# MATH 623: DIFFERENTIAL GEOMETRY II

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## 1. Introduction

This is the second semester of a two-semester graduate course providing an introduction to differential geometry. The second semester is primarily a study of Riemannian manifolds with a focus on curvature. At the end of the course, we may go in different directions depending on the interests of the class. Possible directions include comparison theorems, principal bundles and the Atiyah–Singer index theorem, Lorentzian manifolds, the Hodge theorem, or the Chern–Gauss–Bonnet theorem.

The topics roughly covered (updated later based on course interest) include:

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Make sure to add topics!

## 2. Preliminaries and review

Recall from last semester (or your previous experience) the notions of smooth manifold, the tangent and cotangent bundles of a smooth manifold, and tensor fields. Put in definitions to refresh! Maybe recall how to work with the objects!

Sections. Diffeomorphisms.

In coordinates $(x^1, \dots, x^n)$ on a patch $U$ of $M$, recall that $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ form a basis for $T_pM$ for each $p \in U$.

The following characterization of tensor fields is so useful, let's just get it out of the way.

**Lemma 1.** *Suppose $\overline{T} : \mathcal{X} \times \dots \times \mathcal{X} \times \Omega^1 \times \dots \times \Omega^1 \to C^\infty(M)$ is an $\mathbb{R}$-multillinear function of $k$ vector fields and $\ell$ 1-forms (covector fields). Then $\overline{T}$ arises from a $(\ell, k)$-tensor field $T$ if and only if $T$ is multilinear over $C^\infty(M)$ in each of its arguments, e.g.,*

$$\overline{T}\left(fX_1, \dots, X_k, \omega_1, \dots, \omega_\ell\right)(p) = f(p)\overline{T}\left(X_1, \dots, X_k, \omega_1, \dots, \omega_\ell\right)(p).$$

*Proof.* Given a $(\ell, k)$-tensor field $T$, we form $\overline{T}$ from it by evaluating at each point. We then have

$$\begin{aligned}\overline{T}\left(fX_1, \dots, X_k, \omega_1, \dots, \omega_\ell\right)(p) &= T_p\left(f(p)X_{1,p}, \dots, X_{k,p}, \omega_{1,p}, \dots, \omega_{\ell,p}\right) \\ &= f(p)T\left(X_{1,p}, \dots, X_{k,p}, \omega_{1,p}, \dots, \omega_{\ell,p}\right) \\ &= f(p)\overline{T}\left(X_1, \dots, X_k, \omega_1, \dots, \omega_\ell\right)(p).\end{aligned}$$

For the other direction, let's just do the case of $\overline{T} : \mathcal{X} \to C^\infty(M)$. (This contains the main idea; the general case is an exercise in careful bookkeeping.) Suppose for all $f \in C^\infty(M)$ and $X \in \mathcal{X}(M)$ we have

$$\overline{T}(fX)(p) = f(p)\overline{T}(X)(p).$$

Our aim is to show that $\overline{T}$ arises from a 1-form $\omega$. Let's start by defining the purported 1-form. Given $p \in M$ and $v \in T_pM$, choose an $X \in \mathcal{X}(M)$ so that $X_p = v$. We define

$$\omega_p(v) = \overline{T}(X)(p).$$

We must show that $\omega$ is well-defined, i.e., that it does not depend on the choice of vector field $X$. Suppose $X_1$ and $X_2$ are two vector fields with $X_{1,p} = X_{2,p} = v$. In particular, the vector field $Y = X_1 - X_2$ satisfies $Y_p = 0$ and so there are smooth vector fields $Z_1, \dots, Z_r$ and smooth functions $f_1, \dots, f_r$ with $f_1(p) = \dots = f_r(p) = 0$ so that $Y = f_1 Z_1 + \dots + f_r Z_r$. We then have

$$\overline{T}(Y)(p) = \sum_{j=1}^{r} f_j(p)\overline{T}(Z_j)(p) = 0,$$

so that $\overline{T}(X_1)(p) = \overline{T}(X_2)(p)$ and thus $\omega_p : T_pM \to \mathbb{R}$ is well-defined. Its linearity is clear from the linearity of $\overline{T}$ and its smoothness follows from the mapping properties of $\overline{T}$ so it is indeed a 1-form. □

2.1. **Notation.** Unless explicitly noted, all manifolds in this course will be smooth (i.e., $C^\infty$) manifolds.

We use $\mathcal{X}(M)$ to denote the space of $C^\infty$ vector fields on $M$, i.e., smooth sections of $TM$. For sections of other bundles $E \to M$ we often use $\Gamma(E)$ to denote the space of smooth sections. For vector fields along a curve $\alpha$ we use $\mathcal{X}(\alpha)$ and sections of $E$ above a curve $\alpha$ are denoted $\Gamma(E, \alpha)$.

## 3. Riemannian metrics

Suppose $M$ is a smooth manifold of dimension $n$ (typically $n \geq 2$).

Suppose $g : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$ is a symmetric $(0,2)$-tensor. (In other words, $g$ is a tensor field so that $g_p : T_pM \times T_pM \to \mathbb{R}$ is symmetric at each point $p \in M$.)

**Definition 2.** We say that a symmetric $(0,2)$-tensor is a *Riemannian metric* if $g_p(v,v) > 0$ for all $p \in M$, $v \in T_pM$ with $v \neq 0$. The tensor $g$ is *pseudo-Riemannian* if for all $p \in M$, if $v \in T_pM$ has $g_p(v,w) = 0$ for all $w \in T_pM$, then $v = 0$. (In other words, $g$ is pseudo-Riemannian if it is non-degenerate and Riemannian if it is additionally an inner product as you know it from linear algebra.)

A smooth manifold $M$ equipped with a Riemannian metric $g$ is called a *Riemannian manifold.*

In terms of the coordinate basis induced by a coordinate patch $(x^1, \dots, x^n)$ in $M$, we set

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

so that $g_{ij} \in C^\infty(U)$, $g_{ji} = g_{ij}$, and if $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^j \frac{\partial}{\partial x^j}$, then

$$\langle v, w \rangle_p := g_p(v,w) = \sum_{i,j} g_{ij} v^i w^j |_p.$$

A Riemannian metric $g$ then gives an inner product on each tangent space. It also induces metrics on associated bundles (in what follows $g$ is always a metric on the tangent spaces; notation for the induced metrics varies wildly by source and eventually we'll just use $g$ to denote all of them unless there can be confusion as to where various objects live).

One way to view the mechanism for this induction is that $g$ gives a way to "raise and lower indices". More precisely, a Riemannian metric $g$ provides an isomorphism between the tangent and cotangent spaces at each point. Given $\omega \in T_p^*M$, we associate to it a vector $w_\omega \in T_pM$ by demanding that

$$g_p(w_\omega, v) = \omega(v)$$

for all $v \in T_pM$. Because $g$ is non-degenerate, this uniquely defines the vector $w_\omega$. In local coordinates, the displayed equation reads

$$\sum_{i,j} g_{ij}(w_\omega)^i v^j = \sum_j \omega_j v^j,$$

so that

$$(w_\omega)^i = \sum_j g^{ij}\omega_j,$$

where $g^{ij}$ are the components of the matrix inverse of $(g_{ij})$. Similarly, if $v \in T_pM$, one can identify it with the one-form $\omega_v \in T_p^*M$ so that

$$\omega_v(u) = g_p(v, u)$$

for all $u \in T_pM$. In local coordinates, $(\omega_v)_i = \sum_j g_{ij}v^j$.

(1) Cotangent bundle. Given $\omega, \eta \in T_p^*M$, we define the metric $G$ (sometimes denoted $g^{-1}$, sometimes still just $g$) by

$$G(\omega, \eta) = g(w_\omega, w_\eta),$$

where $w_\omega$ is the vector associated to $\omega$ and $w_\eta$ is the one associated to $\eta$. In coordinates, we have that the $(i, j)$-component of the metric $G$ is the same as the $(i, j)$ component of the matrix $g^{-1} = (g_{k\ell})^{-1}$, i.e., $g^{ij}$. To check this, observe that

$$\begin{aligned} g(w_\omega, w_\eta) &= \sum_{i,j} g_{ij}(w_\omega)^i(w_\eta)^j \\ &= \sum_{i,j,k,\ell} g_{ij}g^{ik}\omega_k g^{j\ell}\eta_\ell \\ &= \sum_{jk\ell} \delta_j^k g^{j\ell}\omega_k\eta_\ell = \sum_{i,j} g^{ij}\omega_i\eta_j. \end{aligned}$$

(2) Tensor bundles. For $T_pM \otimes T_pM$ (and higher powers), say that

$$g^{\otimes}(v_1 \otimes v_2, w_1 \otimes w_2) = g(v_1, w_1)g(v_2w_2),$$

and extend linearly. For factors of $T_p^*M$, also use the raising/lower operator.

(3) Exterior powers. Use that $\Lambda^k(TM)$ is a sub-bundle of $(T^*M)^{\otimes k}$ and use above.

(4) Endomorphism bundle. Identify $\mathrm{End}(TM)$ with $T^*M \otimes TM$.

As an example of a Riemannian metric, suppose $F : M \to \mathbb{R}^N$ is an immersion (so that $dF_p$ is injective for all $p$ and thus has rank $\dim M$). The immersion $F$ (and the ambient inner product on $\mathbb{R}^N$) induces a Riemannian metric on $M$ by

$$g_p(v, w) = \langle dF_p(v), dF_p(w)\rangle_{\mathbb{R}^N}$$

for all $v, w \in T_pM$. Exercise: Check that this is a Riemannian metric on $M$.

**Definition 3.** Two Riemannian manifolds $(M, g_M)$ and $(N, g_N)$ are *isometric* if there is a diffeomorphism $F : M \to N$ so that

$$\langle dF_p(v), dF_p(w)\rangle_{g_N} = \langle v, w\rangle_{g_M}$$

for all $p \in M$ and $v, w \in T_pM$.

In other words, $(M, g_M)$ is isometric to $(N, g_N)$ if there is a diffeomorphism $F : M \to N$ for which $F^* g_N = g_M$.

More examples:

(1) $\mathbb{R}^n$ equipped with the dot product. Here $T_p\mathbb{R}^n \cong \mathbb{R}^n$ canonically, and $g(v, w) = v \cdot w$. Once we have the machinery to make this precise, we'll see that this is our model of a *flat* space.

(2) $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with the metric induced by the inclusion map. Concretely, we can use coordinates $(\theta^1, \dots, \theta^n) \in [0, \pi)^{n-1} \times [0, 2\pi)$ on a large patch of $\mathbb{S}^n$ together with the map $F$ given by

$$F(\theta^1, \dots, \theta^n) = \begin{pmatrix} \cos\theta^1 \\ \sin\theta^1 \cos\theta^2 \\ \sin\theta^1 \sin\theta^2 \cos\theta^3 \\ \vdots \\ \sin\theta^1 \sin\theta^2 \dots \sin\theta^{n-1} \cos\theta^n \\ \sin\theta^1 \sin\theta^2 \dots \sin\theta^{n-1} \sin\theta^n \end{pmatrix}$$

A straightforward computation shows that

$$dF_{(\theta^1,\dots,\theta^n)} = \begin{pmatrix} -\sin\theta^1 & 0 & \dots & 0 \\ \cos\theta^1 \cos\theta^2 & -\sin\theta^1 \sin\theta^2 & \dots & 0 \\ \cos\theta^1 \sin\theta^2 \cos\theta^3 & \sin\theta^1 \cos\theta^2 \cos\theta^3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \cos\theta^1 \sin\theta^2 \dots \sin\theta^n & \sin\theta^1 \cos\theta^2 \dots \sin\theta^n & \dots & \sin\theta^1 \sin\theta^2 \dots \cos\theta^n \end{pmatrix}$$

In particular, we have

$$g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right) = dF\left(\frac{\partial}{\partial\theta^i}\right) \cdot dF\left(\frac{\partial}{\partial\theta^j}\right) = \begin{cases} 0 & i \neq j \\ 1 & i = j = 1 \\ \sin^2\theta^1 \sin^2\theta^2 \dots \sin^2\theta^{j-1} & i = j \neq 1 \end{cases}.$$

(This in fact homogeneous, isotropic, etc., but we haven't defined these terms.) The sphere is the nicest example of a *positively curved* space (to be made more precise later).

(3) Hyperbolic space $\mathbb{H}^n$. We'll describe three models. (But there are more! Hyperbolic space just keeps on giving.) Fix some $R > 0$ (this is a parameter going into the metric, just as we could have changed the radius of our sphere in the previous example).

   (a) The upper half-space $U_R = \{(x, y^1, \dots, y^{n-1}) \in \mathbb{R}^n \mid x > 0\}$ equipped with the metric

$$g = R^2 \frac{dx^2 + (dy^1)^2 + \dots + (dy^{n-1})^2}{x^2},$$

   i.e., we take the inner product of two vectors $v, w \in T_{(x,y)}U_R$ by

$$\langle v, w\rangle_{(x,y)} = R^2 \frac{v \cdot w}{x^2}.$$

(b) The Poincaré ball model: Let $B^n_R \subset \mathbb{R}^n$ denote the ball of radius $R$ and equip it with the metric

$$g = 4R^2 \frac{(du^1)^2 + \cdots + (du^n)^2}{(R^2 - |u|^2)^2}.$$

(c) The hyperboloid model: Consider $\mathbb{R}^{n+1}$ equipped with the pseudo-Riemannian metric $\eta_{(t,x^1,\ldots,x^n)} = -(dt)^2 + \sum (dx^j)^2$ and let $\mathbb{H}^n_R$ denote one sheet of the two sheeted hyperboloid:

$$\mathbb{H}^n_R = \{t > 0\} \cap \left\{ -t^2 + |x|^2 = -R^2 \right\},$$

and let $g = i^*\eta$, where $i : \mathbb{H}^n_R \to \mathbb{R}^{n+1}$ is the inclusion.

**Theorem 4.** *All three of the above models of hyperbolic space are isometric.*

*Proof.* You should fill in most of this proof yourself! I'll give you the maps to show it for the hyperboloid and ball models.

Let $S = (-R, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ and let $P \in \mathbb{H}^n_R$, say $P = (t, x^1, \ldots, x^n)$. Define the map $\pi : \mathbb{H}^n_R \to B^n_R$ by letting $\pi(P) = u \in \mathbb{R}^n$, where $(0, u)$ is the point where the line from $S$ to $P$ intersects $\{t = 0\}$.

Note that this line is given by

$$(-R, 0, \ldots, 0) + s(t + R, x^1, \ldots, x^n),$$

which hits $t = 0$ when $st + sR - R = 0$, i.e., when $s = \frac{R}{t+R}$, so that

$$\pi\left(t, x^1, \ldots, x^n\right) = \frac{R}{t+R}(x^1, \ldots, x^n).$$

As $P \in \mathbb{H}^n_R$, we have that $|x|^2 = t^2 - R^2$ and thus

$$|\pi(P)|^2 = \frac{R^2}{(t+R)^2}|x|^2 = \frac{R^2(t^2 - R^2)}{(t+R)^2} = R^2 \frac{t - R}{t + R} < R^2$$

and thus $\pi(P) \in B^n_R$.

The inverse map $\pi^{-1} : B^n_R \to \mathbb{H}^n_R$ is given by

$$\pi^{-1}(z^1, \ldots, z^n) = \left( R\frac{R^2 + |z|^2}{R^2 - |z|^2}, 2\frac{R^2 z}{R^2 - |z|^2} \right).$$

You should check that this is the correct form of the inverse and that both $\pi$ preserves the inner product. $\square$

## 4. Covariant differentiation and connections

Recall that a (smooth) $k$-dimensional vector bundle over a smooth manifold $M$ consists of the data $\pi : E \to M$ so that

(1) $\pi$ is surjective,
(2) $\pi^{-1}(p)$ is a $k$-dimensional vector space for each $p \in M$, and
(3) for each $p \in M$, there is a chart $(x, U)$ around $p$ in $M$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to x(U) \times \mathbb{R}^k$ that restricts to a vector space isomorphism on each fiber.

One of the first challenges in differential geometry is to determine how to differentiate sections of a vector bundle. For a trivial vector bundle in Euclidean space, you have the "constant sections" and so you just differentiate their coefficients. In general, however, there is no constant section because there is no canonical way of identifying the different fibers of the bundle. One way to get around this is to define the notion of "parallel transport" along a curve. In this view, for each smooth path $\gamma : [a,b] \to M$, we equip the bundle $E$ with linear maps $P(\gamma)_s^t : E_{\gamma(s)} \to E_{\gamma(t)}$ depending smoothly on $s, t \in [a,b]$ (and also on $\gamma$ in an appropriate sense). We further demand that

$$P(\gamma)_r^t \circ P(\gamma)_s^r = P(\gamma)_s^t.$$

The maps $P$ provide a way of performing parallel translation along the curve $\gamma$. (If $E$ were equipped with a way of measuring distance or angles, we'd also demand that the parallel translation preserve this.) Given these maps, we could differentiate a section $Y$ of $E$ at $p \in M$ in the direction $v \in T_pM$ by taking a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$ and then finding

$$\nabla_v Y = \lim_{s\to 0} \frac{P(\gamma)_s^0(Y_{\gamma(s)}) - Y_p}{s}.$$

One can check that this is a derivation, but showing that in fact it depends only on $v$ and not on the extension $\gamma$ doesn't quite follow without a more careful accounting of hypotheses.

Instead of defining parallel translation directly, we instead recover it from one of the other related quantities. As is common in many differential geometry texts (especially those focusing on vector bundles like the tangent and cotangent bundles), we'll use the notion of a *Koszul connection*, which we'll just call a *connection*.

**Definition 5.** A connection $\nabla$ on the vector bundle $\pi : E \to M$ is an $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ so that the product rule holds, i.e.,

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for all smooth functions $f \in C^\infty(M)$ and smooth sections $s \in \Gamma(E)$.

Unwinding this definition, it's the same as providing, for each section $s \in \Gamma(E)$ and $p \in M$, an $\mathbb{R}$-linear map $(\nabla s)_p : T_pM \to E_p$ so that

(1) $(\nabla s)_p$ depends smoothly on $p$,
(2) for all $a, b \in \mathbb{R}$ and $s_1, s_2 \in \Gamma(E)$, $\nabla(as_1 + bs_2)_p = a(\nabla s_1)_p + b(\nabla s_2)_p$, and
(3) $\nabla$ satisfies a product rule, so for all smooth functions $f$ on $M$ and $v \in T_pM$,

$$\nabla(fs)_p(v) = df_p(v)s_p + f(p)(\nabla s)_p(v).$$

From now on we'll drop the $p$ subscript and let it be implicit (as $v \in T_pM$). We also typically write $(\nabla s)(v)$ as $\nabla_v s$. When $v$ is the value of a vector field $X \in \mathcal{X}(M)$, we also write $\nabla_X s$, which is the section with value $\nabla_{X_p} s$ at $p$.

The following lemma tells us that connections are local and so we will not need to worry about whether sections are defined globally or only locally.

**Lemma 6.** *If $\nabla$ is a connection on $E$ and $s_1, s_2 \in \Gamma(E)$ are such that $s_1 \equiv s_2$ in a neighborhood of $p \in M$, then for all $v \in T_pM$, $\nabla_v s_1 = \nabla_v s_2$.*

*Proof.* By linearity it suffices to show that if $s \equiv 0$ in a neighborhood of $p$ then $\nabla_v s = 0$. This statement follows from the product rule. Indeed, for any $f \in C^\infty(M)$ so that $\operatorname{supp} f \subset \{s = 0\}$, we have $fs \equiv 0$ on $M$, and

$$0 = \nabla_v(fs) = df_p(v)s_p + f(p)\nabla_v s,$$

so that $f(p)\nabla_v s = 0$. This is true for any such $f$, so $\nabla_v s = 0$. $\square$

A word of warning: you might think that because a connection eats vector fields and gives you vector fields that it is a tensor, but the product rule (and Lemma 1) tells you that it's not. Connections do, however, lie in an affine space whose underlying linear space is the space of tensors.

**Lemma 7.** *If $\nabla$ and $\widetilde{\nabla}$ are two connections on the tangent bundle $TM$ then the difference $\nabla - \widetilde{\nabla}$ is a $(1,2)$-tensor.*

*Proof.* Let $T = \nabla - \widetilde{\nabla}$ be the $\mathbb{R}$-multilinear object $\mathcal{X} \times \mathcal{X} \times \Omega^1 \to C^\infty$ given by

$$T(X, Y, \omega) = \omega\left(\nabla_X Y - \widetilde{\nabla}_X Y\right).$$

By Lemma 1, we must only check that $T$ is multilinear over $C^\infty$. As it is already multilinear over $C^\infty$ in $X$ and $\omega$, we need only check in $Y$, but this follows from the product rule. $\square$

We can think of $\nabla_v s$ as denoting a directional derivative of $s$ in the direction of $v$. Just as we did in calculus (and in the previous semester of this course), we'd like to also differentiate along curves. Let's fix a curve $\alpha : (a, b) \to M$.

**Definition 8.** A section along the curve $\alpha$ is a map $t \mapsto s(t) \in E_{\alpha(t)}$ depending smoothly on $t$. In an abuse of notation we'll denote the set of smooth sections along $\alpha$ by $\Gamma(E, \alpha)$.

**Proposition 9.** *There is a unique map $\Gamma(E, \alpha) \to \Gamma(E, \alpha)$, denoted $s \mapsto \frac{D}{dt}s$ and called the covariant derivative of $s$ along $\alpha$, so that*

*(1)* $\dfrac{D}{dt}\left(s_1(t) + s_2(t)\right) = \dfrac{D}{dt}s_1(t) + \dfrac{D}{dt}s_2(t)$,

*(2)* $\dfrac{D}{dt}\left(f(t)s(t)\right) = f(t)\dfrac{D}{dt}s(t) + f'(t)s(t)$, *and*

*(3) If $\widetilde{s} \in \Gamma(E)$ satisfies $s(t) = \widetilde{s}_{\alpha(t)} \in E_{\alpha(t)}$, then* $\dfrac{D}{dt}s(t) = \nabla_{\alpha'(t)}\widetilde{s}$.

*Proof.* By localization we can assume that $\alpha(I)$ is contained in a single coordinate chart on which the bundle $E$ is trivial. We then take a local basis $e_1, \ldots, e_k$ for all $E_p$ for $p$ contained in this chart and write $s(t) = \sum_{j=1}^k s^j(t)e_j$.

For uniqueness, we observe that if $\frac{D}{dt}$ satisfies all three conditions, we must have

$$\begin{aligned} \frac{D}{dt}s(t) &= \sum_{j=1}^k \frac{D}{dt}(s^j(t)e_j) \\ &= \sum_{j=1}^k \left((s^j)'(t)e_j + s^j(t)\nabla_{\alpha'(t)}e_j\right), \end{aligned} \tag{1}$$

as $e_j$ are defined in a neighborhood. The right side does not depend on $\frac{D}{dt}$ so we have uniqueness.

For existence, we now have a formula: we write $s$ in terms of a local basis for the sections and use equation (1) to define the covariant derivative along $\alpha$. $\square$

Another notion of connection is called an *Ehresmann connection* and involves a splitting of the tangent bundle of $E$ into "horizontal" and "vertical" sub-bundles. In particular, there is always a canonical sub-bundle $V$ of $TE$ (called the "vertical bundle") given by the kernel of the pushfoward map (i.e., the differential of the projection) $\pi_* : TE \to TM$. An Ehresmann connection is the data of a "horizontal" sub-bundle complementary to the vertical one, i.e., a sub-bundle $H \subset TE$ so that $TE = H \oplus V$. The notion of connection above induces such a splitting. (To ensure that it is equivalent to the definition above involves another condition that we omit here.)

**Proposition 10.** *A connection $\nabla$ on $E$ induces a splitting $TE = H \oplus V$.*

*Proof.* We must show that $\nabla$ defines a horizontal sub-bundle $H \subset TE$ and that $TE$ splits as the direct sum $H \oplus V$. We start by noting that, for each $e \in E$, $V_e = T_e E_{\pi(e)} \cong E_{\pi(e)}$ because the tangent space of a vector space is canonically isomorphic to the vector space.

We now aim to define the horizontal subspace. We first define a map $K : T_e E \to E_{\pi(e)}$ and then define $H_e$ to be the kernel of $K$. Given $e \in E$ and $v \in T_e E$, choose $\gamma : (-\epsilon, \epsilon) \to E$ so that $\gamma(0) = e$ and $\gamma'(0) = v$. We now regard $\gamma$ as a section of $E$ over $\pi \circ \gamma$, i.e., $\gamma \in \Gamma(E, \pi \circ \gamma)$ and set

$$Kv = \frac{D}{dt} \gamma(t)|_{t=0}.$$

We claim that $Kv$ is independent of the choice of $\gamma$. By linearity it suffices to show that if $\gamma : (-\epsilon, \epsilon) \to E$ has $\gamma(0) = e$ and $\gamma'(0) = 0$, then $\frac{D}{dt}\gamma(t)|_{t=0} = 0$. We then note that $(\pi \circ \gamma)'(0) = 0$ and appeal to equation (1) after writing $\gamma$ in terms of a local frame for $E$ to see that indeed $\frac{D}{dt}\gamma(t)|_{t=0} = 0$.

Now, equipped with the map $K : T_e E \to E_{\pi(e)}$, we define $H_e = \ker K$. We now claim that $T_e E \cong H_e + V_e$. Indeed, note that if $v \in V_e$ is a vertical vector, we use the identification $V_e \cong E_{\pi(e)}$ to construct the curve $\gamma : (-\epsilon, \epsilon) \to E_{\pi(e)}$ given by $\gamma(t) = e + tv$. This curve satisfies $\gamma(0) = e$ and $\gamma'(0) = v$ and $\frac{D}{dt}\gamma(t)|_{t=0} = v$, so $Kv = v$ for vertical vectors. The operator $K$ can therefore be regarded as a projection onto $V_e$ and so $T_e E \cong H_e \oplus V_e$.

The smoothness of the sub-bundles follows from the smoothness of the maps $(\pi)_*$ and $K$; that $K$ depends smoothly on $e$ is a consequence of the identity (1). $\square$

We now return to parallel transport. Given a curve $\alpha : [0, 1] \to M$ so that $\alpha(0) = p$ and $\alpha(1) = q$, we can construct the parallel translation of a vector $v \in E_p$ along $\alpha$ in two related ways. One way is by solving a differential equation: we say that a section $s \in \Gamma(E, \alpha)$ is parallel if and only if $\frac{D}{dt}s(t) = 0$ for all $t \in [0, 1]$.

**Lemma 11.** *For every $v \in E_p$, there is a unique $s \in \Gamma(E, \alpha)$ so that $s(0) = v$ and $s$ is parallel.*

*Proof.* Working locally in charts where the bundle is trivial, this again follows from the identity (1), this time interpreted as a linear system of differential equations for the coefficients of the frame. It is not hard to check that existence and uniqueness for ODEs then guarantees a solution. $\square$

We then define the parallel translate of $v$ by $P(\alpha)_0^1 v = s(1)$. Note that this value typically depends on the choice of path!

Another way to define the parallel translate of $v$ along $\alpha$ to lift $\alpha$ to a curve $\widetilde{\alpha} : [0,1] \to E$ so that $\pi \circ \widetilde{\alpha} = \alpha$, $\alpha(0) = (p, v)$ and so that $\widetilde{\alpha}'(t) \in H_{\widetilde{\alpha}(t)}$ for all $t$. The existence of such a lift follows from the observation that $H_e \cong T_{\pi(e)}M$ for all $e \in E$ and the decomposition of $TE = V \oplus H$. (You need $\pi_* \widetilde{\alpha}'(t) = \alpha'(t)$ and you need $\widetilde{\alpha}'(t)$ to be horizontal.)

We'll return to the question of why parallel transport is so called once we start talking about specific connections.

4.1. **Induced connections.** A connection $\nabla$ on a vector bundle $E$ over $M$ induces connections over other bundles formed from $E$. A few examples:

(1) Dual bundles. If $\nabla$ is a connection on $E$, we get a connection $\nabla^*$ (we'll later just call this $\nabla$) on the dual bundle $E^*$ by duality. Indeed, if $\xi \in \Gamma(E^*)$, we define $\nabla_v \xi$ by demanding that, for all $s \in \Gamma(E)$,
$$d\left(\langle \xi, s\rangle\right)(v) = \langle \nabla^*_v \xi, s\rangle + \langle \xi, \nabla_v s\rangle.$$
(2) If $\nabla^E$ and $\nabla^F$ are connections on vector bundles $E$ and $F$ over $M$, then we get a connection $\nabla^E \otimes \nabla^F$ on the vector bundle $E \otimes F$ over $M$ by demanding it satisfy a product rule:
$$\left(\nabla^E \otimes \nabla^F\right)_v (s \otimes t) = \nabla^E_v s \otimes t + s \otimes \nabla^F_v t.$$
(3) A similar construction gives a connection on the exterior powers $\Lambda^k E$ by a product rule:
$$\nabla_v \left(s_1 \wedge \cdots \wedge s_k\right) = \nabla_v s_1 \wedge \cdots \wedge s_k + \cdots + s_1 \wedge \cdots \wedge \nabla_v s_k.$$
(4) Similarly we get a connection $\nabla^E \oplus \nabla^F$ on the direct sum bundle $E \oplus F$ by linearity:
$$\left(\nabla^E \oplus \nabla^F\right)_v (s \oplus t) = \left(\nabla^E_v s\right) \oplus \left(\nabla^F_v t\right).$$
(5) By identifying the endomorphisms of $E$ with $E^* \otimes E$ we also get a connection on the endomorphism bundle $\mathrm{End}(E)$.

As a result, if we have a connection on the vector bundle $TM$ then we in fact have a connection on all of the tensor bundles.

4.2. **The Levi-Civita connection.** We now specialize to the case where $E = TM$ and its associated vector bundles.

**Definition 12.** Given a connection $\nabla$ on $TM$, its *torsion* is given by
$$T(\nabla)(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in C^\infty(M)$$
where $X, Y \in \mathcal{X}(M)$ and $[X, Y]$ is the Lie bracket.

**Lemma 13.** *The torsion of a connection on $TM$ is an antisymmetric $(0,2)$-tensor.*

*Proof.* It's clearly antisymmetric; that it is a tensor follows from the characterization of tensor fields given by Lemma 1. Indeed, we have to check that it is multilinear over $C^\infty(M)$. Take $f \in C^\infty(M)$, $X, Y \in \mathcal{X}(M)$ and consider
$$\begin{aligned} T(\nabla)(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\ &= f\nabla_X Y - (Y(f)X + f\nabla_Y X) - (f[X, Y] - Y(f)X) \\ &= fT(\nabla)(X, Y). \end{aligned}$$
□

**Theorem 14.** *Suppose $(M, g)$ is a Riemannian manifold. There is a unique connection $\nabla$ so that*

*(1) $\nabla$ is torsion-free (i.e., $\nabla_X Y - \nabla_Y X - [X, Y] = 0$), and*
*(2) $\nabla$ is compatible with the metric $g$, i.e., for all $X, Y, Z \in \mathcal{X}(M)$,*

$$X\left(g(Y, Z)\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right).$$

**Definition 15.** The unique torsion-free connection compatible with $g$ is called the *Levi-Civita connection.*

*Proof.* To prove uniqueness, suppose that $\nabla$ is torsion-free and compatible with $g$. Then compatibility yields

$$\begin{aligned}
&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle \\
&\quad = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\
&\qquad - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle,
\end{aligned}$$

while $\nabla$ being torsion-free lets us write, e.g., $\nabla_X Z - \nabla_Z X = [X, Z]$, so that the above expression is equal to (here using that $g$ is symmetric)

$$\begin{aligned}
&\langle \nabla_X Y, Z\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle \\
&\quad + \langle Z, \nabla_X Y + [Y, X]\rangle - \langle X, [Y, Z]\rangle - \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle \\
&= 2\langle \nabla_X Y, Z\rangle .
\end{aligned}$$

In other words, we end up with the equality

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left[X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle\right].$$

Because $g$ is non-degenerate and this relationship must hold for all $Z$, any two choices of torsion-free metric-compatible connection must agree.

Similarly, because $g$ is non-degenerate, the formula above also defines the connection $\nabla$, so we also get existence. (You should check that it is tensorial in $X$ and $Z$ and satisfies a product rule in $Y$ and should also check that it is torsion-free and metric-compatible.) ☐

How does this connection look in local coordinates? Let $(x^1, \dots, x^n)$ be coordinates in a chart on $M$ and $\partial_j$ denote the corresponding basis for the tangent space. We then have

$$\begin{aligned}
\langle \nabla_{\partial_i} \partial_j, \partial_k\rangle &= \frac{1}{2}\left[\partial_i\langle \partial_j, \partial_k\rangle + \partial_j\langle \partial_i, \partial_k\rangle - \partial_k\langle \partial_i, \partial_j\rangle\right] \\
&= \frac{1}{2}\left[\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right].
\end{aligned}$$

If we write $\nabla_{\partial_i}\partial_j = \sum_\ell \Gamma^\ell_{ij}\partial_\ell$, then

$$\begin{aligned}
\sum_{\ell=1}^n \Gamma^\ell_{ij} g_{\ell k} &= \sum_{\ell=1}^n \Gamma^\ell_{ij}\langle \partial_\ell, \partial_k\rangle \\
&= \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right).
\end{aligned}$$

This is an $n \times n$ system of linear equations, which we solve to find

$$\Gamma^k_{ij} = \sum_{\ell=1}^n \frac{1}{2} g^{k\ell}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right),$$

where $g^{k\ell}$ are the components of the inverse metric (i.e., the inverse of the matrix $(g_{ij})$ or the components of the induced metric on the cotangent bundle, etc.).

**Definition 16.** The $\Gamma_{ij}^k$ are called *Christoffel symbols* or *connection coefficients.*

Word of warning: $\Gamma_{ij}^k$ are NOT the components of a tensor (they lie in an affine space).

As described above, $\nabla$ induces a connection on the cotangent bundle, given for $\omega \in \Omega^1(M)$ by

$$(\nabla_v \omega)(X) = v(\omega(X)) - \omega(\nabla_v X).$$

For more general tensors it has an analogous form, with $\nabla_v A$ of a $(k,\ell)$-tensor given by

$$\begin{aligned}(\nabla_v A)(\omega_1, \dots, \omega_k, X_1, \dots, X_\ell) = v\left(A(\omega_1, \dots, \omega_k, X_1, \dots, X_\ell)\right) \\ - \sum_{i=1}^{k} A\left(\omega_1, \dots, \omega_{i-1}, \nabla_v \omega_i, \omega_{i+1}, \dots, \omega_k, X_1, \dots, X_\ell\right) \\ - \sum_{j=1}^{\ell} A\left(\omega_1, \dots, \omega_k, X_1, \dots, X_{j-1}, \nabla_v X_j, X_{j+1}, \dots, X_\ell\right).\end{aligned}$$

**Proposition 17.** *Parallel transport using the Levi-Civita connection is an isometry.*

*Proof.* Suppose $\alpha : [0,1] \to M$ is smooth and $V, W \in \mathcal{X}(\alpha)$. Then

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \left\langle \frac{D}{dt}V(t), W(t)\right\rangle + \left\langle V(t), \frac{D}{dt}W(t)\right\rangle,$$

so that if $V$ and $W$ are parallel then their inner product is preserved. $\square$

## 5. Geodesics and Hamiltonian flows

Most differential geometry textbooks use the connection directly to define and reason about geodesics. We'll instead take a Hamiltonian approach. To do that, we need some preliminaries about the symplectic structure on the cotangent bundle.

### 5.1. Symplectic manifolds.

**Definition 18.** A manifold $(M, \omega)$ is called a symplectic manifold if $M$ is a smooth manifold and $\omega$ is a non-degenerate closed 2-form on $M$.

In other words, at each point $\omega$ is an alternating $(0,2)$-tensor so that $d\omega = 0$ and, if $v \in T_pM$ satisfies

$$\omega(v, u) = 0 \text{ for all } u \in T_pM,$$

then $v = 0$.

**Lemma 19.** *A symplectic manifold must be even-dimensional.*

*Proof.* Working in local coordinates, this reduces to the statement that if $n$ is odd, then any skew-symmetric $n \times n$ real matrix must have a kernel. To see this, a skew-symmetric matrix $A$ has

$$\det A = \det(A^\intercal) = \det(-A) = (-1)^n \det A,$$

so that $\det A = 0$ if $n$ is odd and thus 0 is an eigenvalue of $A$, i.e., $A$ must have a kernel. $\square$

There is a lot to say about symplectic manifolds, most of which we omit here. One of the most famous is Darboux's theorem, which essentially says that there are no local invariants of symplectic manifolds, i.e., one can always find local coordinates $(q, p)$ so that $\omega = \sum_i dq^i \wedge dp_i$. There's a lovely proof of this using Moser's trick which I'm happy to talk about if you like:

**Theorem 20** (Darboux)**.** *Suppose $(M, \omega)$ is a symplectic $2k$-dimensional manifold. Around any point $p \in M$ there is a coordinate chart $U$ and coordinates $(x^1, \dots, x^k, y^1, \dots, y^k)$ so that*

$$\omega|_U = \sum_{i=1}^{k} dx^i \wedge dy^i.$$

*Proof.* You can always find a coordinate system $(\widetilde{x}^1, \dots, \widetilde{x}^k, \widetilde{y}^1, \dots, \widetilde{y}^k)$ achieving this exactly at the point $p$. We claim then that there is a local diffeomorphism $\phi$ of a smaller neighborhood so that

$$\phi^* \left( \sum d\widetilde{x}^i \wedge d\widetilde{y}^i \right) = \omega|_U,$$

and then the desired coordinate system is $x = \widetilde{x} \circ \phi$ and $y = \widetilde{y} \circ \phi$.

It remains to prove the claim. We first claim that if $\omega_t$ is a family of symplectic forms so that $\frac{d}{dt}\omega_t = d\sigma_t$ for some 1-forms $\sigma_t$, then there is a family of diffeomorphisms $\psi_t$ so that $\psi_t^* \omega_t = \omega_0$. Indeed, this follows from Lemma 21 below by using Cartan's magic formula

$$\mathcal{L}_X \omega = d \circ i_X + i_X \circ d,$$

where $i_X$ denotes the interior product (i.e., plugging $X$ into the first slot) and observing that the differential equation in Lemma 21 becomes

$$0 = \frac{d}{dt}\omega_t + d(i_X \omega_t) + i_X(d\omega_t) = d(\sigma_t + i_X \omega_t).$$

As $\omega_t$ is non-degenerate, one can find $X$ so that $\sigma_t = i_X \omega_t$ and so Lemma 21 applies.

Finally, by the Poincaré lemma (which says that on $\mathbb{R}^n$, closed forms are exact), the difference $\omega_t - \omega_0 = d\sigma_t$ and so we can apply the claim. ☐

**Lemma 21** (Moser's trick)**.** *Suppose $\omega_t$, $t \in [0, 1]$ is a family of differential forms on $M$. If there is a solution $X_t \in \mathcal{X}(M)$, $t \in [0, 1]$ to the differential equation*

$$\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = 0,$$

*where $\mathcal{L}_X$ denotes the Lie derivative with respect to $X$, then there exists a family of diffeomorphisms $\psi_t$ on $M$ so that $\psi_t^* \omega_t = \omega_0$ and $\psi_0 = \mathrm{Id}$.*

*Proof.* Given $X_t$, let $\psi_t$ be the flow it generates, so

$$\frac{d}{dt}(\psi_t^* \omega_t) = \psi_t^* \left( \frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t \right) = 0,$$

so $\psi_t^* \omega_t = \psi_0^* \omega_0 = \omega_0$. ☐

Now, back on track. One of the main things we want to use symplectic structures for is to get *Hamilton vector fields*. Just as we had with Riemannian metrics, we can use the nondegenerate 2-form $\omega$ to identify the tangent and cotangent spaces at each point. Indeed, given any 1-form $\sigma$, we can find some vector field $X$ associated to it by demanding that

$$\omega(v, X) = \sigma_p(v)$$

for all vectors $v \in T_pM$. In particular, if we have a real-valued function $H$ (called a Hamiltonian) on $M$, we can find the *Hamilton vector field* of $H$, which we'll denote $X_H$, by demanding that $X_H$ be the vector field associated to $dH$ by $\omega$.

As smooth vector fields yield flows, we therefore also obtain a flow $\phi_t$ from the Hamilton vector field $X_H$.

**Lemma 22.** *The Hamiltonian $H$ is conserved by the flow $\phi_t$.*

*Proof.* This is an exercise in unraveling the definitions and using the chain rule. Indeed, suppose $p \in M$ and let $\gamma(t) = \phi_t(p)$, so that

$$\begin{aligned} \gamma'(t) &= (X_H)_{\gamma(t)}, \\ \gamma(0) &= p. \end{aligned}$$

We then differentiate $H(\gamma(t))$:

$$\begin{aligned} \frac{d}{dt} H(\gamma(t)) &= dH(\gamma'(t)) \\ &= dH(X_H) = \omega(X_H, X_H) = 0, \end{aligned}$$

so that $H(\gamma(t)) = H(\gamma(0))$. $\square$

5.2. **A very brief foray into Hamiltonian mechanics.** One of the most important examples of a symplectic manifold is the cotangent bundle $T^*M$ of a smooth manifold $M$. (Note that the dimension of $T^*M$ is always twice the dimension of $M$ and therefore even.) If $\pi : T^*M \to M$ denotes the projection, we can define the *canonical* 1*-form* $\alpha$ on $T^*M$ by

$$\alpha_{(x,\xi)}(v) = \xi(\pi_* v)$$

where $(x, \xi) \in T^*M$ and $v \in T_{(x,\xi)}(T^*M)$. In other words, the form $\alpha$ acts on a vector $v$ at a point $(x, \xi)$ in the cotangent bundle by evaluating the covector $\xi$ (a covector on $M$) on the pushvorward of $v$. In terms of local coordinates $(x, \xi)$,[1] you can check that

$$\alpha = \sum_j \xi_j dx^j.$$

We then define a symplectic form $\omega$ by $\omega = d\alpha$. In a local coordinate system $\omega$ has the form

$$\omega = \sum_J d\xi_j \wedge dx^j.$$

It is plainly a 2-form, $d\omega = d(d\alpha) = 0$, and there are several ways to check that it is non-degenerate. One way is to observe that $\omega^n$, the $n$-th wedge power of $\omega$, is a non-vanishing volume form. We can also check it directly by taking $v \in T_{(x,\xi)}(T^*M)$ with $\omega(v, \bullet) = 0$. Writing $v$ in terms of the basis given by the coordinate system, we write

$$v = \sum_j v^j \frac{\partial}{\partial x^j} + \sum_j w^j \frac{\partial}{\partial \xi_j},$$

so that

$$\omega(v, \bullet) = -\sum_j v^j d\xi_j + \sum_j w^j dx^j,$$

so that we must have $v^j = w^j = 0$, i.e., $v = 0$.

[1] Recall that a coordinate system $x$ on $M$ induces a coordinate chart $(x, \xi)$ on $T^*M$ by writing covectors in terms of the basis $dx^J$ of each cotangent space; $\xi_j$ are the coefficients here.

From a physical perspective, the symplectic manifold $(T^*M, \omega)$ is thought of as a "phase space" for a physical system taking place on the "configuration space" $M$ while a Hamiltonian $H$ is the total energy of the system. If $M = \mathbb{R}$ and

$$H(x, \xi) = \frac{1}{2m} |\xi|^2 + V(x),$$

(i.e., $H$ is kinetic energy plus potential energy), then

$$dH = \frac{1}{m}\xi \, d\xi + V'(x)dx,$$

so that

$$X_H = \frac{1}{m}\xi \frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial \xi},$$

and thus the integral curves of the flow generated by $X_H$ satisfy

$$\frac{dx}{dt} = \frac{1}{m}\xi, \quad \frac{d\xi}{dt} = -V'(x).$$

In particular, $x$ satisfies the second-order differential equation

$$x''(t) = -\frac{1}{m}V'(x(t)),$$

which you might recognize as Newton's second law for the conservative force given by the potential $V(x)$.

5.3. **Geodesics.** Suppose now that $(M, g)$ is a Riemannian manifold and (in a mild abuse of notation) let $g^{-1}$ denote the induced inner product on each cotangent space. Recall from the last section that $T^*M$ is always a symplectic manifold and consider the Hamiltonian function

$$H(x, \xi) = \frac{1}{2} |\xi|^2_{g^{-1}} = \frac{1}{2} \sum_{i,j} g^{ij}(x)\xi_i \xi_j.$$

By the discussion above, we associate to $H$ a Hamilton vector field $X_H$ and a (very important!) flow $\phi_t$.

**Definition 23.** We say that $\phi_t$ is the *geodesic flow on the cotangent bundle* and the integral curves of $X_H$ are called *lifted geodesics*. If $\gamma(t)$ is a lifted geodesic, its projection $\pi \circ \gamma$ to $M$ is called a *geodesic*.

In local coordinates $(x, \xi)$ on the cotangent bundle, the Hamilton vector field of $H$ is given by

$$\begin{aligned} X_H &= \sum_i g^{ii}(x)\xi_i \partial_{x_i} + \frac{1}{2}\sum_{i \neq j} g^{ij}(x)\xi_i \partial_{x_j} - \frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x_k}\xi_i\xi_j\partial_{\xi_k} \\ &= w_\xi - \frac{1}{2}\frac{\partial g^{ij}(x)}{\partial x_k}\xi_i\xi_j\partial_{\xi_k}, \end{aligned}$$

where, in another abuse of notation, $w_\xi$ is the vector field on $M$ (regarded as a vector field on $T^*M$) associated to $\xi$ by the metric $g$. In particular, the integral curves $(x(t), \xi(t))$ of $X_H$ satisfy

$$\frac{dx}{dt} = w_\xi, \quad \frac{d\xi}{\tilde{d}} = -\frac{1}{2}\sum \frac{\partial g^{ij}(x(t))}{\partial \xi}\xi_i\xi_j.$$

Before we get to examples, let's connect[2] this computation with our discussion of connections from earlier.

**Lemma 24.** *A curve* $\widetilde{\gamma} : (a,b) \to T^*M$ *is a lifted geodesic if and only if* $\gamma = \pi \circ \widetilde{\gamma}$ *satisfies the geodesic equation, namely,*

$$\frac{D}{dt}\gamma'(t) = 0$$

*along* $\gamma$.

In other words, the tangent vector of a geodesic is parallel along the geodesic. This second order equation is the typical way to introduce geodesics; the Hamiltonian formulation is akin to turning a second order equation into a system of first order equations.

*Proof.* The easiest/fastest way to verify the lemma is to write down the differential equations in local coordinates. Indeed, fix a coordinate system $(x^1, \ldots, x^n)$.

The coordinates of the lifted geodesic satisfy the following system of equations:

$$\frac{dx^k}{dt} = \frac{\partial}{\partial \xi_k}\left(\frac{1}{2}\sum_{i,j} g^{ij}\xi_i\xi_j\right),$$
$$\frac{d\xi_k}{dt} = -\frac{\partial}{\partial x_k}\left(\frac{1}{2}g^{ij}\xi_i\xi_j\right).$$

Observe that, for fixed $k$, we may write the sum $g^{ij}\xi_i\xi_j$ as

$$g^{ij}\xi_i\xi_j = \sum_{i,j\neq k} g^{ij}\xi_i\xi_j + \sum_{j\neq k}(g^{jk} + g^{kj})\xi_j\xi_k + g^{kk}\xi_k\xi_k,$$

so that

$$\frac{\partial}{\partial x^k}\left(\frac{1}{2}g^{ij}\xi_i\xi_j\right) = \sum_{j=1}^{n} g^{jk}\xi_j.$$

In particular, this is the $k$-th component of the vector associated to $\xi$. The first half of the differential equation then reads that $\gamma'(t) = v_\xi$. We therefore introduce the "vector variables" $v^j = g^{jk}\xi_k$, so that the first half becomes $\frac{dx^j}{dt} = v^j(t)$.

We now turn our attention to the second half. The right side of the second part of the differential equation has the form

$$-\frac{1}{2}\frac{\partial g^{ij}}{\partial x_k}\xi_i\xi_j,$$

which we now rewrite using the fact that $g^{ij}$ are the components of the inverse matrix of $g$. Recall that for a family of invertible matrices $A(s)$, we have $A(t)A^{-1}(s) = I$, so that $\frac{dA}{ds}A^{-1} + A\frac{dA^{-1}}{ds} = 0$, i.e.,

$$\frac{dA^{-1}}{ds} = -A^{-1}\frac{dA}{ds}A^{-1}.$$

In particular, each component of the matrix must agree. Applying this observation to $g^{ij}$, we find that

$$-\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k} = \frac{1}{2}g^{i\ell}\frac{\partial g_{\ell m}}{\partial x^k}g^{mj},$$

[2] HA!

so that the second half of the equation reads

$$\frac{d\xi_k}{dt} = \frac{1}{2} g^{i\ell}(x(t)) \frac{\partial g_{\ell m}(t)}{\partial x^k} g^{mj}(t) \xi_i \xi_j.$$

We now aim to turn this into an equation in terms of $x$ and $v$. We multiply both sides by $g^{rk}$, sum, and add $\xi_k \frac{g^{rk}}{dt}$ to both sides. Observe that

$$\frac{dg^{rk}}{dt} = \sum_s \frac{\partial g^{rk}}{\partial x^s} \frac{dx^s}{dt} = -g^{ra} \frac{\partial g_{ab}}{\partial x^s} g^{bk} \frac{dx^s}{dt},$$

so that the equation reads (after using the first half for $dx^s/dt$)

$$\frac{d}{dt}\left(g^{rk}\xi_k\right) = \frac{1}{2} g^{rk} \frac{\partial g_{\ell m}}{\partial x^k} g^{i\ell} g^{mj} \xi_i \xi_j - g^{ra} \frac{\partial g_{ab}}{\partial x^s} g^{bk} g^{sj} \xi_j \xi_k.$$

In terms of $v^j$, the equations then read (after re-indexing)

$$\begin{aligned}
\frac{dx^k}{dt} &= v^k, \\
\frac{dv^k}{dt} &= v^i v^j g^{k\ell} \left( \frac{1}{2} \frac{\partial g_{ij}}{\partial x^\ell} - \frac{\partial g_{i\ell}}{\partial x^j} \right) \\
&= -\frac{1}{2} v^i v^j g^{k\ell} \left( \frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right) = -v^i v^j \Gamma^k_{ij}.
\end{aligned}$$

Notice that this equation is the first-order rewriting of the following system of second-order equations:

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x(t)) \left( \frac{dx^i}{dt} \right) \left( \frac{dx^j}{dt} \right) = 0.$$

We now turn to the other system of equations, namely

$$\frac{D}{dt} \gamma'(t) = 0.$$

As $\gamma'(t) = \frac{dx}{dt}$, we can regard this as a second-order equation of the form (where we have written out the basis elements explicitly)

$$\frac{D}{dt} \left( \frac{dx^k}{dt} \frac{\partial}{\partial x^k} \right) = 0.$$

By our construction of the covariant derivative, we have

$$\begin{aligned}
\frac{D}{dt} \left( \frac{dx^k}{dt} \frac{\partial}{\partial x^k} \right) &= \frac{d^2 x^k}{dt^2} \frac{\partial}{\partial x^k} + \frac{dx^k}{dt} \nabla_{\frac{dx^j}{dt} \frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \\
&= \frac{d^2 x^k}{dt^2} \frac{\partial}{\partial x^k} + \frac{dx^k}{dt} \frac{dx^j}{dt} \Gamma^\ell_{jk} \frac{\partial}{\partial x^\ell}.
\end{aligned}$$

After re-indexing, we see that the equation reads

$$\frac{d^2 x^k}{dt^2} \frac{\partial}{\partial x^k} + \frac{dx^i}{dt} \frac{dx^j}{dt} \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

i.e., the same system as above. $\square$

In fact, we can recover the entire connection from the symplectic structure and the Hamiltonian. For convenience we discuss only how to recover the splitting of $T_{(x,\xi)}(T^*M)$ into horizontal and vertical subspaces. (The following discussion is adapted from Robert Bryant's answer here: `https://mathoverflow.net/questions/127319/intuition-for-levi-civita-connection-` Indeed, we regard the Hamiltonian function $g^{ij}\xi_i\xi_j$ in two different respects. The first is that it provides our Hamiltonian vector field $X_H$ and the second is that it induces a quadratic form $\gamma_H$ on each $T_{(x,\xi)}T^*M$. In our Riemannian case, the quadratic form it induces is simply $\pi^*g$, where $g$ is regarded as a quadratic form on each $T_xM$. We then consider the symmetric quadratic form

$$\dot{\gamma}_H = \mathcal{L}_{X_H}\gamma_H.$$

A quick computation in coordinates shows that in terms of the basis $\partial/\partial x^j$, $\partial/\partial\xi_k$, $\dot{\gamma}_H$ corresponds to a matrix

$$\begin{pmatrix} * & I_n \\ I_n & 0 \end{pmatrix},$$

where $I_n$ is the $n \times n$ identity matrix and the entries in the upper left can be computed in terms of Christoffel symbols but are irrelevant here. As a result, $\dot{\gamma}_H$ is non-degenerate and has $n$ positive and $n$ negative eigenvalues. Using this form (and its matrix realization), we can see that at each $(x,\xi)$ there is a unique $n$-plane that is copmlementary to $V_{(x,\xi)}$, null with respect to the symplectic form $\omega$, and also null with respect to $\dot{\gamma}_H$. These $n$-planes fit smoothly into a bundle, which a coordinate computation shows is the same horizontal bundle as the one associated to the Levi-Civita connection.

5.4. **The exponential map.** Back to the main thread, given any point $(x,\xi) \in T^*M$, there is a unique lifted geodesic through that point, namely, $t \mapsto \phi_t(x,\xi)$. From our ODE discussion last semester, we know that it also has a maximal connected interval of existence. By using the Riemannian metric to identify tangent and cotangent vectors, we then have

**Proposition 25.** *Given any point $p \in M$ and any $v \in T_pM$, there is an open interval $I$ containing $0$ and a unique geodesic $\gamma : I \to M$ so that $\gamma(0) = p$ and $\gamma'(0) = v$.*

It's traditional to let the exponential map denote the mapping from the tangent space/bundle (rather than cotangent space/bundle) to the manifold, so we'll do that. For a point $(p,v) \in TM$, let $\gamma_{(p,v)}$ denote the unique geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$, and let $I_{(p,v)}$ denote its maximal interval of existence.[3]

**Definition 26.** Define the exponential map $\exp(p,v) = \gamma_{(p,v)}(1)$ if $1 \in I_{(p,v)}$. Define the restricted exponential map $\exp_p(v) = \gamma_{(p,v)}(1)$ if $1 \in I_{(p,v)}$.

Let's let $\mathcal{E} \subset TM$ denote the domain of exp, i.e., $(p,v) \in \mathcal{E}$ if and only if $1 \in I_{(p,v)}$.

**Proposition 27.** *(1) $\mathcal{E}$ is open, contains the zero section, and is star-shaped.*
*(2) For each $(p,v) \in TM$, the geodesic $\gamma_{(p,v)}$ is given by $\gamma_{(p,v)}(t) = \exp_p(tv)$ whenever either side is defined.*
*(3)* exp *is smooth.*

**Lemma 28.** *For any $(p,v) \in TM$, and $c,t \in \mathbb{R}$, $\gamma_{(p,cv)}(t) = \gamma_{(p,v)}(ct)$ whenever either side is defined.*

[3]In keeping with the view of the author, we should instead define the exponential map from $T^*M$ to $M$ by $\exp(x,\xi) = \phi_1(x,\xi)$ whenever it is defined. The restricted exponential map would then be $\exp_x(\xi) = \phi_1(x,\xi)$, but we do not pursue this here for ease of reading other texts.

*Proof.* It's enough to show that $\gamma_{(p,cv)}(t)$ exists and equality holds whenever $\gamma_{(p,v)}(ct)$ is defined. (Replace $v$ by $c^{-1}v$ and $t$ by $c^{-1}t$ for the other direction.)

Suppose $I$ is the maximal interval of definition for $\gamma = \gamma_{(p,v)}$, and define

$$\widetilde{\gamma}(t) = \gamma(ct).$$

The maximal interval of definition for $\widetilde{\gamma}$ is

$$c^{-1}I = \{t : ct \in I\}.$$

We now observe that $\widetilde{\gamma}$ satisfies the geodesic equation (by the chain rule) with $\widetilde{\gamma}(0) = p$ and $\widetilde{\gamma}'(0) = cv$ (so that $\widetilde{\gamma} = \gamma_{(p,cv)}$). □

*Proof of proposition.* By the lemma we have that

$$\exp(p, cv) = \gamma_{(p,cv)}(1) = \gamma_{(p,v)}(c),$$

which proves the second statement. If $v \in \mathcal{E}_p$ (where $\mathcal{E}_p = \{v \in T_pM : (p,v) \in \mathcal{E}\}$), then $\gamma_{(p,v)}$ is defined at least on $[0,1]$. Then, for $t \in [0,1]$, we have $\exp_p(tv) = \gamma_{(p,tv)}(1) = \gamma_{(p,v)}(t)$ and thus $\mathcal{E}$ is star-shaped.

We still need to show that $\mathcal{E}$ is open and that exp is smooth. For now we use the notation $e : TM \to T^*M$ to denote the bundle isomorphism given by "lowering indices". By Lemma 24, we see that if $\gamma(t)$ is a geodesic, then

$$t \mapsto (\gamma(t), e(\gamma'(t)))$$

is the lifted geodesic that projects to $\gamma$. By the definition of the flow $\phi_t$, we then have

$$\left(\gamma_{(p,v)}(t), e\left(\gamma'_{(p,v)}(t)\right)\right) = \phi_t\left(p, e(v)\right).$$

By the proof last semester[4] of existence and uniqueness of ODEs, there is an open neighborhood $U$ of $\{0\} \times T^*M$ in $\mathbb{R} \times T^*M$ on which $\phi_\bullet(\bullet, \bullet)$ is defined and the smooth dependence on parameters shows that $\phi$ is smooth where it is defined.

So, if $(p,v) \in \mathcal{E}$, the geodesic $\gamma_{(p,v)}$ is defined at least on $[0,1]$, so $(1,(p,e(v))) \in U$ and it has an open neighborhood around it, so there is an open neighborhood around $(p,v)$ for which the flow exists for all $t \in [0,1]$, so $\mathcal{E}$ is open.

Finally, $\exp(p,v) = \pi \circ \phi_1(p, e(v))$, so exp is smooth as a composition of smooth functions. □

**Lemma 29.** *For any $p \in M$, there is a neighborhood $V \subset T_pM$ of $0 \in T_pM$ and a $U \subset M$ so that $\exp_p : V \to U$ is a diffeomorphism.*

*Proof.* This follows from the inverse function theorem as soon as we show that the derivative of the restricted exponential map at 0 is invertible. As $T_pM$ is a vector space, we have $T_0(T_pM) \cong T_pM$ canonically so we can think of

$$(D\exp_p)_0 : T_pM \to T_pM.$$

Let $v \in T_pM$ and take $\tau : (-\epsilon, \epsilon) \to T_0(T_pM) \cong T_pM$ to be $\tau(t) = tv$, so that $\tau(0) = 0$ and $\tau'(0) = v$. We now compute

$$\begin{aligned}(D\exp_p)_0(v) &= \frac{d}{dt}|_{t=0} \exp_p \circ \tau(t) \\ &= \frac{d}{dt}|_{t=0} \gamma_{(p,v)}(t) = v,\end{aligned}$$

[4]Soon to be recreated and improved in Appendix A.

so indeed $(D\exp_p)_0 = \mathrm{Id}_{T_pM}$ and is therefore invertible. $\square$

## 6. Curvature

One question we might ask is to what degree we can make a Riemannian metric "look like" the Euclidean metric at a point. Can we make its value agree at a given point? (Yes.) Its first derivatives? (Again, yes.) What about its second derivatives?

6.1. **Geodesic normal coordinates.** The invertibility of the exponential map at the origin gives a way of picking distinguished coordinates at a point. Given $p \in M$, let $e_1, \ldots, e_n$ be an orthonormal basis for $T_pM$ and consider the map $V \to M$, where $V \subset \mathbb{R}^n$, given by

$$(x^1, \ldots, x^n) \mapsto \exp_p\left(x^1 e_1 + \cdots + x^n e_n\right).$$

This map is a diffeomorphism from a neighborhood of $0 \in \mathbb{R}^n$ to a neighborhood of $p$ and so it gives a local coordinate system called *geodesic normal coordinates*.

What does the metric look like in these coordinates?

We first note that $p \leftrightarrow (0, \ldots, 0)$ and then claim that, in this coordinate system, $g_{ij}(0) = \delta_{ij}$. Indeed, we computed earlier that $(D\exp_p)_0 = \mathrm{Id}_{T_pM}$, so

$$\begin{aligned} g_{ij}(x) &= \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \\ &= \left\langle (D\exp_p)_{\sum x^k e_k}(e_i), (D\exp_p)_{\sum x^k e_k}(e_j) \right\rangle, \end{aligned}$$

so at $x = 0$, this is $g_{ij}(0) = \langle e_i, e_j \rangle = \delta_{ij}$.

We now turn our attention to the Christoffel symbols. Fix $(x^1, x^2, \ldots, x^n) \in V$ and take $\alpha(t) = (tx^1, \ldots, tx^n)$; $\alpha$ is a geodesic in $U$ because $\alpha(t) = \exp_p(t(x^1, \ldots, x^n))$ (in this coordinate system). Because $\alpha$ is a geodesic, we have

$$\frac{d^2}{dt^2}\alpha^k(t) + \sum_{i,j=1}^n \Gamma^k_{ij}(\alpha(t)) \frac{d\alpha^i}{dt}(t) \frac{d\alpha^j}{dt}(t) = 0$$

for all $k$ and all $t$ sufficiently small. As we can see that $\frac{d^2}{dt^2}\alpha(t) = 0$, we see that

$$\sum_{i,j=1}^n \Gamma^k_{ij}(\alpha(t)) \frac{d\alpha^i}{dt}(t) \frac{d\alpha^j}{dt}(t) = 0.$$

Setting $t = 0$, we have that

$$\sum_{ij} \Gamma^k_{ij}(0) x^i x^j = 0$$

for all $k$ and all $x$. As it is true for all $x$, we may apply it to $x = e_i$ to see that $\Gamma^k_{ii}(0) = 0$ for all $i$. We then apply it with $x = e_i + e_j$ and see that $\Gamma^k_{ij}(0) + \Gamma^k_{ji}(0) = 0$ and thus $\Gamma^k_{ij}(0)$ is antisymmetric in $i, j$. On the other hand, we also knew that $\Gamma^k_{ij}(0)$ was symmetric in $i, j$ (indeed, this follows from the coordinate formula for $\Gamma^k_{ij}$), so they must all vanish at 0.

What does this fact about the Christoffel symbols mean for the first derivative of the metric? Well, at $x = 0$, we have $\nabla_{\partial_k}\partial_i = 0$, so $\partial_k g_{ij}|_{x=0} = 0$, i.e., $g_{ij}(x) = \delta_{ij}$+quadratic and higher terms.

What do these quadratic terms represent?

6.2. **Dimension counting.** Let's take a more general approach and try to pick a new system of coordinates $x^1, \dots, x^n$ so that our point $p$ is $x = 0$ and so that we arrange for the vanishing of as many terms in the Taylor series of $g_{ij}$ at zero as possible.

Let's assume we already have a coordinate system $y^1, \dots, y^n$ sending $p$ to $y = 0$ and let $g_{ij} = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle$. If we change coordinates to $x^1, \dots, x^n$, we know that

$$\frac{\partial}{\partial y^j} \rightsquigarrow \sum_i \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i},$$

i.e.,

$$\frac{\partial}{\partial x^i} = \sum \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

Letting $\widetilde{g}_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$, we see that

$$\widetilde{g}_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \sum_{k,\ell} \left( \frac{\partial y^k}{\partial x^i} \right) \left( \frac{\partial y^\ell}{\partial x^j} \right) g_{k\ell}.$$

We therefore use the values of the Jacobian, i.e., the first derivatives of the coordinate changes, to put $\widetilde{g}_{ij}$ into a model form. We have the freedom to choose all $n^2$ values of $\frac{\partial y^k}{\partial x^j}$, so we think of this as $n^2$ unknowns. Now $\widetilde{g}_{ij}$ is always a symmetric matrix, so there are $n(n+1)/2$ independent components of $\widetilde{g}_{ij}$ and so if we want to make $\widetilde{g}_{ij} = \delta_{ij}$, we have a system of $n(n+1)/2$ equations in $n^2$ unknowns. Unless $n = 1$, this is an underdetermined system (there are $n(n-1)/2$ more unknowns than equations), so we should expect to be able to solve it. Indeed, the number of excess degrees of freedom here is the dimension of $SO(n)$ and corresponds to the rotational symmetry enjoyed by Euclidean space.

We'd now like to arrange that the first derivatives of $\widetilde{g}$ vanish at 0. Differentiating the equation above, we have

$$\frac{\partial}{\partial x^k} \widetilde{g}_{ij} = \frac{\partial}{\partial x^k} \left( \sum_{p,q} \frac{\partial y^p}{\partial x^i} \frac{\partial y^q}{\partial x^j} g_{pq} \right),$$

which yields a total of $n$ equations for each component of $g$, i.e., $n^2(n+1)/2$ total equations. We have the freedom to pick the second derivatives of our coordinate changes at the point 0, but these are subject to the constraint that

$$\frac{\partial^2 y^p}{\partial x^i \partial x^j} = \frac{\partial^2 y^p}{\partial x^j \partial x^i},$$

so we have $n(n+1)/2$ choices for each $y^p$, i.e., a total of $n^2(n+1)/2$ unknowns. This set of equations is formally determined (number of equations is the same as number of unknowns), so we'd expect it to have a unique solution. This is essentially what we found in our discussion of geodesic normal coordinates.

What about the second derivatives? Now the equation for the second derivative of the metric tensor involves third derivatives of our coordinate change. The second derivatives

$$\frac{\partial^2}{\partial x^k \partial x^\ell} \widetilde{g}_{ij}$$

are again subject to the constraint that mixed partials commute, so we have $n(n+1)/2$ equations for each component of $\widetilde{g}$, i.e., a total of $n^2(n+1)^2/4$ equations. How many new unknowns do we have? In other words, how many different third partial derivatives $\frac{\partial^3 y^k}{\partial x^k \partial x^\ell \partial x^r}$ does each component $y^k$ have? We consider in cases. When all three of $k, \ell$, and $r$ are distinct, there are $\binom{n}{3} = n(n-1)(n-2)/6$ such possibilities. When two are distinct, we have $n$ choices for which derivative we take twice and then $n-1$ for the third derivative, i.e., $n(n-1)$ choices. Finally, when all three indices agree, we have $n$ choices for their common value. This gives a total of

$$\frac{n(n-1)(n-2)}{6} + n(n-1) + n = \frac{n}{6}\left((n-1)(n-2) + 6n\right) = \frac{n(n+1)(n+2)}{6}.$$

Alternatively, we could have used a counting argument that labels how many indices take each value. More precisely, we think of having three identical elements representing the three derivatives we can take and then divide them up into $n$ buckets (which we realize by $n-1$ dividers). If two elements land in the $j$-th bucket, we take two derivatives in $x^j$. If our derivatives and buckets are distinguishable, then there are $(n-1+p)!$ ways to order them. The dividers are definitely not distinguishable, so we must divide by $(n-1)!$. Similarly, our mixed partials are symmetric, so we should also think that we can't distinguish the order in which we take derivatives, so we should also divide by 3!. This leaves

$$\binom{n-1+3}{3}$$

degrees of freedom for each $y^k$. (In general you'd have $\binom{n-1+p}{p}$ degrees of freedom for $p$-th derivatives.) As we have $n$ different functions $y^k$, we then have $n^2(n+1)(n+2)/6$ total degrees of freedom.

The thing to note here is that the problem of annihilating the second derivatives at a point is a formally overdetermined problem, as there are

$$\frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)(n+2)}{6} = \frac{n^2(n+1)}{12}\left(3(n+1) - 2(n+2)\right) = \frac{n^2(n^2-1)}{12}$$

more equations than unknowns. We therefore do not expect be able to solve this problem generally. Encoding this obstruction is the *curvature tensor.*

6.3. **The curvature tensor.** As before we suppose $(M, g)$ is a Riemannian manifold equipped with the Levi-Civita connection.

**Definition 30.** For $X, Y, Z \in \mathcal{X}(M)$, define

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

so that $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$.

We call $R$ "the" curvature tensor; we should verify that it is a tensor.

**Proposition 31.** *$R$ is a $(1,3)$-tensor.*

*Proof.* As before, we need to show that for $X, Y, Z \in \mathcal{X}(M)$ and $f \in C^\infty(M)$, that

(i) $R(fX, Y)Z = fR(X, Y)Z$,
(ii) $R(X, fY)Z = fR(X, Y)Z$, and
(iii) $R(X, Y)(fZ) = fR(X, Y)Z$.

In fact, $R$ is clearly antisymmetric in $X$ and $Y$, so the first two are equivalent. We now compute:

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z \\ &= f\nabla_X\nabla_Y Z - \nabla_Y\left(f\nabla_X Z\right) - \nabla_{f[X,Y]-Y(f)X}Z \\ &= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z \\ &= fR(X,Y)Z. \end{aligned}$$

Similarly,

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X\left(f\nabla_Y Z + Y(f)Z\right) - \nabla_Y\left(f\nabla_X Z + X(f)Z\right) - f\nabla_{[X,Y]}Z - [X,Y](f)Z \\ &= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z \\ &\qquad - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z - Y(X(f))Z - f\nabla_{[X,Y]}Z - [X,Y](f)Z \\ &= fR(X,Y)Z + \left(X(Y(f)) - Y(X(f)) - [X,Y](f)\right)Z = fR(X,Y)Z. \end{aligned}$$

□

In local coordinates, we write

$$R(\partial_i,\partial_j)\partial_k = \sum_{\ell=1}^{n} R_{ijk}^{\ell}\partial_\ell.$$

Since $R$ is a tensor, these components tell us all of the information about $R$, and so

$$R(X,Y)Z = \sum_{i,j,k,\ell} X^i Y^j Z^k R_{ijk}^{\ell}\partial_\ell.$$

The tensor $R$ is often called the Riemann curvature tensor and it is independent of coordinate choices (indeed, we defined it intrinsically). It's often convenient to turn $R$ from a $(1,3)$-tensor into a $(0,4)$-tensor using the metric and we use the same letter $R$ to denote this. Indeed, we define

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle,$$

so that

$$R_{ijk\ell} = \sum_p R_{ijk}^{p} g^{p\ell}.$$

The Riemann curvature tensor has several symmetries, and these are easiest to state for the $(0,4)$-tensor.

**Proposition 32.** *(i)* $R(Y,X,Z,W) = -R(X,Y,Z,W)$,
*(ii)* $R(X,Y,W,Z) = -R(X,Y,Z,W)$,
*(iii)* $R(Z,W,X,Y) = R(X,Y,Z,W)$, *and*
*(iv)* $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$.

The last identity (the cyclic one) is called the first Bianchi identity.

*Proof.* The first identity is obvious from the definition, which is antisymmetric in $X$ and $Y$.

For the second identity, we let $f$ denote the smooth function $f = \langle Z, W\rangle$, so that

$$X(Y(f)) - Y(X(f)) - [X,Y](f) = 0.$$

We now use that the Levi-Civita connection is compatible with the metric to write

$$\begin{aligned}
0 &= X\left(Y\langle Z,W\rangle\right) - Y\left(X\langle Z,W\rangle\right) - [X,Y]\langle Z,W\rangle \\
&= X\left(\langle\nabla_Y Z,W\rangle + \langle Z,\nabla_Y W\rangle\right) - Y\left(\langle\nabla_X Z,W\rangle + \langle Z,\nabla_X W\rangle\right) - \left\langle\nabla_{[X,Y]}Z,W\right\rangle - \left\langle Z,\nabla_{[X,Y]}W\right\rangle \\
&= \langle\nabla_X\nabla_Y Z,W\rangle + \langle\nabla_Y Z,\nabla_X W\rangle + \langle\nabla_X Z,\nabla_Y W\rangle + \langle Z,\nabla_X\nabla_Y W\rangle - \langle\nabla_Y\nabla_X Z,W\rangle \\
&\quad - \langle\nabla_X Z,\nabla_Y W\rangle - \langle\nabla_Y Z,\nabla_X W\rangle - \langle Z,\nabla_Y\nabla_X W\rangle - \left\langle\nabla_{[X,Y]}Z,W\right\rangle - \left\langle Z,\nabla_{[X,Y]}W\right\rangle \\
&= \langle R(X,Y)Z,W\rangle - \langle Z,R(X,Y)W\rangle = R(X,Y,Z,W) - R(X,Y,W,Z),
\end{aligned}$$

as desired.

To prove the third identity, we'll use the fourth identity (the Bianchi identity), so let's prove the fourth one now.

$$\begin{aligned}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X \\
&\quad - \nabla_{[Y,Z]}X + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\
&= \nabla_X\left(\nabla_Y Z - \nabla_Z Y\right) - \nabla_{[Y,Z]}X + \nabla_Y\left(\nabla_Z X - \nabla_X Z\right) - \nabla_{[Z,X]}Y \\
&\quad + \nabla_Z\left(\nabla_X Y - \nabla_Y X\right) - \nabla_{[X,Y]}Z.
\end{aligned}$$

Because the Levi-Civita connection is torsion-free, we know that for any $X$, $Y$, we have $\nabla_X Y - \nabla_Y X = [X,Y]$, so that the above sum is given by

$$\begin{aligned}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_X[Y,Z] - \nabla_{[Y,Z]}X + \nabla_Y[Z,X] - \nabla_{[Z,X]}Y \\
&\quad + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,
\end{aligned}$$

by the Jacobi identity, thus proving the Bianchi identity.

We finally turn to the third identity. We use the first two identities and the Bianchi identity to write $R(Z,W,X,Y)$ in two ways:

$$\begin{aligned}
R(Z,W,X,Y) &= -R(W,Z,X,Y) = R(Z,X,W,Y) + R(X,W,Z,Y), \\
R(Z,W,X,Y) &= -R(Z,W,Y,X) = R(W,Y,Z,X) + R(Y,Z,W,X),
\end{aligned}$$

so that

$$2R(Z,W,X,Y) = R(Z,X,W,Y) + R(X,W,Z,Y) + R(W,Y,Z,X) + R(Y,Z,W,X).$$

An identical calculation gives

$$2R(X,Y,Z,W) = R(X,Z,Y,W) + R(Z,Y,X,W) + R(Y,W,X,Z) + R(W,X,Y,Z).$$

Using antisymmetry twice on each term (e.g., writing $R(W,X,Y,Z) = -R(X,W,Y,Z) = R(X,W,Z,Y)$) shows that these two sums agree, establishing the third identity. □

6.3.1. *Another counting argument.* How many degrees of freedom does $R$ have? In other words, how many independent components $R_{ijk\ell}$ can be prescribed without the above symmetries forcing our hands?

To be more precise, we ask for the dimension of the space of $(0, 4)$ tensors satisfying the following four identities:

$$\begin{aligned} R_{ijk\ell} &= R_{k\ell ij}, \\ R_{ijk\ell} &= -R_{jik\ell}, \\ R_{ijk\ell} &= -R_{ij\ell k}, \\ R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} &= 0. \end{aligned}$$

We'll now count by cases.[5] Antisymmetry demands vanishing if all indices are repeated, i.e., $R_{aaaa} = 0$. For two distinct indices, we have three possibilities, namely $R_{aaab}$, $R_{aabb}$, and $R_{abab}$, and all others can be put into one of these forms by the identities. Of these, only $R_{abab}$ might be non-vanishing, giving us $\binom{n}{2} = n(n+1)/2$ components with two distinct indices.

For three distinct indices, terms of the form $R_{aabc}$ must vanish, and all other terms can be put into the form $R_{abac}$. For these, we have $n$ choices for the repeated index $a$ and $\binom{n-1}{2} = n(n-1)/2$ choices for the other two indices, giving a total of $3\binom{n}{3}$ of these components.

Finally, for four distinct indices, we start by noticing that we have

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{badc} = R_{cdab} = -R_{dcab} = -R_{cdba} = R_{dcba},$$

so, given a choice of $a, b, c, d$ (with, say $a < b < c < d$ for definiteness), there are only three independent components with these indices and we can take as their representatives $R_{abcd}$, $R_{bcad}$, and $R_{cabd}$. The Bianchi identity tells us that the third of these is determined by the other two, so we have in fact two components for each choice of four distinct indices, i.e., a total of

$$2\binom{n}{4} = \frac{1}{12}n(n-1)(n-2)(n-3)$$

independent components.

Summing these up, we have a total of

$$\begin{aligned} &\frac{1}{12}n(n-1)(n-2)(n-3) + \frac{1}{2}n(n-1)(n-2) + \frac{1}{2}n(n+1) \\ =&\frac{1}{12}n(n-1)\left(n^2 - 5n + 6 + 6n - 12 + 6\right) = \frac{1}{12}n^2(n^2-1) \end{aligned}$$

independent components. It is no coincidence that this is the same number we found in our counting argument earlier in this section, as the curvature tensor is the obstruction to looking metrically Euclidean.

In particular, note this size in low dimensions. In one dimension, there are no independent components; the curvature tensor always vanishes there. Indeed, you've seen that any curve can be parametrized by arc length, which provides you a local isometry with $\mathbb{R}$.

In two dimensions, the curvature tensor has a single independent component, which you've seen in another form as the Gaussian curvature. In this sense the curvature tensor generalizes the curvature we defined for surfaces.

In three dimensions, the curvature has 6 independent components and in four dimensions it has 20.

[5]If you have a cleaner way to do this please let me know.

6.4. **Sectional curvature.** Another generalization of the Gaussian curvature is called the sectional curvature, which essentially is the Gaussian curvature of a submanifold.

**Definition 33.** Let $p \in M$ and let $\Pi \subset T_pM$ be a two-dimensional subspace with basis $v$ and $w$. The *sectional curvature of* $\Pi$ is

$$K(\Pi) = \frac{R(v, w, w, v)}{|v|^2 |w|^2 - \langle v, w\rangle^2}.$$

We start by claming that $K(\Pi)$ depends only on $\Pi$ and not on the choice of basis $v, w$. Indeed, suppose $\widetilde{v}$ and $\widetilde{w}$ is another basis, so

$$\begin{aligned}\widetilde{v} &= av + bw,\\ \widetilde{w} &= cv + dw,\end{aligned}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible. We then use multilinearity and antisymmetry to see that

$$R(\widetilde{v}, \widetilde{w}, \widetilde{w}, \widetilde{v}) = (ad - bc)R(v, w, \widetilde{w}, \widetilde{v}) = (ad - bc)^2 R(v, w, w, v),$$

while we also have

$$|\widetilde{v}|^2 |\widetilde{w}|^2 - \langle \widetilde{v}, \widetilde{w}\rangle^2 = (ad - bc)^2 \left(|v|^2 |w|^2 - \langle v, w\rangle^2\right),$$

so that $K(\Pi)$ is well-defined.

One interpretation of $K(\Pi)$ is as the Gaussian curvature at $p$ of the surface $\exp_p(\Pi)$. In two dimensions, there is only one sectional curvature $K$ at each point and the Riemann tensor is given by

$$R(X, Y, Z, W) = K \left(\langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, W\rangle\right).$$

It's also true that in higher dimensions the entire Riemann tensor can be recovered from the sectional curvatures, but the reconstruction formula is long and we probably won't use it in this course.

**Definition 34.** A Riemannian manifold $(M, g)$ has constant sectional curvature $k$ if $K(\Pi) = k$ for all $p \in M$ and all $\Pi \subset T_pM$.

As examples, you should check that $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$, equipped with the pullback of the Euclidean metric, has constant sectional curvature 1.

6.5. **Curvature as an operator on tensors.** Given an $(r, s)$-tensor $T$, for $X, Y \in \mathcal{X}(M)$, we obtain another $(r, s)$-tensor that we call $R(X, Y)T$ by

$$R(X, Y)T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]}T.$$

**Proposition 35.** *If $\theta \in \Omega^1(M)$, and $X, Y, Z \in \mathcal{X}(M)$, then*

$$\left(R(X, Y)\theta\right)(Z) = -\theta\left(R(X, Y)Z\right).$$

*Proof.* By definition of $\nabla_Y \theta$, we have

$$Y\left(\theta(Z)\right) = \nabla_Y \theta(Z) + \theta(\nabla_Y Z),$$

so that

$$\begin{aligned}X\left(Y\left(\theta(Z)\right)\right) &= X\left(\nabla_Y \theta(Z) + \theta(\nabla_Y Z)\right)\\ &= \left(\nabla_X \nabla_Y \theta\right)(Z) + \nabla_Y \theta(\nabla_X Z) + \nabla_X \theta(\nabla_Y Z) + \theta(\nabla_X \nabla_Y Z).\end{aligned}$$

Now, applying $X(Y(f)) - Y(X(f)) - [X,Y](f) = 0$ with the smooth function $f = \theta(Z)$, we have

$$\begin{aligned} 0 &= X(Y(\theta(Z))) - Y(X(\theta(Z))) - [X,Y](\theta(Z)) \\ &= \left(\nabla_X \nabla_Y \theta - \nabla_Y \nabla_X \theta - \nabla_{[X,Y]}\theta\right)(Z) + \theta\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right), \end{aligned}$$

finishing the proof. □

In general, for an $(r,s)$-tensor $T$, we have

$$\begin{aligned} &\left(R(X,Y)T\right)(\theta_1, \dots, \theta_r, Z_1, \dots, Z_s) \\ &\quad = -\sum_{k=1}^{r} T\left(\theta_1, \dots, R(X,Y)\theta_k, \dots, \theta_r, Z_1, \dots, Z_s\right) \\ &\qquad + \sum_{k=1}^{s} T\left(\theta_1, \dots, \theta_r, Z_1, \dots, R(X,Y)Z_k, \dots, Z_s\right). \end{aligned}$$

Applying this to the metric tensor $g$, we already know $\nabla_X g = 0$, so $R(X,Y)g = 0$ and thus

$$\begin{aligned} 0 &= [R(X,Y)g](Z,W) = -g\left(R(X,Y)Z, W\right) - g\left(Z, R(X,Y)W\right) \\ &= -R(X,Y,Z,W) - R(X,Y,W,Z), \end{aligned}$$

which yields one of our earlier properties (nothing new here).

There is, however, a nontrivial property of $\nabla R$, though:

**Proposition 36** (Second Bianchi identity)**.** *For $X, Y, Z, V, W \in \mathcal{X}(M)$,*

$$(\nabla_X R)(Y,Z,V,W) + (\nabla_Y R)(Z,X,V,W) + (\nabla_Z R)(X,Y,V,W) = 0.$$

*Proof.* One (tedious) way to check this is to use the definnition and express the curvature in terms of commutators and eventually appeal to the Jacobi identity as we did for the first Bianchi identity. It's convenient to take a shortcut.

The equation above is tensorial so it is enough to show it in a convenient choice of coordinates. Fix $p \in M$; we want to check the identity at $p$. We work in geodesic normal coordinates at $p$, so $\Gamma_{ij}^k(p) = 0$. It's enough to check it at $p$ for $(X,Y,Z,V,W) = (\partial_i, \partial_j, \partial_k, \partial_\ell, \partial_m)$. Note that because we are working in geodesic normal coordinates, we have $\nabla_{\partial_*}\partial_* = 0$ at $p$. We therefore conclude that

$$\nabla_X R(Y,Z,V,W) = X\left(R(Y,Z,V,W)\right)$$

at $p$ because all other terms vanish.

We now compute. One shortcut we use is to keep two of the terms that vanish at $p$. Indeed, we know that at $p$,

$$-R(\partial_j, \partial_k, \nabla_{\partial_i}\partial_\ell, \partial_m) - R(\partial_j, \partial_k, \partial_\ell, \nabla_{\partial_i}\partial_m) = 0.$$

Our other shortcut is that we are using coordinate vector fields, and so $[\partial_*, \partial_*] = 0$. We use this in two ways. First, there is no third term in the expression for $R(\partial_j, \partial_k)\partial_\ell$. Second, because the metric is torsion-free, we get to conclude that $\nabla_{\partial_a}\partial_b = \nabla_{\partial_b}\partial_a$. We then see that, at $p$,

$$\begin{aligned}X\left(R(Y,Z,V,W)\right) &= \partial_i \left\langle R(\partial_j,\partial_k)\partial_\ell, \partial_m\right\rangle - R(\partial_j,\partial_k,\nabla_{\partial_i}\partial_\ell,\partial_m) - R(\partial_j,\partial_k,\partial_\ell,\nabla_{\partial_i}\partial_m)\\ &= \left\langle \nabla_{\partial_i}\left(\nabla_{\partial_j}\nabla_{\partial_k}\partial_\ell - \nabla_{\partial_k}\nabla_{\partial_j}\partial_\ell\right), \partial_m\right\rangle + \left\langle \nabla_{\partial_j}\nabla_{\partial_k}\partial_\ell - \nabla_{\partial_k}\nabla_{\partial_j}\partial_\ell, \nabla_{\partial_i}\partial_m\right\rangle\\ &\quad - \left\langle \nabla_{\partial_j}\nabla_{\partial_k}\nabla_{\partial_i}\partial_\ell - \nabla_{\partial_k}\nabla_{\partial_j}\nabla_{\partial_i}\partial_\ell, \partial_m\right\rangle - \left\langle \nabla_{\partial_j}\nabla_{\partial_k}\partial_\ell - \nabla_{\partial_k}\nabla_{\partial_j}\partial_\ell, \nabla_{\partial_i}\partial_m\right\rangle\\ &= \left\langle \left[\nabla_{\partial_i}, \left[\nabla_{\partial_j},\nabla_{\partial_k}\right]\right]\partial_\ell, \partial_m\right\rangle.\end{aligned}$$

Now we take the cyclic sum over $i,j,k$ and use the Jacobi identity (which is a theorem about letters and so valid for commutators) to finish the proof. ☐

6.6. **Ricci and scalar curvature.** Given the Riemann tensor $R$, we can define a $(0,2)$-tensor called the Ricci tensor by taking a trace. You should check for yourself that the following definition is in fact the trace of $R(\bullet, X)Y$ regarded as a linear transformation $T_pM \to T_pM$.

**Definition 37.** The Ricci curvature is the $(0,2)$-tensor given in any coordinate system by

$$\mathrm{Ric}(X,Y) = \sum_{j=1}^{n} g^{ij} R\left(X, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, Y\right).$$

The scalar curvature is a function given by contracting the Ricci curvature:

$$R_p = \sum_{k,\ell} g^{k\ell}\,\mathrm{Ric}\left(\partial_k,\partial_\ell\right) = \sum_{i,j,k,\ell} g^{ij} g^{k\ell} R\left(\partial_k,\partial_i,\partial_j,\partial_\ell\right).$$

Note that the Ricci tensor is a symmetric $(0,2)$-tensor.

The Ricci tensor and the scalar curvature satisfy a nice relationship called the contracted second Bianchi identity.[6]

**Theorem 38** (Contracted second Bianchi identity)**.** *For $p \in M$ and $X \in T_pM$, we have, for any basis $v_i$ of $T_pM$,*

$$X(R) = 2\sum_{i,j=1}^{n} g^{ij}\left(\nabla_{v_i}\mathrm{Ric}\right)(X, v_i).$$

*where $R$ denotes the scalar curvature and* Ric *the Ricci tensor.*

*Proof.* The key idea is that the covariant derivative commutes with contraction. We'll prove it only for the two cases we need here, but it is true in general. [7] Geodesic normal coordinates make this much easier to see.

Fix $p \in M$ and let $x^1,\dots,x^n$ be geodesic normal coordinates at $p$ (so that $\Gamma_{ij}^k(p) = 0$).

Recall that $\nabla_X g = 0$, so that, in particular,

$$\partial_k g_{\ell m} - \Gamma^r_{k\ell} g_{rm} - \Gamma^r_{km} g_{\ell r} = 0.$$

As we are working in geodesic normal coordinates, we have that at $p$, $\partial_k g_{\ell m} = 0$. In particular, because $g_{\ell m} g^{\ell m} = n$, we also have $\partial_k g^{\ell m} = 0$ at $p$. We then have, for a $(0,2)$-tensor $T$,

[6]This relationship comes up in relativity and tells you that the left hand side of the Einstein equations (which describe the curvature of the spacetime) must be divergence free and therefore impose a constraint on the right hand side (which describes the contribution from the matter fields).

[7]The more general statement follows by an inductive procedure together with the base cases of a $(2,0)$ and $(1,1)$-tensor.

that at $p$ (again, we are using geodesic normal coordinates; ordinarily there would be more terms!)

$$\partial_k \left( g^{ij} T_{ij} \right) = g^{ij} \partial_k T_{ij} = g^{ij} (\nabla_{\partial_k} T)_{ij}.$$

Similarly, for a $(0,4)$-tensor $T$, we have at $p$

$$\partial_k \left( g^{ij} T_{ij\ell m} \right) = g^{ij} \partial_k T_{ij\ell m} = g^{ij} (\nabla_{\partial_k} T)_{ij\ell m}.$$

We now turn to the identity. By the second Bianchi identity, we have

$$0 = (\nabla_{\partial_i} R)(\partial_j, \partial_k, \partial_\ell, \partial_m) + (\nabla_{\partial_j} R)(\partial_k, \partial_i, \partial_\ell, \partial_m) + (\nabla_{\partial_k} R)(\partial_i, \partial_j, \partial_\ell, \partial_m),$$

so we contract by multiplying by $g^{i\ell}$ and $g^{jm}$ and then summing. Because contraction commutes with covariant derivatives, we know that, e.g., at $p$

$$(\nabla_{\partial_i} R)(\partial_j, \partial_k, \partial_\ell, \partial_m) = \partial_i \left( R(\partial_j, \partial_k, \partial_\ell, \partial_m) \right),$$

so that we obtain at $p$

$$0 = g^{i\ell} \partial_i \left( g^{jm} R(\partial_j, \partial_k, \partial_\ell, \partial_m) \right) + g^{jm} \partial_j \left( g^{i\ell} R(\partial_k, \partial_i, \partial_\ell, \partial_m) \right) + \partial_k \left( g^{i\ell} g^{jm} R(\partial_i, \partial_j, \partial_\ell, \partial_m) \right).$$

Using the definition of the Ricci and scalar curvatures, we see that this equation yields

$$\begin{aligned} \partial_k R &= g^{jm} \partial_j \left( \mathrm{Ric}(\partial_k, \partial_m) \right) + g^{i\ell} \partial_i \left( \mathrm{Ric}(\partial_k, \partial_\ell) \right) \\ &= 2 g^{ij} \partial_j \left( \mathrm{Ric}(\partial_k, \partial_i) \right) = 2 g^{ij} \left( \nabla_{\partial_j} \mathrm{Ric} \right) (\partial_k, \partial_i), \end{aligned}$$

as desired. □

A significant application of the contracted second Bianchi identity is Schur's theorem:

**Theorem 39.** *Suppose $n = \dim M \geq 3$ and $M$ is connected.*

*(1) Assume there is a smooth function $f \in C^\infty(M)$ so that*

$$\mathrm{Ric}_p(X, Y) = f(p) g(X, Y)$$

*for all $p \in M$ and $X, Y \in T_p M$. Then $f$ is constant.*

*(2) Assume there is a smooth $f \in C^\infty(M)$ so that*

$$R(X, Y, Z, W) = f(p) \left( g(X, W) g(Y, Z) - g(X, Z) g(Y, W) \right)$$

*for all $p \in M$ and $X, Y, Z, W \in T_p M$. (In other words, the sectional curvatures at $p$ are all $f(p)/(n-1)$.) Then $f$ is constant.*

*Proof.* (1) We note that $\mathrm{Ric} = fg$, so

$$\nabla_{v_i} \mathrm{Ric} = \nabla_{v_i} (fg) = v_i(f) g.$$

The scalar curvature is the trace of Ric, so $R = nf$. By the contracted second Bianchi identity, we have

$$nX(f) = 2 \sum g^{ij} \nabla_{v_j} \mathrm{Ric}(X, v_i) = 2X(f).$$

As $n \neq 2$, we conclude $X(f) = 0$ and so $f$ is constant.

(2) Now observe

$$\begin{aligned}
\mathrm{Ric}_p(X,Y) &= \sum_{i,j} g^{ij} R(X, v_i, v_j, Y) \\
&= \sum_{i,j} f(p) g^{ij} \left( g(X,Y) g_{ij} - g(X, v_i) g(Y, v_j) \right) \\
&= \sum_{i,j} f(p) \left( n g(X,Y) - \sum_{k,\ell} g^{ij} g_{ik} X^k g_{j\ell} Y^\ell \right) \\
&= \sum_{i,j} f(p) \left( n g(X,Y) - g(X,Y) \right) = (n-1) f(p) g,
\end{aligned}$$

putting us in the setting of the first part.

□

Not so relevant here, but an important tensor in relativity is the Einstein tensor, given by $\mathrm{Ric} - \frac{1}{2} R g$. It is divergence-free by the contracted second Bianchi identity.

## 7. Riemannian distance

**Definition 40.** For a piecewise smooth path $\gamma : [0, t_0] \to M$, define its length

$$L(\gamma) = \int_0^{t_0} \left|\gamma'(t)\right|_g \, dt = \int_0^{t_0} \sqrt{g(\gamma'(t), \gamma'(t))} \, dt$$

and its energy

$$E(\gamma) = \int_0^{t_0} \left|\gamma'(t)\right|_g^2 \, dt = \int_0^{t_0} g(\gamma'(t), \gamma'(t)) \, dt.$$

A quick exercise in the chain rule proves the following:

**Lemma 41.** *$L(\gamma)$ is independent of orientation-preserving parametrization.*

*Proof.* Suppose $\gamma : [0, t_0] \to M$ is smooth (for piecewise smooth, break into pieces) and $\alpha : [0, s_0] \to [0, t_0]$ is an orientation-preserving diffeomorphism. Let $\tilde{\gamma} = \gamma \circ \alpha$, so then $\tilde{\gamma}'(s) = \gamma'(\alpha(s)) \alpha'(s)$, and

$$\begin{aligned}
L(\tilde{\gamma}) &= \int_0^{s_0} \left|\tilde{\gamma}'(s)\right|_g \, ds = \int_0^{s_0} \left|\gamma'(\alpha(s))\right| \alpha'(s) \, ds \\
&= \int_0^{s_0} \left|\gamma'(\alpha(s))\right| \alpha'(s) \, ds = \int_0^{t_0} \left|\gamma'(t)\right| \, dt = L(\gamma).
\end{aligned}$$

□

We now make our Riemannian manifold into a metric space.

**Definition 42.** For $(M, g)$ Riemannian, and $p, q \in M$, define

$$d(p,q) = \inf \left\{ L(\gamma) \mid \gamma : [0,1] \to M \text{ piecewise smooth}, \gamma(0) = p, \gamma(1) = q \right\}.$$

Note that we could equivalently minimize the energy over paths from $p$ to $q$ – one inequality is Cauchy–Schwarz and the other is reparametrizing by arc length.

Our aims for this section include the following:

(1) $d$ is a metric,

(2) If the infimum is attained, the minimizer is a geodesic,
(3) Conditions so that the infimum is attained, and
(4) A second derivative test for minimization.

**Lemma 43** (Gauss lemma)**.** *Let $P \in M$ and $v \in T_pM$, $v \neq 0$. For any $w \in T_v(T_pM) \cong T_pM$, we have*

$$\left\langle (D\exp_p)_v(v), (D\exp_p)_v(w)\right\rangle = \langle v, w\rangle .$$

*Proof.* Let $\tilde{F}(t,s) = t(v+sw)$ in $T_pM$ and $F(t,s) = \exp_p(\tilde{F}(t,s))$. Observe that

$$\frac{\partial \tilde{F}}{\partial t}(0,0) = v, \qquad \frac{\partial \tilde{F}}{\partial s}(0,0) = 0,$$
$$\frac{\partial \tilde{F}}{\partial t}(1,0) = v, \qquad \frac{\partial \tilde{F}}{\partial s}(1,0) = w,$$

so that

$$\frac{\partial F}{\partial t}(0,0) = v, \qquad \frac{\partial F}{\partial s}(0,0) = 0,$$
$$\frac{\partial F}{\partial t}(1,0) = (D\exp_p)_v(v), \qquad \frac{\partial F}{\partial s}(1,0) = (D\exp_p)_v(w).$$

The curve $t \mapsto F(t,s)$ is a geodesic with initial velocity vector $v + sw$, so

$$\frac{D}{dt}\left(\frac{\partial F}{\partial t}\right) = 0 \quad \text{and} \quad \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle = \langle v+sw, v+sw\rangle .$$

We now claim that

$$\frac{D}{dt}\left(\frac{\partial F}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial F}{\partial t}\right).$$

Assuming this claim, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right\rangle &= \left\langle \frac{D}{dt}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right\rangle + \left\langle \frac{\partial F}{\partial t}, \frac{D}{dt}\frac{\partial F}{\partial s}\right\rangle \\ &= 0 + \left\langle \frac{\partial F}{\partial t}, \frac{D}{ds}\frac{\partial F}{\partial t}\right\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle = \langle v, w\rangle .\end{aligned}$$

So, since the value of the inner product at $(0,0)$ is 0, its value at $(1,0)$ must be $\langle v, w\rangle$.

We now prove the claim. Suppose $F : [0,t_0] \times [0,s_0] \to M$ is smooth. Note that

$$\frac{D}{ds}\left(\frac{\partial F}{\partial t}\right) = \nabla_{\frac{\partial F}{\partial s}}\frac{\partial F}{\partial t} = \nabla_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial t} + \left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right] = \frac{D}{dt}\left(\frac{\partial F}{\partial s}\right) + \left[\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right],$$

so it remains to see that the commutator term vanishes, which can be checked in coordinates. (Alternatively, we could go immediately to coordinates to see that

$$\left(\frac{D}{ds}\left(\frac{\partial F}{\partial t}\right)\right)^k = \frac{\partial^2 F^k}{\partial s \partial t} + \sum_{i,j}\Gamma^k_{ij}(F(t,s))\frac{\partial F^i}{\partial s}\partial F^j \partial t,$$

which is symmetric in the roles of $s$ and $t$.) ∎

Now we construct a "local position vector". Fix $p \in M$, and for $q$ close to $p$ we write $q = \exp_p(v)$ for some $v \in T_pM$ (in the neighborhood of 0 where $\exp_p$ is a diffeomorphism). Let

$$P = (D\exp_p)_v(v) \in T_{\exp_p(v)}M = T_qM.$$

Note that $P$ is a smooth vector field defined in a neighborhood of $p$, and define the function $Q$ in this neighborhood by

$$Q(q) = |v|_g^2,$$

where $v = \exp_p^{-1}(q)$. We claim that the gradient of $Q$ is $2P$.

*Proof of claim.* Define $\tilde{Q} : T_pM \to \mathbb{R}$ by $\tilde{Q}(v) = |v|_g^2$, so that in a neighborhood of $p$, $\tilde{Q} = Q \circ \exp_p$.

Take $q$ close to $p$ and $w \in T_qM$, and let $v = \exp_p^{-1}(q)$, and $\tilde{w} \in T_v(T_pM) \cong T_pM$ be defined by $(D\exp_p)_v(\tilde{w}) = w$.

Since $\tilde{Q} = \exp_p^* Q$, we have

$$\begin{aligned}
\langle (\operatorname{grad} Q)_q, w \rangle &= w(Q) = (D\exp_p)_v(\tilde{w})(Q) \\
&= \tilde{w}\left(\exp_p^* Q\right) = \tilde{w}(\tilde{Q}) \\
&= \left\langle (\operatorname{grad} \tilde{Q})_v, \tilde{w} \right\rangle = 2\langle v, \tilde{w} \rangle .
\end{aligned}$$

Now by Gauss we also have $\langle P, w \rangle = \langle v, \tilde{w} \rangle$, so we must have $2P = \operatorname{grad} Q$ because $w$ (and hence $\tilde{w}$) was arbitrary. ☐

**Proposition 44.** *Let $p \in M$ and $B_r(0) = \{v \in T_pM \mid |v|_g^2 < r^2\}$. Assume $r > 0$ is sufficiently small that $\exp_p|_{B_r(0)}$ is a diffeomorphism and let $U = \exp_p(B_r(0))$. For $q \in U$, the radial geodesic $\gamma$ from $p$ to $q$ is the unique shortest curve (up to reparametrization) in $U$ from $p$ to $q$.*

*Proof.* We have to show that for $\alpha : [0, t_0] \to U$ with $\alpha(0) = p$ and $\alpha(t_0) = q$, we have

(1) $L(\alpha) \geq L(\gamma)$, and
(2) If $L(\alpha) = L(\gamma)$, then $\alpha$ is a reparametrization of $\gamma$.

In the notation from above, define a function $R = \sqrt{Q}$ on $U$, so that $V = P/R$ is the outward radial unit vector on $U \setminus \{p\}$, and $\operatorname{grad} r = \frac{1}{2R}\operatorname{grad} Q = P/R = V$.

Without loss of generality we can assume that $\alpha(t) \neq p$ for $t > 0$ (or else we could start later with a path that was no longer). Write $\alpha'$ in terms of its radial and orthogonal components, i.e.,

$$\alpha' = \langle \alpha', V \rangle V + N,$$

where $N$ is orthogonal to $V$. (At $t = 0$, it doesn't really matter what you do, so maybe take $V = \alpha'(0)$ and $N = 0$.) We compute, for $t > 0$,

$$|\alpha'(t)| = \sqrt{\langle \alpha', \alpha' \rangle} = \sqrt{\langle \alpha', V \rangle^2 + |N|^2} \geq |\langle \alpha', V \rangle| .$$

$V$ is the gradient of $R$, so $\langle \alpha', V \rangle = \frac{d}{dt}(R \circ \alpha)$, and thus

$$L(\alpha) = \int_0^{t_0} |\alpha'(t)|\, dt \geq \int_0^{t_0} \frac{d}{dt}(R \circ \alpha)\, dt = R(\alpha(t_0)) = R(q).$$

On the other hand, we compute the length of $\gamma$:

$$L(\gamma) = \int_0^1 |\gamma'(t)|\ dt = \int_0^1 |v|\ dt = |v| = R(q),$$

so $L(\alpha) \geq L(\gamma)$. To get equality in the above inequality, we must have $|N| = 0$ and $|\langle \alpha', V\rangle| = \langle \alpha', V\rangle$, so that $\alpha'(t) = \left(\frac{d}{dt}(R \circ \alpha)\right) V$, and thus $\alpha$ travels monotonically along $\gamma$ and is a reparametrization. $\square$

**Proposition 45.** *Let $U = \exp_p(B_r(0))$ be as above. For any $q \in U$, the radial geodesic $\gamma$ from $p$ to $q$ is the unique (up to reparametrization) shortest curve in $M$ from $p$ to $q$, i.e., $d(p, q) = L(\gamma)$.*

*Proof.* We know it's the shortest curve within $U$ from $p$ to $q$, and $L(\gamma) < r$, so it remains to show that if $\alpha$ is a curve in $M$ starting at $p$ and leaving $U$, then $L(\alpha) \geq r$. Since $\alpha$ leaves $U$, it meets every geodesic sphere $S(a) = \exp_p(\partial B_a(0))$ with $a < r$. If $\alpha_1$ is the shortest inital sement from $p$ to $S(a)$, then, because $\alpha$ lies in $U$ initially, we must have $L(\alpha) \geq L(\alpha_1) = a$ for all $a < r$, i.e., $L(\alpha) \geq r$. $\square$

**Proposition 46.** *If $M$ is connected then $d$ is a metric, i.e.,*

*(i) $d(p, q) = 0$ if and only if $p = q$,*
*(ii) $d(p, q) = d(q, p)$, and*
*(iii) $d(p, q) \leq d(p, z) + d(z, q)$ for any $z \in M$.*

*Proof.* Note that if $M$ is connected then it is path connected (because it is a manifold), so $d(p, q) < \infty$. The second statement is easy by reversing paths, while the third follows by concatenating them. One implication in the first statement is trivial, so it remains to show that if $d(p, q) = 0$, then $p = q$. Let $p, q \in M$, $p \neq q$. Let $U = \exp_p(B_r(0))$ be as before. If $q \in U$, then we write $q = \exp_p(v)$ and then $d(p, q) = |v| \neq 0$, while if $q \notin U$, we saw that $d(p, q) \geq r$, so we're done. $\square$

**Proposition 47.** *If $(M, g)$ is Riemannian, then the distance function $d$ induces a topology on $M$ that agrees with the original one. In other words, $U$ is open in $M$ if and only if for all $p \in U$, there is an $\epsilon > 0$ so that $d(p, q) \geq \epsilon$ for all $q \notin U$.*

*Proof.* Need to check:

(1) Given any open $U \subset M$ and $p \in U$, there is some $r > 0$ so $d(p, q) \geq r$ for all $q \notin U$, and
(2) Given any $r > 0$ and $p \in M$, there is an open set $U$ so that $d(p, q) < r$ for all $q \in U$.

The second statement follows from the fact that $\exp_p$ is a local diffeomorphism around 0. The first statement follows from the proof of Proposition 45. $\square$

**Proposition 48.** *Let $p, q \in M$ and suppose that $\alpha : [0, 1] \to M$ is a path from $p$ to $q$ with $L(\alpha) = d(p, q)$. Then $\alpha$ is a geodesic (up to reparametrization).*

You might try to reparametrize $\alpha$ by arc length, but the worry is that it might no longer be smooth. (Picture sharp corners you can slow down enough for.) Instead we'll cut it into small pieces and use Proposition 44.

*Proof.* Split $[0, 1]$ into finitely many intervals $[t_i, t_{i+1}]$ so that $\alpha([t_i, t_{i+2}]) \subset U_i$, where $U_i$ is a normal coordinate neighborhood of $\alpha(t_{i+1})$. Because $\alpha$ minimizes distance between its endpoints it must also minimize distance between the points in between by the triangle inequality, so that

$$L\left(\alpha|_{[t_i,t_{i+2}]}\right) = d(\alpha(t_i), \alpha(t_{i+2})),$$

and thus $\alpha|_{[t_i,t_{i+2}]}$ is a reparametrization of a geodesic. Write

$$\alpha|_{[t_i,t_{i+2}]} = \gamma_i \circ \varphi_i,$$

where $\gamma_i$ is a unit speed geodesic and $\varphi_i$ is monotone. Say $\gamma_i : [0, T_i] \to U_i$ and $\varphi_i : [t_i, t_{i+2}] \to [0, T_i]$ and let $\tau_i = \varphi_i(t_{i+1}) \in [0, T_i]$, $\gamma_i(\tau_i) = \alpha(t_{i+1})$.

Consider, for $s \in [0, 1]$, the geodesics $s \mapsto \gamma_i((T_i - \tau_i)s + \tau_i)$ and $s \mapsto \gamma_{i+1}(\tau_{i+1}s)$. Note that these have the same endpoints and are both contained in the normal coordinate neighborhood $U_{i+1}$, and are both radial geodesics, so the must agree. We can then define $\gamma$ by

$$\gamma(s) = \gamma_i\left(s - \sum_{j=1}^{i-1} \tau_j\right)$$

for all $s$ so that $\sum_{j=1}^{i-1} \tau_j \leq s \leq \sum_{j=1}^{i} \tau_i$, so that $\gamma$ is a unit speed geodesic connecting $p$ and $q$ and a reparametrization of $\alpha$. □

7.1. **Existence of minimizing geodesics.** We can't always expect that there are minimizing geodesics. Take, for example, $M = (\mathbb{R}^n \setminus \{0\}, \delta_{ij})$, which has no geodesic joining $x$ and $-x$.

**Proposition 49.** *Suppose $(M, g)$ is Riemannian and connected, $p \in M$, and the domain of $\exp_p$ is all of $T_pM$. Then for all $q \in M$ there is a geodesic $\gamma$ from $p$ to $q$ with $L(\gamma) = d(p, q)$.*

*Proof.* The idea is to build a geodesic piecemeal to find a minimizing curve. Now, given $p, q$, choose $r > 0$ so that $\exp_p|_{B_r(0)}$ is a diffeormorphism.

If $d(p, q) \leq r$, we're done by earlier results, so we can assume that $d(p, q) > r$. Define $S \subset M$ as the geodesic sphere around $p$ of distance $r$, i.e.,

$$S = \{z \in M \mid d(z, p) = r\} = \left\{\exp_p(v) \mid v \in T_pM, |v| = r\right\}.$$

We need to find a direction to start. As $S$ is compact and $d(\cdot, \cdot)$ is continuous,[8] we can find a minimizer $m \in S$ so that

$$d(m, q) \leq d(z, q), \quad \text{for all } z \in S.$$

We now claim that $d(p, m) + d(m, q) = d(p, q)$. Note that one inequality follows immediately by concatenation of paths, so we must show only that $d(p, m) + d(m, q) \leq d(p, q)$. Indeed, given any $\epsilon > 0$, we can find an $\alpha : [0, 1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $L(\alpha) \leq d(p, q) + \epsilon$. The intermediate value theorem implies that there is some $\tau \in [0, 1]$ with $d(p, \alpha(\tau)) = r$, so $\alpha(\tau) \in S$. As $m \in S$, we have $d(p, m) = r = d(p, \alpha(\tau))$. Because $m$ is a minimizer, we have $d(m, q) \leq d(\alpha(\tau), q)$, so that

$$d(p, m) + d(m, q) \leq d(p, \alpha(\tau)) + d(\alpha(\tau), q) \leq L(\alpha|_{[0,\tau]}) + L(\alpha|_{[\tau,1]}) = L(\alpha) \leq d(p, q) + \epsilon.$$

As this inequality holds for all $\epsilon > 0$, we conclude that $d(p, m) + d(m, q) = d(p, q)$.

Moving on, we let $\gamma$ be a unit speed geodesic with $\gamma(0) = p$ and $\gamma(r) = m$. (As $m = \exp_p(v)$ for some $v$ with $|v| = r$, put $\gamma(t) = \exp_p(tv/r)$.) By hypothesis, $\gamma$ is defined for all $t$, so we let $\ell = d(p, q)$. We aim to show that $\gamma(\ell) = q$. Let

$$t_0 = \sup\{t \in [0, \ell] \mid t + d(\gamma(t), q) = \ell\}.$$

[8] You can see this by playing with the definition of $d$.

Note that the set on the right is non-empty because $0 + d(\gamma(0), q) = d(p, q) = \ell$. If $t_0 = \ell$, then we're done as this shows that $d(\gamma(\ell), q) = 0$ and hence $\gamma(\ell) = q$.

Suppose now that $t_0 < \ell$ and let

$$\tilde{S} = \{z \in M \mid d(\gamma(t_0), z) = \tilde{r}\},$$

where $0 < \tilde{r} < \ell - t_0$ is sufficiently small that $\exp_{\gamma(t_0)}$ is s diffeomorphism on the ball of radius $\tilde{r}$. Choose $\tilde{m} \in \tilde{S}$ with $d(\tilde{m}, q) \leq d(z, q)$ for all $z \in \tilde{S}$. The same argument as before shows that $d(\gamma(t_0), \tilde{m}) + d(\tilde{m}, q) = d(\gamma(t_0), q)$, and so

$$t_0 + d(\gamma(t_0), \tilde{m}) + d(\tilde{m}, q) = t_0 + d(\gamma_{t_0}, q) = d(p, q) \leq d(p, \tilde{m}) + d(\tilde{m}, q),$$

so that

$$t_0 + d(\gamma(t_0), \tilde{m}) \leq d(p, \tilde{m}).$$

On the other hand,

$$d(p, \tilde{m}) \leq d(p, \gamma(t_0)) + d(\gamma(t_0), \tilde{m}) = t_0 + d(\gamma(t_0), \tilde{m}),$$

so that $t_0 + d(\gamma(t_0), \tilde{m}) = d(p, \tilde{m})$, and

$$d(p, \gamma(t_0)) + d(\gamma(t_0), \tilde{m}) = d(p, \tilde{m}),$$

so that all three points lie on the same geodesic and thus $\tilde{m} = \gamma(t_0 + \tilde{r})$, so that

$$t_0 + \tilde{r} + d(\gamma(t_0 + \tilde{r}), q) = d(p, q),$$

a contradiction. Thus $t_0 = \ell$ and we are done. □

**Theorem 50** (Hopf–Rinow)**.** *The following are equivalent for $(M, g)$ connected:*

*(i) $(M, d)$ is complete as a metric space,*
*(ii) There is some $p \in M$ so that $\exp_p$ is defined on all of $T_pM$,*
*(iii) For all $p \in M$, $\exp_p$ is defined on all of $T_pM$, and*
*(iv) Every closed and bounded subset of $M$ is compact.*

*Proof.* The plan is to show (i) implies (iii) implies (ii) implies (iv) implies (i). One implication is honestly trivial ((iii) implies (ii)), while another is "trivial" ((iv) implies (i), which follows from your undergraduate analysis course). We therefore need to show (i) implies (iii) and (ii) implies (iv).

To show (i) implies (iii), we start by fixing $p \in M$ and $v \in T_pM$. Let $\gamma$ be the unique geodesic starting at $p$ with $\gamma'(0) = v$. Assume $\gamma$ cannot be extended to $[0, \infty)$ and let $[0, T)$ be the maximal interval of definition. Take $t_k \uparrow T$, then

$$d(\gamma(t_k), \gamma(t_{k+n})) \leq |v| (t_{k+n} - t_k) \leq |v| (T - t_k) \to 0,$$

so that $\gamma(t_k)$ is a Cauchy sequence. As $(M, d)$ is a complete metric space, $\gamma(t_k) \to q$ as $k \to \infty$ and hence $\gamma(T) = q$. By short-time existence for ODEs, we can extend beyond $T$.[9]

To show that (ii) implies (iv), we assume that $\exp_p$ is defined on all of $T_pM$ and that $A \subset M$ is closed and bounded. Let $R > 0$ be so that $d(p, q) \leq R$ for all $q \in A$, so that $A \subset \exp_p(\overline{B_R(0)})$ where $\overline{B_R(0)} \subset T_pM$ is the closed ball of radius $R$. (This follows by the existence of minimizing geodesics.) As $\overline{B_R(0)}$ is compact and $\exp_p$ is continuous, $A$ is therefore a closed subset of a compact set in a Hausdorff space and is thus compact. □

[9]This really needs an additional lemma from ODEs, which would say that if $\gamma$ is continuously extendible, then it is smoothly extendible.

7.2. **Stability of geodesics.** Recall the length and energy of a curve $\alpha : I \to M$:

$$L(\alpha) = \int_I |\alpha'(t)|\, dt, \quad E(\alpha) = \int_I |\alpha'(t)|^2\, dt.$$

Recall further that $L(\alpha)$ is invariant under reparametrization but $E(\alpha)$ is not.

**Proposition 51.** *If $\alpha : [0, t_0] \to M$, then $E(\alpha) \geq L(\alpha)^2/t_0$, with equality if and only if $\alpha$ is parametrized by a multiple of arc length.*

*Proof.* Observe that

$$L(\alpha) = \int_0^{t_0} |\alpha'(t)|\, dt \leq \left(\int_0^{t_0} |\alpha'(t)|^2\, dt\right)^{1/2} \left(\int_0^{t_0} dt\right)^{1/2} = E(\alpha)^{1/2} t_0^{1/2},$$

and equality holds if and only if $|\alpha'(t)|$ and 1 are linearly dependent functions, i.e., if $\alpha$ is parametrized by a multiple of arc length. □

**Corollary 52.** $d(p,q)^2 = \inf\{E(\alpha) \mid \alpha : [0,1] \to M, \alpha(0) = p, \alpha(1) = q\}$.

7.2.1. *First variation formula.* Suppose $\alpha : [0,1] \to M$.

**Definition 53.** A variation of $\alpha$ is a smooth map $F : [0,1] \times (-\epsilon, \epsilon) \to M$ so that $F(t, 0) = \alpha(t)$. We typically write $\alpha_s(t) = F(t,s)$. Sometimes $F$ is called a homotopy. If $F(0,s) = F(0,0)$ and $F(1,s) = F(1,0)$ for all $s$, then $F$ is a variation fixing the endpoints.

Set $E(s) = E(\alpha_s)$ and define $V \in \mathcal{X}(\alpha)$ by $V = \frac{\partial F}{\partial s}|_{s=0}$.

The following proposition is a straightforward computation (after knowing, as we showed before, that $\frac{D}{ds}\left(\frac{\partial F}{\partial t}\right) = \frac{D}{dt}\left(\frac{\partial F}{\partial s}\right)$).

**Proposition 54.** *Geodesics are critical points of $E$ for variations fixing the endpoints.*

*Proof.* We compute

$$\begin{aligned}
\frac{1}{2}E'(0) = \frac{1}{2}\frac{d}{ds}|_{s=0}E(\alpha_s) &= \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\left\langle \alpha_s'(t), \alpha_s'(t)\right\rangle|_{s=0}\, dt \\
&= \int_0^1 \left\langle \frac{D}{ds}\left(\frac{\partial F}{\partial t}\right), \alpha_s'(t)\right\rangle|_{s=0}\, dt \\
&= \int_0^1 \left\langle \frac{D}{dt}\left(\frac{\partial F}{\partial s}\right), \alpha_s(t)\right\rangle|_{s=0}\, dt \\
&= -\int_0^1 \left\langle V(t), \frac{D}{dt}\alpha'(t)\right\rangle dt + \left[\left\langle V(t), \alpha'(t)\right\rangle\right]_{t=0}^{t=1},
\end{aligned}$$

where the last equality follows by integrating by parts (or, using the compatibility of $D/dt$ with the metric). If the variation fixes the endpoints, the second term vanishes and so the critical points of $E$ are when $\frac{D}{dt}\alpha'(t) = 0$, i.e., when $\alpha$ is a geodesic. □

7.2.2. *Second variation formula.* Let's start by calculating the second derivative of $E$:

**Proposition 55.**

$$\frac{1}{2}E''(0) = \int_0^1 \left\langle \frac{D}{dt}V(t), \frac{D}{dt}V(t) \right\rangle dt - \int_0^1 R(\alpha'(t), V(t), \alpha'(t), V(t))\, dt$$
$$+ \left[ \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s}, \alpha'(t) \right\rangle \right]_{t=0,s=0}^{t=1,s=0} - \int_0^1 \left\langle \frac{D}{dt}\alpha'(t), \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s} \right\rangle dt$$

*Proof.* The first variation formula gave us

$$\begin{aligned}
\frac{1}{2}E'(0) &= \frac{1}{2}\int_0^1 \frac{\partial}{pds} \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle |_{s=0}\, dt \\
&= \int_0^1 \left\langle \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t} \right\rangle |_{s=0}\, dt \\
&= \int_0^1 \left\langle \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \right\rangle |_{s=0}\, dt \\
&= \int_0^1 \left\langle \alpha'(t), \nabla_{\frac{\partial}{\partial t}} V(t) \right\rangle dt \\
&= \int_0^1 \left[ \frac{\partial}{\partial t} \langle \alpha'(t), V(t) \rangle - \left\langle \nabla_{\frac{\partial}{\partial t}} \alpha'(t), V(t) \right\rangle \right] dt.
\end{aligned}$$

The second derivative is given by

$$\begin{aligned}
\frac{1}{2}E''(t) &= \int_0^1 \frac{\partial^2}{\partial s^2} \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle |_{s=0}\, dt \\
&= \int_0^1 \left\langle \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t} \right\rangle |_{s=0}\, dt + \int_0^1 \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t} \right\rangle |_{s=0}\, dt \\
&= \int_0^1 \left\langle \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \right\rangle |_{s=0}\, dt + \int_0^1 \left| \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \right|_{s=0}^2 dt \\
&= \int_0^1 \left\langle \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s} \right\rangle |_{s=0}\, dt - \int_0^1 \left\langle \frac{\partial F}{\partial t}, R\left( \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s} \right) \frac{\partial F}{\partial s} \right\rangle |_{s=0}\, dt + \int_0^1 \left| \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \right|_{s=0}^2 dt \\
&= \int_0^1 \left[ \frac{\partial}{\partial t} \left\langle \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s} \right\rangle \right]_{s=0} dt \\
&\quad - \int_0^1 \langle \alpha'(t), R(\alpha'(t), V(t))V(t) \rangle\, dt + \int_0^1 \left\langle \frac{D}{dt} V(t) \right\rangle^2 dt.
\end{aligned}$$

Applying the fundamental theorem of calculus finishes the proof. $\square$

We make several observations:

(1) If $F$ is a variation fixing the endpoints, then the term in brackets in the proposition vanishes.
(2) If $\alpha$ is a geodesic, then $\nabla_{\partial/\partial t}\alpha'(t) = 0$ and so the last term in the proposition vanishes.
(3) If both of the above hold and we integrate by parts (and use a symmetry of $R$), we obtain

$$\frac{1}{2}E''(0) = \left[ \left\langle \frac{D}{dt}V(t), V(t) \right\rangle \right]_{t=0}^{t=1} - \int_0^1 \left\langle V(t), \frac{D^2}{dt^2}V(t) + R(V(t), \alpha'(t))\alpha'(t) \right\rangle dt.$$

(4) If $(M, g)$ has non-positive sectional curvature, then $E''(0) > 0$ for variations of geodesics fixing the endpoints, i.e., geodesics are minimizers of energy.
(5) Similarly, if $\alpha$ is a geodesic and $L(\alpha)$ is sufficiently small, then the second integral in the proposition is small and so short geodesics are also stable.

In fact, let's give this a name (and write it as a bilinear form for later):

$$I(V,V) = \int_0^1 \left|\frac{D}{dt}V(t)\right|^2 dt - \int_0^1 R\left(V(t), \alpha'(t), \alpha'(t), V(t)\right) dt. \tag{2}$$

7.3. **Jacobi fields.** The third item in the note at the end of the last section motivates a definition:

**Definition 56.** For a geodesic $\alpha$, a vector field $V \in \mathcal{X}(\alpha)$ is a Jacobi field if

$$\frac{D^2}{dt^2}V(t) + R\left(V(t), \alpha'(t)\right)\alpha'(t) = \nabla_{\alpha'(t)}\nabla_{\alpha'(t)}V(t) + R\left(V(t), \alpha'(t)\right)\alpha'(t) = 0;$$

for all $t$.

Standard results from (linear!) ordinary differential equations tell us that the initial value problem is well-posed, i.e., given $v, w \in T_{\alpha(0)}M$, there is a unique Jacobi field $V \in \mathcal{X}(\alpha)$ with $V(0) = v$ and $\frac{D}{dt}V(0) = w$.

**Proposition 57.** *If $\alpha$ is a geodesic and $V \in \mathcal{X}(\alpha)$ is a Jacobi field, then there are $a, b \in \mathbb{R}$ and a normal Jacobi field $W$ (i.e. a Jacobi field $W$ with $W(t) \perp \alpha'(t)$) so that*

$$V(t) = W(t) + (a + bt)\alpha'(t).$$

*Proof.* Observe that because $\alpha$ is a geodesic we have

$$\frac{d^2}{dt^2}\left\langle V(t), \alpha'(t)\right\rangle = \left\langle \frac{D^2}{dt^2}V(t), \alpha'(t)\right\rangle = -R\left(V(t), \alpha'(t), \alpha'(t), \alpha'(t)\right) = 0,$$

so that

$$\langle V(t), \alpha'(t)\rangle = a + bt$$

for some constants $a, b \in \mathbb{R}$. Let $W(t) = V(t) - (a + bt)\alpha'(t)$, so $\langle W(t), \alpha'(t)\rangle = 0$ and thus $W$ is normal. Now $(a + bt)\alpha'(t)$ is a Jacobi field, and the Jacobi equation is linear, so $W$ is also a Jacobi field. ☐

One way Jacobi fields arise is via families of geodesics:

**Theorem 58.** *Let $\alpha : [0, 1] \to M$ be a geodesic.*

*(i) If $F : [0, 1] \times (-\epsilon, \epsilon) \to M$ is a variation of $\alpha$ through geodesics (i.e., $t \mapsto F(t, s)$ is a geodesic for each $s$ and $F(t, 0) = \alpha(t)$), then $\frac{\partial F}{\partial s}|_{s=0} \in \mathcal{X}(\alpha)$ is a Jacobi field.*
*(ii) All Jacobi fields along $\alpha$ arise in this way.*

*Proof.* The first statement is a calculation:

$$\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s} = \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t} = \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t} - R\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t} = -R\left(\frac{\partial F}{\partial s}|_{s=0}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t},$$

where in last equality we use that $t \mapsto F(t, s)$ is a geodesic for each $s$ to conclude that $\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t} = 0$. Restricting to $s = 0$ we see that $\frac{\partial F}{\partial s}|_{s=0}$ is a Jacobi field along $\alpha$.

For the other part, assume that $V \in \mathcal{X}(\alpha)$ is a Jacobi field and let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = \alpha(0)$ and $\gamma'(0) = V(0)$. Let $X \in \mathcal{X}(\gamma)$ be a vector field with $X(0) = \alpha'(0)$ and

$$\nabla_{\frac{\partial}{\partial s}} X(s)|_{s=0} = \nabla_{\frac{\partial}{\partial t}} V(t)|_{t=0}.$$

Let $F(t,s) = \exp_{\gamma(s)}(tX(s))$, so that

$$\begin{aligned} F(t,0) &= \exp_{\alpha(0)}(t\alpha'(0)) = \alpha(t), \\ F(0,s) &= \gamma(s), \end{aligned}$$

so in particular

$$\frac{\partial F}{\partial s}|_{s=0} = \gamma'(0) = V(0).$$

We now observe that

$$\frac{\partial F}{\partial t}(0,s) = X(s),$$

and so

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}|_{s=t=0} = \nabla_{\frac{\partial}{\partial s}} X(s)|_{s=0} = \nabla_{\frac{\partial}{\partial t}} V(t)|_{t=0} = \nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}|_{s=t=0}.$$

Let $\tilde{V} = \frac{\partial F}{\partial s}|_{s=0}$. As $F$ is a variation of $\alpha$ through geodesics, $\tilde{V}$ is a Jacobi field arising in the claimed way. We now see by the above computation that

$$\begin{aligned} \tilde{V}(0) &= V(0), \\ \nabla_{\frac{\partial}{\partial t}} \tilde{V}(t)|_{t=0} &= \nabla_{\frac{\partial}{\partial t}} V(t)|_{t=0}, \end{aligned}$$

so then uniqueness of solutions of linear differential equations implies that $\tilde{V} = V$, i.e., that $V$ arises by a variation of $\alpha$ through geodesics. □

**Corollary 59.** *If $\alpha : [0,1] \to M$ is a geodesic and $v \in T_{\alpha(0)}M$, then $V(t) = \left(D \exp_{\alpha(0)}\right)_{t\alpha'(0)}(tv)$ is the unique Jacobi field along $\alpha$ with $V(0) = 0$ and $\frac{D}{dt}V(0) = v$.*

*Proof.* Let $F(t,s) = \exp_{\alpha(0)}\left(t(\alpha'(0) + sv)\right)$. Note that

$$\frac{\partial F}{\partial s}|_{s=0} = \left(D \exp_{\alpha(0)}\right)_{t\alpha'(0)}(tv) = V(t),$$

so $V$ is indeed a Jacobi field. It is straightforward to check it satisfies the claimed initial conditions. □

The theorem above motivates one of the main uses of Jacobi fields, which is a use more akin to boundary value problems than to initial value problems:

**Definition 60.** If $\alpha : I \to M$ is a geodesic and $t_0, t_1 \in I$ with $t_0 \neq t_1$, then $\alpha(t_0)$ and $\alpha(t_1)$ are conjugate along $\alpha$ if there is a nontrivial Jacobi field $V \in \mathcal{X}(\alpha)$ with $V(t_0) = V(t_1) = 0$.

**Theorem 61.** *Let $\alpha : [0,1] \to M$ be a geodesic.*

*(i) Suppose there is no point conjugate to $\alpha(0)$ is not conjugate to $\alpha(t)$ for all $t \in [0,1]$. There is a number $\epsilon > 0$ so that if $\gamma : [0,1] \to M$ is a piecewise smooth curve with $\gamma(0) = \alpha(0)$, $\gamma(1) = \alpha(1)$, and $d(\gamma(t), \alpha(t)) < \epsilon$ for all $t \in [0,1]$, then $L(\gamma) \geq L(\alpha)$ with equality holding if and only if $\gamma = \alpha$ up to reparametrization.*

*(ii) Assume $\alpha(0)$ is conjugate to $\alpha(\tau)$ along $\alpha$ for some $\tau \in (0,1)$. Then there is a variation $F : (-\epsilon, \epsilon)_s \times [0,1]_t \to M$ of $\alpha$ fixing the endpoints so that $L(\alpha_s) < L(\alpha)$ for sufficiently small $s$.*

The intuition (and idea of proof) is that the conjugate points come from zero eigenvalues of the bilinear form $I$ arising as the Hessian of the energy functional under variations fixing the endpoints; having a negative eigenvalue at the end (so a direction in which one can decrease length) forces a zero eigenvalue somewhere in the middle.

You should also note that the two parts of the theorem are not quite complementary: you can think of this in analogy with the failure of the second derivative test in freshman calculus. Indeed, the first part of the theorem tells us that the second derivative of energy is positive, so a small variation will increase energy, while the second tells us that there is a direction in which the second derivative is negative, so you can move that way to decrease energy.

Before we prove the theorem, there's a useful proposition about solving a boundary value problem:

**Proposition 62.** *Suppose $\alpha : I \to M$ is a geodesic and $t_0, t_1 \in I$ with $t_0 \neq t_1$. If $\alpha(t_0)$ and $\alpha(t_1)$ are not conjugate along $\alpha$, then for all $v_0 \in T_{\alpha(t_0)}M$ and $v_1 \in T_{\alpha(t_1)}M$, there is a unique Jacobi field $V$ along $\alpha$ so that $V(t_0) = v_0$ and $V(t_1) = v_1$.*

*Proof.* This is essentially an elementary linear algebra statement in disguise. For $w \in T_{\alpha(t_0)}M$, let $J_w$ denote the unique Jacobi field along $\alpha$ with $J_w(t_0) = 0$ and $\frac{D}{dt}J_w(t_0) = w$. Consider the map $T_{\alpha(t_0)}M \to T_{\alpha(t_1)}M$ given by

$$w \mapsto J_w(t_1).$$

This is a linear transformation and must be invertible because any nonzero element of the kernel would give a Jacobi field vanishing at $\alpha(t_0)$ and $\alpha(t_1)$ but these points are not conjugate along $\alpha$.

Let $V_0(t)$ denote the Jacobi field along $\alpha$ with $V(t_0) = v_0$ and $\frac{D}{dt}V(t_0) = 0$. If $L$ is the linear map defined in the previous paragraph, let $w = L^{-1}(v_1 + V(t_1))$, then one can check that $J_w + V_0$ is the unique Jacobi field satisfying the given boundary conditions. ☐

*Proof of theorem.* (i) If $p = \alpha(0)$ is not conjugate to $\alpha(t)$ for all $t \in [0,1]$, then by the previous proposition, (writing $v = \alpha'(0)$)

$$\left(D \exp_p\right)_{tv}$$

is invertible for all $t$. By the inverse function theorem, $\exp_p$ is a local diffeomorphism in a neighborhood of $tv$ for each $t \in [0,1]$. We then cover the compact set $[0,1]v \subset T_pM$ by a finite collection $U_i$ of neighborhoods on which $\exp_p$ is a diffeomorphism. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition of $[0,1]$ so that $[t_{i-1}, t_i]v \subset U_i$, and let $\epsilon > 0$ be sufficiently small so that for all $\gamma$ as in the theorem statement, there is a lift $\beta : [0,1] \to T_pM$ with $\gamma = \exp_p \circ \beta$ and $\beta(t) \in \cup_i U_i$ for each $t \in [0,1]$.

By the Gauss lemma, we know

$$\left\langle \left(D \exp_p\right)_{\beta(t)} (\beta(t)), \left(D \exp_p\right)_{\beta(t)} (\beta'(t)) \right\rangle = \langle \beta(t), \beta'(t) \rangle . \tag{3}$$

By Cauchy–Schwarz, the left side is bounded above by

$$\left|\left(D \exp_p\right)_{\beta(t)} (\beta(t))\right| \left|\left(D \exp_p\right)_{\beta(t)} (\beta'(t))\right| = |\beta(t)| \left|(D \exp_p)_{\beta(t)} (\beta'(t))\right| ,$$

where the equality also holds by the Gauss lemma. On the other hand, the right side of equation (3) is equal to

$$\frac{1}{2}\frac{d}{dt}\left|\beta(t)\right|^2 = \left|\beta(t)\right| \frac{d}{dt}\left|\beta(t)\right|.$$

Dividing by $|\beta(t)|$, we therefore have $|\gamma'(t)| \geq \frac{d}{dt}|\beta(t)|$, and $\exp_p(\beta(1)) = \gamma(1) = \alpha(1) = \exp_p(v)$, so $\beta(1) = v$ because it's close to $v$. We therefore have

$$L(\gamma) = \int_0^1 |\gamma'(t)|\ dt \geq \int_0^1 \frac{d}{dt}|\beta(t)|\ dt = |\beta(1)| = |v| = L(\alpha).$$

The equality statement follows from arguments analogous to those we gave ealier about minimizing geodesics. (These are locally minimizing.)

(ii) We sketch a proof of the seonc statement. Let $V(t)$ denote the nontrivial Jacobi field along $\alpha$ with $V(0) = 0$ and $V(\tau) = 0$. Let $W(t)$ be a vector field along $\alpha$ so that $W(0) = 0$, $W(1) = 0$, and

$$W(\tau) = -\frac{D}{dt}V(t)|_{t=\tau}.$$

For $\delta \in \mathbb{R}$, we define a piecewise smooth vector field $X$ along $\alpha$ by

$$X(t) = \begin{cases} V(t) + \delta W(t) & t \in [0,\tau] \\ \delta W(t) & t \in [\tau, 1] \end{cases}.$$

Note that $X$ is continuous because $V$ vanishes at $\tau$. Now we calculate, using the bilinear form $I$ defined above (2):

$$\begin{aligned} I(X,X) &= \int_0^\tau \left|\frac{D}{dt}V(t)\right|^2 dt + 2\delta \int_0^\tau \left\langle \frac{D}{dt}V(t), \frac{D}{dt}W(t)\right\rangle dt + \delta^2 \int_0^1 \left|\frac{D}{dt}W(t)\right|^2 dt \\ &\quad - \int_0^\tau R\left(V,\alpha',\alpha',V\right)\ dt - 2\delta\int_0^\tau R\left(V,\alpha',\alpha',W\right)\ t - \delta^2 \int_0^1 R\left(W,\alpha',\alpha',W\right)\ dt \\ &= \delta^2 I(W,W) + 2\delta\left\langle \frac{D}{dt}V(\tau), W(\tau)\right\rangle \end{aligned}$$

where in the last equality we used that $V$ satisfies the Jacobi equation,

$$\int_0^\tau \left\langle \frac{D}{dt}V(t), \frac{D}{dt}W(t)\right\rangle dt = \left[\left\langle \frac{D}{dt}V(t), W(t)\right\rangle\right]_{t=0}^{t=\tau} - \int_0^\tau \left\langle \frac{D^2}{dt^2}V(t), W(t)\right\rangle dt,$$

and then that $V$ satisfies the Jacobi equation again. Given our choice of $W$, we obtain that

$$I(X,X) = \delta^2 I(W,W) - 2\delta\left|W(\tau)\right|^2.$$

As $W(\tau) \neq 0$ because $V$ is nontrivial and vanishes at $\tau$, $I(X,X) < 0$ for sufficiently small $\delta > 0$. We can then take

$$F(s,t) = \exp_{\alpha(t)}(sX(t))$$

to find a continuous variation through piecewise smooth curves that decreases length. Approximating it by a smooth variation then finishes the proof.

$\square$

7.3.1. *Another interpretation of Jacobi fields.* Recall that the connection gives a splitting (at each $(p, v) \in TM$) of $T_{(p,v)}(TM)$ into horizontal and vertical subspaces, i.e.,

$$T_{(p,v)}(TM) = H_{(p,v)} \oplus V_{(p,v)},$$

where $V_{(p,v)} = \ker \pi_*$ (so $\pi_*$ gives an isomorphism $H_{(p,v)} \to T_pM$) and $H_{(p,v)} = \ker K$ (so $K$ gives an isomorphism $V_{(p,v)} \to T_pM$).[10]

We now let $\phi_t : TM \to TM$ denote the geodesic flow map (sometimes called geodesic spray). In other words, we let

$$\phi_t(p, v) = (\gamma(t), \gamma'(t)),$$

where $\gamma$ is the unique geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$.

Fix a geodesic $\alpha : I \to M$ and a Jacobi field $J$ along $\alpha$. Let $\widetilde{\alpha} : I \to TM$ denote the lift of $\alpha$, i.e.,

$$\widetilde{\alpha}(t) = (\alpha(t), \alpha'(t)).$$

We use the splitting above to encode the initial data of $J$ at $(p, v) = \widetilde{\alpha}(0)$. In particular, we take $\xi \in T_{(p,v)}(TM)$ so that

$$K\xi = \frac{D}{dt}|_{t=0} J(t), \quad \pi_*\xi = J(0).$$

Such a $\xi$ is guaranteed to exist by the splitting of $T_{(p,v)}(TM)$ into horizontal and vertical subspaces. We now define $Y(t) \in \mathcal{X}(\widetilde{\alpha})$ by

$$Y(t) = (\phi_t)_* \xi \in T_{\widetilde{\alpha}(t)}(TM).$$

**Proposition 63.** *With $J$ a Jacobi field along $\alpha$ and $Y$ defined as above, we have*

$$J(t) = \pi_* Y(t),$$

*i.e., the Jacobi field $J$ is given by the horizontal part of $Y$.*

*Proof.* We appeal again to the uniqueness of solutions of differential equations; we demonstrate equality by showing that $\pi_*Y(t)$ is a Jacobi field along $\alpha$ with the same initial conditions as $J$.

Choose a curve $\widetilde{\gamma} : (-\epsilon, \epsilon)_s \to TM$ with $\widetilde{\gamma}(0) = (p, v)$ and $\widetilde{\gamma}'(0) = \xi$ (where $\xi \in T_{(p,v)}(TM)$ is given as above). We then have

$$(\phi_t)_* \xi = \frac{d}{ds} (\phi_t \circ \widetilde{\gamma})(s)|_{s=0},$$

and

$$\pi_*Y(t) = \pi_* (\phi_t)_* \xi = \frac{d}{ds} (\pi \circ \phi_t \circ \widetilde{\gamma})(s)|_{s=0}.$$

We now write $\widetilde{\gamma}(s) = (\gamma(s), \gamma'(s))$ and set

$$F(s, t) = (\pi \circ \phi_t \circ \gamma)(s) = \exp_{\gamma(s)} (t\gamma'(s)),$$

so that $F$ is a variation of $\alpha$ through geodesics and thus $\frac{\partial F}{\partial s} = \pi_*Y(t)$ is a Jacobi field along $\alpha$.

[10] Recall that $K : T_{(p,v)}(TM) \to T_pM$ was the "Ehresman connection". We defined it in terms of the covariant derivative by taking $\xi \in T_{(p,v)}(TM)$ and choosing $\gamma : (-\epsilon, \epsilon) \to TM$ with $\gamma(0) = (p, v)$ and $\gamma'(0) = \xi$. We then set $\alpha = \pi \circ \gamma$ and regarded $\gamma \in \mathcal{X}(\alpha)$ to define $K\xi \in T_pM$ by $K\xi = \frac{D}{ds}\gamma(s)|_{s=0}$. We showed (in coordinates) that $K$ was independent of the choice of $\gamma$.

We now compute its initial data. Note that
$$\pi_* Y(0) = \pi_*(\phi_0)_*\xi = \pi_*\xi = J(0).$$
We regard $\widetilde{\gamma}$ as a vector field over $\gamma$, so by definition
$$\frac{D}{ds}\gamma'(s)|_{s=0} = K\xi = \frac{D}{dt}|_{t=0}J(t).$$
But the other piece of initial data is
$$\frac{D}{dt}\pi_* Y(t)|_{t=0} = \nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s}|_{s=0,t=0} = \nabla_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t}|_{s=0,t=0} = \frac{D}{ds}\gamma'(s)|_{s=0},$$
finishing the proof. □

**Corollary 64.** *If $\alpha : [0,1] \to M$ is a geodesic and $v \in T_{\alpha(0)}M$, the unique Jacobi field $V$ along $\alpha$ with $V(0) = 0$ and $\frac{D}{dt}V(0) = v$ is given by*
$$V(t) = \left(D\exp_{\alpha(0)}\right)_{t\alpha'(0)}(tv).$$

*Proof.* Note that
$$F(s,t) = \exp_{\alpha(0)}\left(t(\alpha'(0) + sv)\right)$$
is a variation through geodesics giving the Jacobi field $V$. You can check that it has the right initial conditions. □

7.4. **Applications of variational formulae.** The main applications of Section 7.3 are the following results connecting curvature and topology:

**Theorem 65** (Synge)**.** *If $M$ is a compact, oriented, even-dimensional Riemannian manifold with strictly positive sectional curvatures then $\Pi_1(M) = 0$.*

**Theorem 66** (Myers)**.** *If $M$ is a complete Riemannian manifold with $\mathrm{Ric}(X,X) \geq \lambda |X|^2$ everywhere for a fixed $\lambda > 0$, then*
$$\mathrm{diam}(M) = \sup\{d(p,q) \mid p,q \in M\} \leq \pi\sqrt{\frac{n-1}{\lambda}}$$
*with equality holding on the sphere.*

**Corollary 67.** *Under the hypotheses of Myers's theorem, $M$ is compact by Hopf–Rinow.*

**Corollary 68.** *Under the same hypotheses, the universal cover of $M$ is also compact and thus $\Pi_1(M)$ is finite.*

*Proof of Synge's theorem.* Assume that $\Pi_1(M) \neq 0$ and pick a noncontractible closed curve $\gamma$. We may assume that $\gamma$ is smooth and parametrized by $[0,1]$.

Find $\alpha \in [\gamma]$ that minimizes length. You can do this by looking at concatenations of broken geodesics to tend to the infimum (which is positive because $\gamma$ is not contractible and $M$ is compact). A technicality that we're not addressing here is that you need an upper bound on the number of break points, but you can aensure that each segment has some minimum length because $M$ is compact (so the size of balls on which you have normal coordinate charts is bounded below). Without loss of generality, we can assume that $\alpha : [0,1] \to M$ is a geodesic loop (so that $\alpha(0) = \alpha(1)$ and $\alpha'(0) = \alpha'(1)$).

Let $P_0^t : T_{\alpha(0)}M \to T_{\alpha(t)}M$ denote parallel transport along $\alpha$. Because parallel transport preserves angles and lengths, $P_0^1 : T_{\alpha(0)}M \to T_{\alpha(1)}M = T_{\alpha(0)}M$ is an orientation-preserving isometry, i.e., $P_0^1 \in \mathrm{SO}(n)$.

Because $P_0^t\alpha'(0) = \alpha'(t)$, $P_0^1\alpha'(0) = \alpha'(1) = \alpha'(0)$, so $\alpha'(0)$ is an eigenvector of $P_0^1$ with eigenvalue 1. We want to find another one.

Parallel transport preserves angles, so $P_0^1$ preserves the subspace

$$V = \left\{v \in T_{\alpha(0)}M \mid \langle v, \alpha'(0)\rangle = 0\right\} \subset T_{\alpha(0)}M.$$

The dimension of $V$ is odd, so $P_0^1 : V \to V$ has a real eigenvalue (as odd degree polynomials have at least one real root); because $P_0^1 \in \mathrm{SO}(n)$, at least one of these real roots must be $+1$, i.e., there is some $v \in T_{\alpha(0)}M$ with $v \neq 0$, $\langle v, \alpha'(0)\rangle = 0$ with $P_0^1 v = v$.

Let $V \in \mathcal{X}(\alpha)$ be $V(t) = P_0^t v$, so $V$ is parallel, and

$$I(V,V) = \int_0^1 \left|\nabla_{\partial/\partial t}V(t)\right|^2 \, dt - \int_0^1 R(\alpha'(t), V(t), V(t), \alpha'(t))\, dt,$$

which must be strictly negative because the first term vanishes (as $V$ is parallel) and the sectional curvatures of $M$ are bounded below away from zero. The variation

$$F(s,t) = \exp_{\alpha(t)}\left(sV(t)\right) = \alpha_s(t)$$

is a homotopy of $\alpha$ to a curve with strictly shorter length, contradicting that $\alpha$ minimized length in its homotopy class. $\square$

*Proof of Myers's theorem.* Fix $p, q \in M$, we aim to bound $d(p,q)$ from above by the purported bound. (It is easy to check that the sphere of radius $\lambda^{-1}$ satisfies the claimed bound with equality.)

As $M$ is complete, there is a minimizing geodesic $\alpha : [0,1] \to M$ from $p$ to $q$ with $L(\alpha) = d(p,q)$. Note that because $\alpha$ is minimizing, we have $I(V,V) \geq 0$ for all vector fields along $\alpha$ with $V(0) = 0$ and $V(1) = 0$.

Choose $X_1, \ldots, X_{n-1} \in \mathcal{X}(\alpha)$ so that $X_1(0), \ldots, X_{n-1}(0)$, $\frac{1}{|\alpha'(0)|}\alpha'(0)$ form an orthonormal basis for $T_pM$ and all $X_j$ are parallel along $\alpha$. In particular, at each $t$, they continue to form an orthornomal basis for $T_{\alpha(t)}M$.

Define

$$Y_j(t) = \sin(\pi t)X_j(t),$$

so $Y_j(0) = 0$, $Y_j(1) = 0$, and thus $I(Y_j, Y_j) \geq 0$.

We compute

$$\begin{aligned} I(Y_j, Y_j) &= \int_0^1 \left|\frac{D}{dt}Y_j(t)\right| - \int_0^1 R\left(Y_j(t), \alpha'(t), \alpha'(t), Y_j(t)\right)\, dt \\ &= \pi^2 \int_0^1 \cos^2(\pi t)\, dt - \int_0^1 \sin^2(\pi t) R\left(X_j(t), \alpha'(t), \alpha'(t), X_j(t)\right)\, dt. \end{aligned}$$

Sum over $i = 1, \ldots, n-1$ and observe that the last term gives a contraction of the Riemann tensor (since $X_j$ are orthonormal and the remaining direction is parallel to $\alpha'(t)$ and thus the term is zero) to obtain

$$\begin{aligned} 0 \leq \sum_{j=1}^{n-1} I(Y_j, Y_j) &= (n-1)\pi^2 \int_0^1 \cos^2(\pi t)\, dt - \int_0^1 \sin^2(\pi t)\,\mathrm{Ric}(\alpha'(t), \alpha'(t))\, dt \\ &\leq (n-1)\pi^2 \int_0^1 \cos^2(\pi t) - \lambda \int_0^1 |\alpha'(t)|^2 \sin^2(\pi t)\, dt. \end{aligned}$$

As $\alpha$ is a geodesic parameterized on a unit interval, we have $|\alpha'(t)| = L(\alpha)$, so

$$L(\alpha)^2 \int_0^1 \sin^2(\pi t)\, dt \leq \frac{n-1}{\lambda}\pi^2 \int_0^1 \cos^2(\pi t)\, dt,$$

i.e.,

$$L(\alpha) \leq \pi\sqrt{\frac{n-1}{\lambda}}.$$

□

## 8. Submanifold geometry

Our aim in this section is to generalize the results from last semester about immersions of surfaces in $\mathbb{R}^3$. Recall that if $M$ is a smooth surface and $F : M \to \mathbb{R}^3$ is an immersion, it induces a metric $g$ on $M$ by

$$g_{ij}(x) = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle.$$

(This is sometimes called the first fundamental form.)

The second fundamental form is given by

$$h_{ij}(x) = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right\rangle = -\left\langle \frac{\partial^2}{\partial x^i \partial x^j}, \nu \right\rangle,$$

where $\nu$ is a unit normal to the hypersurface. (The second equality holds because $\left\langle \frac{\partial F}{\partial x^i}, \nu \right\rangle = 0$.) This definition requires an orientation on $M$ to make global (as you need a global choice of normal vector), but the Gaussian curvature (defined locally) does not:

$$\kappa = R(\partial_1, \partial_2, \partial_2, \partial_1) = h_{11}h_{22} - h_{12}^2 = \det h.$$

8.1. **General set-up.** Suppose now that $M$ is a submanifold of $\overline{M}$ and that $(\overline{M}, \overline{g})$ is Riemannian.[11] The inclusion induces a Riemannian metric on $M$ by restriction:

$$g_p(v, w) = \overline{g}_p(v, w) \left(= \overline{g}_p(\iota_* v, \iota_* w)\right).$$

For $p \in M$, we write the tangent space to $\overline{M}$ in terms of tangential (to $M$) and normal (to $M$) components:

$$T_p\overline{M} = T_pM \oplus (T_pM)^{\perp}, \quad v = \tan(v) + \mathrm{nor}(v).$$

We define three spaces of vector fields on $M$ (sitting inside $\overline{M}$):

- $\overline{\mathcal{X}}(M)$ is the space of vector fields on $M$ taking values in $T\overline{M}$ (i.e., $V : M \to T\overline{M}$, $V_p \in T_p\overline{M}$),
- $\mathcal{X}(M) \subset \overline{\mathcal{X}}(M)$ is the space of vector fields on $M$ taking values in $TM$ (i.e,. $\mathrm{nor}(V) = 0$), and
- $\mathcal{X}^{\perp}(M) \subset \overline{\mathcal{X}}(M)$ is the space of vector fields normal to $M$ (i.e., $\tan(V) = 0$).

Let $\nabla$ denote the Levi-Civita connection on $(M, g)$ and $\overline{\nabla}$ denote the Levi-Civita connection on $(\overline{M}, \overline{g})$. Suppose $X \in \overline{\mathcal{X}}(M)$ and $V \in \mathcal{X}(M)$. We'd like to define $\overline{\nabla}_V X$. To do this, we pick extensions of $X$ and $V$ to $\overline{M}$, which we call $\overline{X}$ and $\overline{V}$, i.e., $\overline{X}$ is a vector field on $\overline{M}$ so that $\overline{X}|_M = X$.

[11]More precisely, you have an inclusion map $\iota : M \to \overline{M}$. Nearly everything below should have pull-backs/pushforwards by iota, but I won't write them.

**Lemma 69.** $\overline{\nabla}_{\overline{V}}\overline{X}|_M$ *is independent of extension.*

*Proof.* The proof is essentially the same as when we defined the covariant derivative along a curve. Suppose $\overline{X}$ restricts to $X$ and that $\overline{V}$ restricts to $V$. Fix a point $p \in M$ and take local coordinates $(x^1, \dots, x^{n+k})$ so that $M$ is locally defined by $x^{n+1} = \cdots = x^{n+k} = 0$. We then have that $\frac{\partial}{\partial x^i}$ are a local frame for $T\overline{M}$ and the first $n$ of them are a local frame for $TM$.

We then compute

$$\begin{aligned}\overline{\nabla}_{\overline{V}}\overline{X}|_M &= \sum_{j=1}^{n+k} \left(\overline{\nabla}_{\overline{V}}\overline{X}\right)^j \frac{\partial}{\partial x^j}|_M \\ &= \sum_{i,j=1}^{n+k} \overline{V}^i \frac{\partial \overline{X}^j}{\partial x^i} \frac{\partial}{\partial x^j}|_M + \sum_{i,j,\ell=1}^{n+k} \overline{V}^i \overline{X}^\ell \overline{\Gamma}^j_{i\ell}(x) \frac{\partial}{\partial x^j}|_M,\end{aligned}$$

which restricts to (as $\overline{V}^{n+i}|_M = 0$ for $i = 1, \dots, k$)

$$\sum_{i,j=1}^{n} V^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i,j,\ell} V^i X^\ell \Gamma^j_{i\ell} \frac{\partial}{\partial x^j},$$

and this expression is clearly independent of how $X$ and $V$ were extended. □

**Proposition 70.** *Suppose* $V, W \in \mathcal{X}(M)$, $X, Y \in \overline{\mathcal{X}}(M)$, *and* $f \in C^\infty(M)$.

*(i)* $\overline{\nabla}_{V+W}X = \overline{\nabla}_V X + \overline{\nabla}_W X$, $\overline{\nabla}_{fV}X = f\overline{\nabla}_V X$,
*(ii)* $\overline{\nabla}_V(X+Y) = \overline{\nabla}_V X + \overline{\nabla}_V Y$, $\overline{\nabla}_V(fX) = f\overline{\nabla}_V X + V(f)X$,
*(iii)* $\overline{\nabla}_V W - \overline{\nabla}_W V = [V, W] \in \mathcal{X}(M)$, *and*
*(iv)* $V\langle X, Y\rangle = \left\langle \overline{\nabla}_V X, Y\right\rangle + \left\langle X, \overline{\nabla}_V Y\right\rangle$.

*Proof.* All but (iii) follow immediately from properties of the Levi-Civita connection and the fact that the definition of $\overline{\nabla}$ is independent of extendion. The third follows from $M$ being a submanifold as well; if $\iota$ denotes the inclusion map, we know immediately that

$$\overline{\nabla}_V W - \overline{\nabla}_W V = [\overline{V}, \overline{W}],$$

where $\overline{V}$ and $\overline{W}$ are extensions of $V$ and $W$. In particular they are $\iota$-related to $V$ and $W$, so their Lie bracket is $\iota$-related to $[V, W]$ and thus is equal to $[V, W]$. □

8.2. **The second fundamental form.** We now want to compare our two covariant derivatives. Suppose $V, W \in \mathcal{X}(M)$ and define

$$\mathrm{II}(V, W) = \overline{\nabla}_V W - \nabla_V W \in \overline{\mathcal{X}}(M),$$

the *second fundamental form.* A quick calculation shows that $\mathrm{II}$ is bilinear over $C^\infty(M)$ and so is a $(1, 2)$-tensor.

**Proposition 71.** *Suppose* $V, W \in \mathcal{X}(M)$.

*(1)* $\mathrm{II}(V, W) \in \mathcal{X}^\perp(M)$ *(so* $\nabla_V W = \tan(\overline{\nabla}_V W)$ *and* $\mathrm{II}(V, W) = \mathrm{nor}(\overline{\nabla}_V W)$*), and*
*(2)* $\mathrm{II}(V, W) = \mathrm{II}(W, V)$.

*Proof.* The second one follows almost immediately from the previous proposition:

$$\mathrm{II}(V, W) - \mathrm{II}(W, V) = \overline{\nabla}_V W - \overline{\nabla}_W V - \nabla_V W + \nabla_W V = [V, W] - [V, W] = 0.$$

For the first, suppose that $V, W, X \in \mathcal{X}(M)$; it suffices to show that $\left\langle \overline{\nabla}_V W, X \right\rangle = \langle \nabla_V W, X\rangle$, as then $\langle \mathrm{I\!I}(V, W), X\rangle = 0$ (so $\mathrm{I\!I}(V, W)$ is orthogonal to all vectors tangent to $M$).

Extend $V, W, X$ to $\overline{V}, \overline{W}, \overline{X} \in \mathcal{X}(\overline{M})$. Recall our formula showing the existence and uniqueness of the Levi-Civita connection:

$$\left\langle \overline{\nabla}_{\overline{V}} \overline{W}, \overline{X} \right\rangle = \frac{1}{2} \left[ \overline{V} \left\langle \overline{W}, \overline{X} \right\rangle + \overline{W} \left\langle \overline{V}, \overline{X} \right\rangle - \overline{X V}, \overline{W} - \left\langle \overline{V}, [\overline{W}, \overline{X}] \right\rangle - \left\langle \overline{W}, [\overline{X}, \overline{V}] \right\rangle + \left\langle \overline{X}, [\overline{V}, \overline{W}] \right\rangle \right].$$

Observe that the restriction of each term to $M$ is the same expression without bars and therefore yield $\langle \nabla_V W, X\rangle$. (Strictly speaking, you need to do some symbol-pushing with the inclusion map here; the last three terms follow from the Lie bracket of $\iota$-related vector fields being $\iota$-related to the Lie bracket of the originals, while the first three follow almost by definition.) $\square$

We therefore have $\mathrm{I\!I} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}^{\perp}(M)$.

We introduce another piece of notation: for $V \in \mathcal{X}(M)$ and $X \in \mathcal{X}^{\perp}(M)$, we let

$$D_V^{\perp} X = \mathrm{nor}(\overline{\nabla}_V X).$$

**Theorem 72** (Gauss equation). *Let $V, W, X, Y \in \mathcal{X}(M)$ and let $R$ denote the Riemann tensor for $(M, g)$ and $\overline{R}$ denote the Riemann tensor for $(\overline{M}, \overline{g})$. $R$ and $\overline{R}$ are related via the second fundamental form:*

$$R(V, W, X, Y) = \overline{R}(V, W, X, Y) + \langle \mathrm{I\!I}(V, Y), \mathrm{I\!I}(W, X)\rangle - \langle \mathrm{I\!I}(V, X), \mathrm{I\!I}(W, Y)\rangle.$$

**Corollary 73.** *If $\Pi \subset T_p M$ is spanned by $v$ and $w$, then*

$$K(\Pi) = \overline{K}(\Pi) + \frac{\langle \mathrm{I\!I}(v, v), \mathrm{I\!I}(w, w)\rangle - |\mathrm{I\!I}(v, w)|^2}{|v|^2 |w|^2 - \langle v, w\rangle^2}.$$

*In particular, if $\dim M = 2$ and $\dim \overline{M} = 3$, then you get the determinant of the second fundamental form!*

*Proof of Gauss equation.* Observe that

$$\begin{aligned} \overline{\nabla}_V \overline{\nabla}_W X &= \overline{\nabla}_V \left( \nabla_W X + \mathrm{I\!I}(W, X) \right) \\ &= \nabla_V \nabla_W X + \mathrm{I\!I}\left( V, \overline{\nabla}_W X \right) + \overline{\nabla}_V \left( \mathrm{I\!I}(W, X) \right). \end{aligned}$$

Using this identity twice yields

$$\begin{aligned} \overline{R}(V, W)X &= \overline{\nabla}_V \overline{\nabla}_W X - \overline{\nabla}_W \overline{\nabla}_V X - \overline{\nabla}_{[V,W]} X \\ &= \nabla_V \nabla_W X - \nabla_W \nabla_V X - \nabla_{[V,W]} X \\ &\quad + \mathrm{I\!I}\left( V, \nabla_W X \right) - \mathrm{I\!I}\left( W, \nabla_V X \right) - \mathrm{I\!I}\left( [V, W], X \right) \\ &\quad + \overline{\nabla}_V \left( \mathrm{I\!I}(W, X) \right) - \overline{\nabla}_W \left( \mathrm{I\!I}(V, X) \right) \\ &= R(V, W)X + \mathrm{I\!I}(V, \nabla_W X) - \mathrm{I\!I}(W, \nabla_V X) - \mathrm{I\!I}([V, W], X) \qquad (4) \\ &\quad + \overline{\nabla}_V \left( \mathrm{I\!I}(W, X) \right) - \overline{\nabla}_W \left( \mathrm{I\!I}(V, X) \right). \end{aligned}$$

Now we take the inner product with $Y \in \mathcal{X}(M)$; since $\mathrm{I\!I} \in \mathcal{X}^{\perp}(M)$, the second, third, and fourth terms vanish. We then then use the compatibility of $\overline{\nabla}$ with the metric on $\overline{M}$ (and the fact that $\mathrm{I\!I}$ and $Y$ are orthogonal):

$$\begin{aligned}\overline{R}(V,W,X,Y) &= R(V,W,X,Y) + \left\langle \overline{\nabla}_V\left(\mathrm{I\!I}(W,X)\right), Y\right\rangle - \left\langle \overline{\nabla}_W\left(\mathrm{I\!I}(V,X)\right), Y\right\rangle \\ &= R(V,W,X,Y) - \left\langle \mathrm{I\!I}(W,X), \overline{\nabla}_V Y\right\rangle + \left\langle \mathrm{I\!I}(V,X), \overline{\nabla}_W Y\right\rangle \\ &= R(V,W,X,Y) + \langle \mathrm{I\!I}(V,X), \mathrm{I\!I}(W,Y)\rangle - \langle \mathrm{I\!I}(W,X), \mathrm{I\!I}(V,Y)\rangle,\end{aligned}$$

with the last equality holding because $\mathrm{I\!I}(V,Y)$ is the normal part of $\overline{\nabla}_V Y$. □

To state the Codazzi equation, we need to discuss what we mean by the covariant derivative of the second fundamental form. More precisely, we regard $\mathrm{I\!I}$ as a a section of the vector bundle

$$T^*M \otimes T^*M \otimes (TM)^\perp,$$

equipped with the connection

$$\nabla^\perp = \nabla^{T^*M} \otimes \nabla^{T^*M} \otimes D^\perp.$$

In particular, for $V, W, X \in \mathcal{X}(M)$, we have

$$(\nabla^\perp_V \mathrm{I\!I})(W,X) = D^\perp_V\left(\mathrm{I\!I}(W,X)\right) - \mathrm{I\!I}\left(\nabla_V W, X\right) - \mathrm{I\!I}\left(W, \nabla_V X\right).$$

**Theorem 74** (Codazzi equation)**.** *For $V, W, X \in \mathcal{X}(M)$, the second fundamental form satisfies*

$$(\nabla^\perp_V \mathrm{I\!I})(W,X) - (\nabla^\perp_W \mathrm{I\!I})(V,X) = \operatorname{nor}\left(\overline{R}(V,W)X\right).$$

*Proof.* Take the normal components of the identity (4) used in the proof of the Gauss equation to obtain

$$\begin{aligned}\operatorname{nor}\left(\overline{R}(V,W)X\right) &= 0 + \mathrm{I\!I}\left(V, \nabla_W X\right) - \mathrm{I\!I}\left(W, \nabla_V X\right) - \mathrm{I\!I}\left([V,W], X\right) \\ &\quad + D^\perp_V\left(\mathrm{I\!I}(W,X)\right) - D^\perp_W\left(\mathrm{I\!I}(V,X)\right) \\ &= \mathrm{I\!I}\left(V, \nabla_W X\right) - \mathrm{I\!I}\left(W, \nabla_V X\right) - \mathrm{I\!I}\left([V,W], X\right) \\ &\quad + \left(\nabla^\perp_V \mathrm{I\!I}\right)(W,X) + \mathrm{I\!I}\left(\nabla_V W, X\right) + \mathrm{I\!I}\left(W, \nabla_V X\right) \\ &\quad - \left(\nabla^\perp_W \mathrm{I\!I}\right)(V,X) - \mathrm{I\!I}\left(\nabla_W V, X\right) - \mathrm{I\!I}\left(V, \nabla_W X\right) \\ &= \left(\nabla^\perp_V \mathrm{I\!I}\right)(W,X) - \left(\nabla^\perp_W \mathrm{I\!I}\right)(V,X) + \mathrm{I\!I}\left(\nabla_V W - \nabla_W V - [V,W], X\right),\end{aligned}$$

which finishes the proof because the Levi-Civita connection is torsion-free. □

8.3. **Example curvature computations.** We consider a few main examples.

8.3.1. *The round sphere.* Consider the sphere $\mathbb{S}^n_R$ of radius $R$ in $\mathbb{R}^{n+1}$ and $n \geq 2$. Note that at $x \in \mathbb{R}^{n+1}$, $|x| = R$, the outward pointing unit normal to $\mathbb{S}^n_R$ is

$$\nu = \frac{1}{R}x,$$

where $x$ is also regarded as the position vector. For a vector $v \in T_xM$, note that the $j$-th component of the directional derivative of the position vector is

$$\left(\overline{\nabla}_v x\right)^j = \sum v^i \frac{\partial}{\partial x^i} x^j = v^j,$$

i.e., $\overline{\nabla}_v x = v$. We compute

$$\begin{aligned}\langle \mathrm{I\!I}(V,W), \nu\rangle &= \left\langle \operatorname{nor}\left(\overline{\nabla}_V W\right), \nu\right\rangle = \left\langle \overline{\nabla}_V W, \nu\right\rangle \\ &= V\langle W, \nu\rangle - \left\langle W, \overline{\nabla}_V \nu\right\rangle = -\frac{1}{R}\left\langle W, \overline{\nabla}_V x\right\rangle = -\frac{1}{R}\langle W, V\rangle .\end{aligned}$$

We now use the Gauss equation and the fact that $\mathbb{R}^{n+1}$ is flat (i.e., $\overline{R} = 0$) to see that

$$\begin{aligned}R(V,W,X,Y) &= \langle \mathrm{I\!I}(V,Y), \mathrm{I\!I}(W,X)\rangle - \langle \mathrm{I\!I}(V,X), \mathrm{I\!I}(W,Y)\rangle \\ &= \frac{1}{R^2}\langle V,Y\rangle\langle W,X\rangle - \frac{1}{R^2}\langle V,X\rangle\langle W,Y\rangle ,\end{aligned}$$

i.e., all sectional curvatures of $\mathbb{S}^n_R$ are equal to $1/R^2$.

8.3.2. *Level sets of smooth functions.* Suppose $(\overline{M}, \overline{g})$ is a Riemannian manifold, $f \in C^\infty(\overline{M})$, and 0 is a regular value of $f$ (i.e., $(Df)_x$ is of maximal rank for all $x \in f^{-1}(0)$), so that $f^{-1}(0)$ is a smooth submanifold of $M$ (of codimension one).

Note that $\operatorname{grad} f$ is orthogonal to $T_pM$, so $\mathrm{I\!I}$ is a multiple of $\operatorname{grad} f$. We then compute

$$\begin{aligned}\langle \mathrm{I\!I}(V,W), \operatorname{grad} f\rangle &= \left\langle \overline{\nabla}_V W, \operatorname{grad} f\right\rangle = V\langle W, \operatorname{grad} f\rangle - \left\langle W, \overline{\nabla}_V \operatorname{grad} f\right\rangle \\ &= -\left(\operatorname{Hess} f\right)(V,W),\end{aligned}$$

where the Hessian of $f$ is defined by

$$\operatorname{Hess} f(V,W) = \left\langle W, \overline{\nabla}_V \nabla f\right\rangle .$$

(Note that the Hessian is a $(0,2)$-tensor.)

We then have that

$$\mathrm{I\!I}(V,W) = -\frac{1}{|\operatorname{grad} f|^2}\operatorname{Hess} f(V,W)\operatorname{grad} f.$$

8.3.3. *Graphs in* $\mathbb{R}^{n+1}$. Suppose $u : \Omega \to \mathbb{R}$ is a smooth function, where $\Omega \subset \mathbb{R}^n$ is open, and consider its graph

$$M = \left\{(x^1, \dots, x^n, u(x^1, \dots, x^n)) \mid (x^1, \dots, x^n) \in \mathbb{R}^n\right\} \subset \mathbb{R}^{n+1}.$$

For convenience we also define the functions $F : \Omega \to \mathbb{R}^{n+1}$ by

$$F(x^1, \dots, x^n) = (x^1, \dots, x^n, u(x^1, \dots, x^n)),$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f(x^1, \dots, x^n, x^{n+1}) = u(x^1, \dots, x^n) - x^{n+1}.$$

Observe that 0 is a regular value of $f$ and that $M = f^{-1}(0)$; $M$ is also the image of $F$.

At each point the tangent space of $M$ is spanned by $v_i = F_* e_i$, where $e_i$ are the standard basis vectors in $\mathbb{R}^n$. Concretely, we have

$$v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \partial_i u \end{pmatrix} = \begin{pmatrix} e_i \\ \partial_i u \end{pmatrix},$$

where $e_i$ is the $i$th standard basis vector in $\mathbb{R}^n$.

The normal vector to $M$ is given by

$$\nu = \frac{1}{|\nabla f|}\nabla f,$$

where we have reverted to using $\nabla$ instead of grad for the gradient because we are working in $\mathbb{R}^{n+1}$ instead of a general manifold. By the previous subsection, we know that

$$\mathrm{II}(v_i, v_j) = -\frac{1}{|\nabla f|^2}\,\mathrm{Hess}(v_i, v_j)\nabla f = \frac{-1}{\sqrt{1+|\nabla u|^2}}\frac{\partial^2 u}{\partial x^i \partial x^j}\nu.$$

(The induced metric is $\delta_{ij} + \partial_i u \partial_j u$; after we define the mean curvature vector in the following section you can compute that we end up with the famous *minimal surface equation* here after taking the trace.)

8.3.4. *Hyperbolic space.* Consider now the Minkowski space $\mathbb{R}^{1,n}$ equipped with linear coordinates $(t, x^1, \dots, x^n)$ and the Lorentzian (pseudo-Riemannian of signature $(1, n)$) metric

$$-dt^2 + dx \cdot dx.$$

Let $M$ be one sheet of the two-sheeted hyperboloid, i.e.,

$$M = \left\{-t^2 + |x|^2 = -R^2 \mid t > 0\right\}.$$

Observe that the metric $g$ induced by the Minkowski metric is in fact Riemannian (though the Minkowski metric is not). Indeed, we can find an explicit parametrization of $M$ by $F : \mathbb{R}^n \to M$,

$$F(x) = \left(\sqrt{|x|^2 + R^2}, x\right).$$

Given $p = (t, x) \in M$ and writing $F : \mathbb{R}^n \to M$, the tangent space is given by

$$T_pM = (DF)_p(\mathbb{R}^n) = \left\{\begin{pmatrix} v^t \\ v^x \end{pmatrix} \mid v^t = \frac{1}{t} x \cdot v^x\right\}.$$

On such a vector, we have

$$g(v, v) = -(v^t)^2 + v^x \cdot v^x = -\frac{(x \cdot v^x)^2}{t^2} + v^x \cdot v^x \geq -\frac{|x|^2 |v^x|^2}{t^2} + |v^x|^2.$$

As $|x|^2 = t^2 - R^2$, we have $|x|^2 / t^2 < 1$, so $g(v, v)$ is strictly positive for $v \neq 0$.

One vector normal to $M$ is given by the position vector

$$\begin{pmatrix} t \\ x \end{pmatrix},$$

which has inner product $-R^2$ with itself. In particular, the "unit" normal (normalized to have inner product $-1$ with itself) is given by

$$\nu = \frac{1}{R}\begin{pmatrix} t \\ x \end{pmatrix}.$$

By the same argument as with a sphere, we have that

$$\langle \mathrm{II}(V, W), \nu\rangle = -\left\langle W, \overline{\nabla}_V \nu\right\rangle = -\frac{1}{R}\langle W, V\rangle.$$

Now by the Gauss equation, we have

$$\begin{aligned} R(V,W,X,Y) &= \langle \mathrm{I\!I}(V,Y), \mathrm{I\!I}(W,X)\rangle - \langle \mathrm{I\!I}(V,X), \mathrm{I\!I}(W,Y)\rangle \\ &= \frac{1}{R^2}\langle V,Y\rangle \langle W,X\rangle \langle \nu,\nu\rangle - \frac{1}{R^2}\langle V,X\rangle\langle W,Y\rangle\langle \nu,\nu\rangle \\ &= \frac{-1}{R^2}\left(\langle V,Y\rangle\langle W,X\rangle - \langle V,X\rangle\langle W,Y\rangle\right), \end{aligned}$$

i.e., the sectional curvatures of $M$ are all $-1/R^2$. (Exercise: Check that this copy of hyperbolic space is isometric to the others you've seen so far.)

8.4. **Parallel transport and geodesics.** As before, assume $M \subset \overline{M}$ is a submanifold, $\overline{g}$ is a metric on $\overline{M}$, and $g$ is the induced metric on $M$.

Suppose $\alpha : I \to M$ is a curve and $Y \in \mathcal{X}(\alpha)$ is tangent to $M$. Using bars as before to denote the extrinsic quantities and a lack of bars to denote the intrinsic ones, we have

$$\frac{D}{dt}Y(t) = \tan\left(\frac{\overline{D}}{dt}Y(t)\right),$$

so that $Y$ is parallel along $\alpha$ in $M$ if and only if $\frac{\overline{D}}{dt}Y(t) \perp T_{\alpha(t)}M$ for all $t$.

By the discussion within the second fundamental form section (and our definition of the covariant derivative along a curve), we have

$$\frac{\overline{D}}{dt}Y(t) = \frac{D}{dt}Y(t) + \mathrm{I\!I}(Y(t), \alpha'(t)),$$

so in particular, $\alpha$ is a geodesic in $M$ if and only if $\alpha'(t)$ is parallel in $M$ if and only if $\frac{\overline{D}}{dt}\alpha'(t) \perp T_{\alpha(t)}M$.

**Proposition 75.** *The following are equivalent:*

*(i) Geodesics in $M$ are also geodesics in $\overline{M}$,*
*(ii) $\mathrm{I\!I} = 0$, and*
*(iii) If $\alpha$ is a curve in $M$ and $v \in T_{\alpha(0)}M$, then the parallel transport of $v$ along $\alpha$ in $M$ agrees with the parallel transport of $v$ along $\alpha$ in $\overline{M}$.*

*Proof.* The third implies the first because geodesics are characterized by the fact that they parallel transport their tangent vectors. The second implies the third by the formula relating covariant derivatives in $M$ and $\overline{M}$.

For the remaining implication, take $p \in M$, $v \in T_pM$ and take a geodesic $\alpha$ in $M$ with $\alpha(0) = p$, $\alpha'(0) = v$. Since $\alpha$ is a geodesic in $M$, it must be a geodesic in $\overline{M}$ (by hypothesis), so that

$$\frac{\overline{D}}{dt}\alpha'(t) = \frac{D}{dt}\alpha'(t) = 0,$$

and therefore $\mathrm{I\!I}(\alpha'(t), \alpha'(t)) = 0$, i.e., $\mathrm{I\!I}(v,v) = 0$ for all vectors $v \in T_pM$. By our work last semester,[12] $\mathrm{I\!I}$ must be skew-symmetric. As we already knew $\mathrm{I\!I}$ was symmetric, we conclude $\mathrm{I\!I} = 0$. ∎

**Definition 76.** If any of these conditions holds, then $M$ is totally geodesic.

Now let's restrict our attention to the setting where $M$ is a hypersurface, so $\dim M = n$ and $\dim \overline{M} = n+1$.

[12] It follows from the identity $\mathrm{I\!I}(v+w, v+w) = \mathrm{I\!I}(v,v) + \mathrm{I\!I}(v,w) + \mathrm{I\!I}(w,v) + \mathrm{I\!I}(w,w)$.

**Definition 77.** $M$ is totally umbilic if $\mathrm{II}(V,W) = \langle V, W\rangle \left(\frac{1}{n}H\right)$, where $H$ is the trace of $\mathrm{II}$, i.e.,

$$H = \sum_{i,j=1}^{n} g^{ij}\mathrm{II}(v_i, v_j).$$

Let's look at the situation when $\overline{M} = \mathbb{R}^{n+1}$ and $\overline{g}$ is the Euclidean metric. Assume $M \subset \mathbb{R}^{n+1}$ is a connected hypersurface and $n \geq 2$. Let $\nu$ denote a choice of unit normal for $M$.

**Proposition 78.** *$M$ is totally umbilic if and only if there is some $\lambda \in \mathbb{R}$ with $\mathrm{II}(V,W) = \lambda \langle V, W\rangle \nu$ if and only if $M$ is a piece of a sphere or a plane.*

*Proof.* First note that $\overline{R} = 0$, so the Codazzi equation implies that

$$(\nabla^{\perp}_V \mathrm{II})(W, X) = (\nabla^{\perp}_W \mathrm{II})(V, X).$$

We've already used before that the covariant derivative commutes with the trace, so, setting $h(V,W) = \langle \mathrm{II}(V,W), \nu\rangle$ and $\tau = \sum_{i,j} g^{ij}h_{ij}$ the trace of $h$, we have, for any $W$

$$W(\tau) = \sum_{i,j} g^{ij}\left(\nabla_W h\right)(v_i, v_j),$$

which is then equal to

$$\sum_{i,j} g^{ij}\left(\nabla_{v_j} h\right)(v_i, W)$$

by the consequence of the Codazzi equation. Note also that $H = \tau\nu$.

We turn to the proof. If $M$ is totally umbilic, then $h(V,W) = \frac{1}{n}\langle V, W\rangle \tau$. We apply this identity with $V = v_i$, take the derivative in $v_j$, and then contract with the metric to get

$$n\sum_{i,j} g^{ij}\left(\nabla_{v_j} h\right)(v_i, W) = \sum_{i,j} g^{ij} v_j(\tau)\langle v_j, W\rangle = W(\tau),$$

but our earlier observation shows that the left side is also $nW(\tau)$. As $n \geq 2$, we conclude $W(\tau) = 0$. As $W$ was arbitrary, $\tau$ is lcoally constant, so $\tau = n\lambda$, i.e., $h = \lambda g$ and $\mathrm{II} = \lambda g \nu$.

We treat the next part in two cases. First suppose $\lambda = 0$, so $\mathrm{II}(V,W) = 0$, so

$$0 = \left\langle \overline{\nabla}_V W, \nu\right\rangle = V\langle W, \nu\rangle - \left\langle W, \overline{\nabla}_V \nu\right\rangle = -\left\langle W, \overline{\nabla}_V\nu\right\rangle.$$

In particular, $\overline{\nabla}_V \nu$ is orthogonal to $M$. As it is a unit normal vector, none of its derivatives have normal components and hence $\overline{\nabla}_V\nu = 0$. This is true for all $V$ tangent to $M$ and so $\nu$ is constant, i.e., $M$ is part of a hyperplane.

If $\lambda \neq 0$, the same argument shows that

$$\left\langle \overline{\nabla}_V \nu, W\right\rangle = -\lambda\langle V, W\rangle.$$

Recalling that the positiion vector $x$ satisfies $\overline{\nabla}_V x = V$ for all $V$, we then have

$$\left\langle \overline{\nabla}_V(\nu + \lambda x), W\right\rangle = 0$$

for all vectors $V, W$ tangent to $M$. Varying $W$ shows that $\overline{\nabla}_V(\nu + \lambda x)$ is orthogonal to $T_xM$; again $\overline{\nabla}_V\nu$ has no normal component and $\overline{\nabla}_V x = V$ also has no normal component so indeed $\overline{\nabla}_V(\nu + \lambda x) = 0$ for all vectors $V$ tangent to $M$. In particular, there is a constant $a \in \mathbb{R}^{n+1}$

so that $\nu + \lambda x = a$ for all $x \in M$ (as $M$ is connected). As $\nu$ is a unit vector, this implies that

$$\left| x - \frac{a}{\lambda} \right| = \frac{1}{|\lambda|},$$

for all $x \in M$, i.e., $M$ is a piece of a sphere.

The final implication is an exercise (and indeed we did it above!): you compute the second fundamental form for a plane ($\mathrm{I\!I} = 0$) or a sphere (done above) to see that it has the desired form. $\square$

## Appendix A. Smooth dependence on parameters

Last semester we had a more-difficult-than-necessary discussion of how solutions of ODEs depend smoothly on parameters.