $\tau_0 < 0$, while near NP we allow asymptotics as in (40) with $\tau_0 < 0$ and exclude those with $\tau_0 > 0$. We set

$$\begin{aligned}\mathcal{B}_{V,\mathrm{src}} &:= \{(\mathrm{f},\tau)\colon \tau \in \mathcal{B}_{V,+}, \tau < 0\} \cup \{(\mathrm{p},\tau)\colon \tau \in \mathcal{B}_{V,-}, \tau > 0\}\,,\\ \mathcal{B}_{V,\mathrm{snk}} &:= \{(\mathrm{f},\tau)\colon \tau \in \mathcal{B}_{V,+}, \tau > 0\} \cup \{(\mathrm{p},\tau)\colon \tau \in \mathcal{B}_{V,-}, \tau < 0\}\,.\end{aligned}$$

To quantify this inclusion/exclusion statement, we introduce the Fourier localizing bound state projectors from [2, Sect. 8.2]. Specifically, following [2, Eq. 8.21], we let $K^{\mathrm{f}}_{\tau_0} = K_{\tau_0}(t,z,t',z')$ denote the integral kernel of the operator which: (1) cuts off to large positive times, (2) projects onto the space of bound states $E_+(\lambda(\tau_0))$, and (3) localizes in τ Fourier space near τ_0. Specifically

$$K^{\mathrm{f}}_{\tau_0}(t,z,t',z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t-t')\tau}\chi_{\geq t_0}(t)\chi_{\tau_0}(\tau)\cdot \Pi_{\tau_0}(z,z')\chi_{\geq t_0}(t')d\tau. \tag{41}$$

(Here $\chi_{\geq t_0}(t)$ is a cutoff to $t \geq t_0$, Π_{τ_0} is the projection onto $E_+(\lambda(\tau_0))$, and χ_{τ_0} is a cutoff to a small neighborhood of τ_0.) Near SP we use $K^{\mathrm{p}}_{\tau_0}$, defined similarly but with $\chi_{\geq t_0}$ replaced by $\chi_{\leq t_0}$ which cut off to negative times.

Importantly, we have:

Lemma 7.1 (cf. [2, Lemma 8.9]). *For χ_{τ_0} with sufficiently small support near τ_0,*

$$K^{\mathrm{f/p}}_{\tau_0} \in {}^{3\mathrm{sc}}\Psi^{-\infty,0}.$$

To ensure that localizers $Q \in {}^{3\mathrm{sc}}\Psi_{\mathrm{src},\delta}$ as defined in Definition 4.5 do not modify the eigenspaces $E_\pm(\lambda(\tau))$, we will assume throughout this section that $\mathrm{WF}'_{\mathrm{ff}}(QG_\phi)$ is disjoint from the set of eigenvalues of the limiting Hamiltonian, i.e.,

$$\mathrm{WF}'_{\mathrm{ff}}(QG_\varphi) \cap \mathcal{B}_{V,\pm} = \varnothing\,.$$

The construction in Lemma 4.6 can be easily modified by shrinking the support of f to satisfy this additional condition.

Our spaces now include both the condition that the distributions be above threshold at the sources, but also that the projections onto the appropriate bound states be above threshold. Specifically, if again we have $s, \ell_0, \ell_+ \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+ < 1/2$ and we let $Q_{\mathrm{src}} G_\phi$ denotes the source microlocalizer from Section 4 with the modification just mentioned, and $u \in H^{s,\ell_0}$ then Feynman-type distributions should satisfy

$$Q_{\mathrm{src}} G_\phi u \in H_{\mathrm{sc}}^{s,\ell_+}, \;\; K_\tau^\kappa u \in H_{\mathrm{sc}}^{s,\ell_+} \text{ for all } (\kappa, \tau) \in \mathcal{B}_{V,\mathrm{src}} . \tag{42}$$

These conditions can be encapsulated in a H_A^{s,ℓ_0,ℓ_+} space for appropriate A. Indeed, let

$$\tilde{Q}_{\mathrm{src},\phi} = Q_{\mathrm{src}} G_\phi + \mathcal{K}_{\mathrm{src}} \tag{43}$$

where

$$\mathcal{K}_{\mathrm{src}} := \sum_{(\kappa,\tau)\in\mathcal{B}_{\mathrm{src}}} K_\tau^\kappa . \tag{44}$$

If we (easily) arrange that all of the terms on the RHS have disjoint operator wavefront set, then $\tilde{Q}_{\mathrm{src},\psi} u \in H_{\mathrm{sc}}^{s,\ell_+}$ if and only if the condition in (42) holds.

We now define our spaces as in the previous sections. Since we use the same method for estimating the $Q'_{\mathrm{src}} G_\psi u$ terms, we also use the notation $\tilde{Q}'_{\mathrm{src},\phi}$ for the operator as in (43) whose RHS has $Q'_{\mathrm{src}} G_\psi$ replacing $Q_{\mathrm{src}} G_\phi$. Thus we assume again that $\mathrm{WF}'(Q_{\mathrm{src}}) \subset \mathrm{Ell}(Q'_{\mathrm{src}})$, $\phi, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$ are bump functions supported at 0 with $\phi\psi = \phi$, and $B_{\mathrm{src}} \in {}^{3\mathrm{sc}}\Psi^{0,0}$ with $\mathcal{R}_{\mathrm{ff},\mathrm{src}} \subset \mathrm{Ell}_{\mathrm{ff}}(B_{\mathrm{src}})$. Define

$$\begin{aligned} \mathcal{Y}^{s,\ell_0,\ell_+} &:= \left\{ v \in H_{\mathrm{sc}}^{s-1,\ell_0+1} \colon \tilde{Q}'_{\mathrm{src},\psi} v \in H_{\mathrm{sc}}^{s-1,\ell_++1}, B_{\mathrm{src}} v \in H_{\mathrm{sc}}^{s-2,\ell_+} \right\}, \\ \mathcal{X}^{s,\ell_0,\ell_+} &:= \left\{ u \in H_{\tilde{Q}_{\mathrm{src},\phi}}^{s,\ell_0,\ell_+} \colon P_V u \in \mathcal{Y}^{s,\ell_0,\ell_+} \right\}. \end{aligned} \tag{45}$$

In our treatment of bound states, for technical reasons, we assume that $V - V_\pm$ decays one order faster than the assumption used above. We have the following main theorems in the presence of bound states.

Theorem 7.2. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+ < 1/2$ and $r \geq \max\{2, \ell_+ - \ell_0\}$. Let $V \in \rho_{\mathrm{mf}} {}^{3\mathrm{sc}}\Psi^{1,0}$ be asymptotically static of order r and assume the limiting Hamiltonians have purely absolutely continuous spectrum in $[m^2, \infty)$, and finitely many bound states in $(-\infty, m^2)$ and (38) holds. Moreover, assume that*

$$V - V^* \in {}^{3\mathrm{sc}}\Psi^{0,-2}(\mathbb{R}^{n+1}) .$$

Then

$$P_V : \mathcal{X}^{s,\ell_0,\ell_+} \to \mathcal{Y}^{s,\ell_0,\ell_+}$$

is Fredholm.

Theorem 7.3. *Let $s, \ell_0, \ell_+, r \in \mathbb{R}$ with $\ell_0 < -1/2 < \ell_+ < 1/2$ and $r \geq \max\{2, \ell_+ - \ell_0\}$. Let $V \in \rho_{\mathrm{mf}} \mathrm{Diff}_{3\mathrm{sc}}^{1,0}$ be asymptotically static of order r and*

- *V is self-adjoint, $V = V^*$,*

- *limiting Hamiltonians have purely absolutely continuous spectrum in* $[m^2, \infty)$*, and finitely many bound states in* $(0, m^2)$*, in particular* $H_{V_\pm} > 0$*.*

then $P_V : \mathcal{X}^{s,\ell_0,\ell_+} \to \mathcal{Y}^{s,\ell_0,\ell_+}$ *is invertible. We denote the inverse by* $(P_V)^{-1}_{\mathrm{Fey}}$ *and for* $f \in \dot{\mathcal{C}}^{-\infty}$ *the function* $(P_V)^{-1}_{\mathrm{Fey}} f \in \mathcal{C}^{-\infty}$ *is independent of the choice of parameters* s, ℓ_0, ℓ_+ *and microlocal cutoff* QG_ϕ*. For any* $f \in \dot{\mathcal{C}}^{-\infty}$*, we have*

$$
{}^{3\mathrm{sc}}\mathrm{WF}((P_V)^{-1}_{\mathrm{Fey}} f) \subset {}^{3\mathrm{sc}}\mathrm{WF}(f) \cup \bigcup_{s \geq 0} {}^{3\mathrm{sc}}\Phi_s \left({}^{3\mathrm{sc}}\mathrm{WF}(f) \cap {}^{3\mathrm{sc}}\mathrm{Char}(P_0) \right) \cup {}^{3\mathrm{sc}}\mathcal{R}_{\mathrm{snk}} \cup \mathcal{B}_{V,\mathrm{snk}} \,.
$$

The same conclusion holds if $V = V^* \in \rho_{\mathrm{mf}} \operatorname{Diff}^1_{3\mathrm{sc}}$ *is globally static, i.e.,* $V = V(x)$*. In particular the positivity condition* $H_V > 0$ *is not needed in the static case, though we still assume* $0 \notin \mathcal{B}_V$*.*

To prove the Fredholm result we again use a global Fredholm estimate, and the main result we use from our previous work regarding the $K_\tau^{\mathrm{f/p}}$ is the following lemma.

Lemma 7.4. *Let* $\ell \in \mathbb{R}$*,* $\tau_0 \in \mathcal{B}_{V,\pm}$*. Below we state the estimate near* NP*, but the same is true near* SP *with appropriate* $+$*'s replaced by* $-$*'s.*

Below threshold: Assume $\ell < -1/2$*. Then for any* $s, M, N \in \mathbb{R}$ *there is* $C > 0$ *such that, provided* $K^{\mathrm{f}}_{\tau_0} P_V u \in H^{s,\ell+1}_{\mathrm{sc}}$*, then we have* $K^{\mathrm{f}}_{\tau_0} u \in H^{s,\ell}_{\mathrm{sc}}$

$$
\|K^{\mathrm{f}}_{\tau_0} u\|_{s,\ell} \leq C(\|K^{\mathrm{f}}_{\tau_0} P_V u\|_{s,\ell+1} + \|u\|_{-N,-M}) \,. \tag{46}
$$

Above threshold: Assume $\ell > -1/2$*. For any* $s, \ell', M, N \in \mathbb{R}$ *with* $\ell > \ell' > -1/2$*, there is* C *such that, if* $K^{\mathrm{f}}_{\tau_0} u \in H^{s,\ell'}_{\mathrm{sc}}$ *and* $P_V K^{\mathrm{f}}_{\tau_0} u \in H^{s,\ell+1}_{\mathrm{sc}}$*, then* $K^{\mathrm{f}}_{\tau_0} u \in H^{s,\ell}_{\mathrm{sc}}$ *and*

$$
\|K^{\mathrm{f}}_{\tau_0} u\|_{s,\ell} \leq C(\|K^{\mathrm{f}}_{\tau_0} P_V u\|_{s,\ell+1} + \|K^{\mathrm{f}}_{\tau_0} u\|_{-N,\ell'} + \|u\|_{-N,-M}) \,. \tag{47}
$$

Proof of Fredholm property. To get the closed range and finite dimensional kernel statement, we can use a global Fredholm estimate in Lemma 5.2 but for the $\mathcal{X}, \mathcal{Y}$ spaces in (45), where now

$$
\|v\|_{\mathcal{Y}^{s,\ell_0,\ell_+}} \sim \|v\|_{s-1,\ell_0+1} + \|Q'_{\mathrm{src}} G_\psi v\|_{s-1,\ell_++1} + \|B_{\mathrm{src}} v\|_{s-2,\ell_+} + \|\mathcal{K}_{\mathrm{src}} v\|_{s-1,\ell_++1} \,.
$$

As in the case of the causal propagators [2, p. 91], we modify the partition of unity by introducing additional microlocalizers $B_{5,\pm}$ and $B_{6,\pm}$ such that $B_{5,\pm}$ is elliptic on $(-m + \varepsilon', \delta') \subset W_\pm^\perp$ and $B_{6,\pm}$ is elliptic on $(-\delta', m - \varepsilon')$. For the microlocalizers $B_{5,\pm}$ and $B_{6,\pm}$, we use the estimates from Lemma 7.4. We arrive at the estimate

$$
\|u\|_{\mathcal{X}^{s,\ell_0,\ell_+}} \lesssim \|P_V u\|_{\mathcal{Y}^{s,\ell_0,\ell_+}} + \|u\|_{-N,-M}
$$

from which we deduce finite dimensionality of the kernel and that the range is closed.

By the same argument as in Lemma 5.4, we obtain that cokernel is finite dimensional as well, which implies that P_V is Fredholm as claimed. □

Proposition 7.5. *Let V satisfy the assumptions in Theorem 7.3. Then, for some $\delta > 0$,*

$$\ker_{\mathcal{X}^{s,\ell_0,\ell_+}} P_V \subset H_{\mathrm{sc}}^{s,-1/2+\delta}.$$

Proof. The proof is identical to that of Proposition 5.6 with the modification that, for the operator $B = \mathrm{Op}_L(\tau^{1/2}b(\tau))G_\psi$, we must have $b \equiv 1$ on $\mathcal{B}_{V,\mathrm{snk}} \cup \mathcal{R}_{\mathrm{ff},\,\mathrm{snk}}$ and $\operatorname{supp} b \cap (-\infty, 0] = \varnothing$. This is easily arranged, for example by choosing $c > 0$ with $c < \min\{|\tau_0| : \tau_0 \in \mathcal{B}_{V,\mathrm{snk}}\}$. Then for example at NP we can take $b(\tau) \equiv 1$ for $c \leq \tau \leq m + c$ and $b(\tau) = 0$ for $\tau < c/2$ and $\tau > m + 2c$. □

Proof of invertibility. We now assume that the spatial Hamiltonians satisfy

$$H_{V_\pm} > 0. \tag{48}$$

The proof that the kernel and cokernel are trivial is the same as for the case with no bound states. By Proposition 7.5, we know that every $u \in \ker P_V$ is a Schwartz function and therefore satisfies the above threshold decay rate on the entire radial set. Hence u is an element in a causal Sobolev space $\mathcal{X}^{s,\boldsymbol{\ell}}$ for any choice of $s, \boldsymbol{\ell}$ and by the theorem for causal propagators [2, Theorem 8.3] the operator $P_V : \mathcal{X}^{s,\boldsymbol{\ell}} \to \mathcal{Y}^{s,\boldsymbol{\ell}}$ is invertible and therefore $u \equiv 0$.

The uniqueness and the wavefront property follow from the same arguments as in the case of no bound states.

In the static case one sees directly that there are no global elements in the kernel or cokernel as their projection onto the bound states would have the excluded asymptotics at one of the two time-like infinities. □

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