EQUIPARTITION OF ENERGY IN GEOMETRIC SCATTERING THEORY

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Abstract. In this note, we use an elementary argument to show that the existence and unitarity of radiation fields implies asymptotic partition of energy for the corresponding wave equation. This argument establishes the equipartition of energy for the wave equation on scattering manifolds, asymptotically hyperbolic manifolds, asymptotically complex hyperbolic manifolds, and the Schwarzschild spacetime. It also establishes equipartition of energy for the energy-critical semilinear wave equation on $\mathbb{R}^3$.

1. Introduction

In this note, we use the radiation fields of Friedlander [Fri80], Sá Barreto [SB05, SB05], and Guillarmou and Sá Barreto [GSB08] to demonstrate the equipartition of energy in the context of geometric scattering theory. Our method also provides an alternate proof for equipartition of energy for the energy-critical semilinear wave equation on $\mathbb{R}^3$ and a proof of the equipartition of energy on the Schwarzschild exterior.

Asymptotic equipartition of energy for the wave equation on $\mathbb{R}^n$ was first observed by Brodsky [Bro67], whose proof relied on the Fourier transform. Using a Paley–Wiener theorem, Duffin [Duf70] showed equipartition of energy after a finite time for the wave equation on $\mathbb{R}^3$. The intervening decades have seen many proofs of equipartition of energy for equations of mathematical physics in many contexts. As there are too many to provide a comprehensive list, we list a few examples: Dassios and Grillakis [DG83] showed asymptotic equipartition for the wave equation on an exterior domain, Vega and Visciglia [VV08] proved an integrated version of the statement for a class of critical semilinear wave equations on $\mathbb{R}^n$, and Gang and Weinstein [GW11] proved an equipartition of mass theorem for nonlinear Schrödinger and Gross–Pitaevskii equations (though the methods discussed in this note do not apply in the Schrödinger setting).

In the model setting, we consider an initial value problem of the form

$$\begin{gathered} \left(D_t^2 - H\right) u = 0 \\ (u(0,z), \partial_t u(0,z)) = (u_0(z), u_1(z)). \end{gathered} \tag{1}$$

Here $D = \frac{1}{i}\partial$ and H is a time-independent Hamiltonian so that the initial value problem (1) has a conserved energy E. We suppose that the energy splits (in a time-dependent manner) into kinetic and potential energies $E_K(t)$ and $E_P(t)$, and

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that this decomposition extends to the level of energy densities e_K and e_P. For the wave equation on $\mathbb{R}^n$, these energy densities are given by $e_K = \frac{1}{2}|\partial_t u|^2$ and $e_P = \frac{1}{2}|\nabla u|^2$ and are given explicitly below in other contexts.

In a more general Lorentzian setting, this splitting (and, indeed, conservation of energy) does not occur. In the example below of the Schwarzschild spacetime from general relativity, the splitting is possible because the spacetime is *static*, i.e., ∂_t is a Killing vector field orthogonal to the Cauchy hypersurface.

We consider the following geometric contexts (in settings (1)–(3) below, $n = \dim X$):

(1) Scattering manifolds (X, g) (in the sense of Melrose [Mel94]), with $H = \Delta_X$, the Laplacian with positive spectrum, with
$$e_K = |\partial_t u|^2 , \quad e_P = |\nabla_g u|_g^2 ,$$
(2) Asymptotically hyperbolic manifolds (X, g) (in the sense of Mazzeo–Melrose [MM87]), with $H = \Delta_X - \frac{(n-1)^2}{4}$ restricted to the orthocomplement of the eigenfunctions in the pure point spectrum, with
$$e_K = |\partial_t u|^2 , \quad e_P = |\nabla_g u|^2 - \frac{(n-1)^2}{4}|u|^2,$$
(3) asymptotically complex hyperbolic manifolds (X, g) (in the sense of Epstein–Melrose–Mendoza [EMM91]), with $H = \Delta_X - \frac{n^2}{4}$ and again restricted to the orthocomplement of the eigenfunctions in the pure point spectrum, with
$$e_K = \frac{1}{2}|\partial_t u|^2 , \quad e_P = \frac{1}{2}|\nabla_g u|^2 - \frac{n^2}{4}|u|^2 ,$$
(4) the Schwarzschild spacetime (here the Killing vector field ∂_t provides the splitting as well as conservation of energy), with
$$e_K = \frac{1}{2}\left(1 - \frac{2M}{r}\right)^{-1} |\partial_t u|^2 , \quad e_P = \frac{1}{2}\left(1 - \frac{2M}{r}\right)|\partial_r u|^2 + \frac{1}{2r}^2 |\nabla_\omega u|^2 ,$$
(5) and the critical semilinear wave equation on $\mathbb{R}^3$, with H a nonlinear Hamiltonian, and
$$e_K = \frac{1}{2}|\partial_t u|^2 , \quad e_P = \frac{1}{2}|\nabla u|^2 + \frac{1}{6}|u|^6$$

The main result of this note is the following theorem:

Theorem 1. *If u is a finite energy solution of the initial value problem* (1) *in one of the contexts above, then u exhibits asymptotic equipartition of energy, i.e.,*
$$\lim_{t\to\infty} E_K(t) - E_P(t) = 0.$$
Here $E_K(t)$ and $E_P(t)$ are given by integrals of the kinetic and potential energy densities, respectively.

One feature that the contexts above share is the existence and unitarity of radiation fields. Radiation fields are restrictions of (rescaled) solutions of wave equations to null infinity and have interpretations both as Lax–Phillips translation representations of wave equations and as generalizations of the Radon transform. Heuristically, the statement that the radiation field is a unitary operator is a statement that all energy radiates to null infinity, i.e., a form of non-quantitative local energy decay.

We start by considering the model setting of the wave equation on $\mathbb{R}^n$:

$$\Box u = 0$$
$$(u, \partial_t u)|_{t=0} = (\phi, \psi).$$

Suppose for now that ϕ and ψ are smooth and compactly supported. For $x \in (0, \infty)$, $\theta \in \mathbb{S}^{n-1}$, and $s \in (-\infty, \infty)$, let us define a new function v_+ by

$$v_+(x, s, \theta) = x^{-\frac{n-1}{2}} u\left(s + \frac{1}{x}, \frac{1}{x}\theta\right).$$

(In other words, $x = |z|^{-1}$ and $s = t - |z|$.) A relatively straightforward calculation shows that v_+ is smooth past $x = 0$ and so we can define the *forward radiation field* $\mathcal{R}_+$ by

$$\mathcal{R}_+(\phi, \psi)(s, \theta) = \partial_s v_+(0, s, \theta).$$

Friedlander [Fri80] observed that this radiation field is a translation representation of the wave group, i.e., it is a unitary map $\dot{H}^1 \times L^2 \to L^2(\mathbb{R} \times \mathbb{S}^{n-1})$ that intertwines wave evolution with translation. Moreover, for the flat wave equation, the radiation field can be written in terms of the Radon transform. For $X = \mathbb{R}^3$, the relationship is given by

$$\mathcal{R}_+(\phi, \psi)(s, \theta) = -\frac{1}{4\pi}\left(R\psi(s, \theta) + \partial_s\left(sR\phi\right)(s, \theta)\right),$$

where the Radon transform R is given by

$$Rf(s, \theta) = \int_{\langle z, \theta\rangle = s} f(z)\, d\sigma(z).$$

The existence and unitary of the radiation field have been established for many other settings in geometric scattering theory, including the settings enumerated above. They were first observed by Friedlander [Fri80, Fri01] in Euclidean and asymptotically Euclidean spaces. Sá Barreto [SB05, SB03, SB08] proved support theorems and unitarity for them on asymptotically hyperbolic and asymptotically Euclidean manifolds, while Guillarmou and Sá Barreto [GSB08] extended these results to asymptotically complex hyperbolic manifolds. Sá Barreto and Wunsch [SBW05] showed that the radiation field is a Fourier integral operator with canonical relation given by a "sojourn relation." In a nonlinear context, Sá Barreto and the author [BSB12] proved a support theorem for the radiation field for the semilinear wave equation on $\mathbb{R}^3$ and showed it is norm-preserving, while Wang [Wan11] studied the mapping properties of the radiation field for the Einstein vacuum equations on perturbations of Minkowski space. In recent work, Wang and the author [BW] showed the existence and unitarity of radiation fields on the Schwarzschild exterior.

In Section 2 we discuss the motivating case of the one-dimensional wave equation. In Section 3 we prove a general proposition implying the equipartition of energy. In the remaining sections, we summarize known results for the radiation field in various contexts and check that they satisfy the conditions of Section 3.

In what follows, dg denotes the volume form of the relevant metric g.

2. The one-dimensional wave equation

In this section we discuss the illuminating example of the radiation field for the one-dimensional wave equation and discuss its connection with equipartition of energy.

Consider now the one-dimensional wave equation:

$$\begin{aligned} &\left(D_t^2 - D_x^2\right) u = 0, \\ &(u, \partial_t u) = (\phi, \psi) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R}). \end{aligned} \tag{2}$$

The solution of equation (2) is given in terms of left- and right-moving waves:

$$\begin{aligned} u(t,x) &= F(x+t) + G(x-t) \\ F(s) &= \frac{1}{2}\phi(s) + \frac{1}{2}\int_0^s \psi(r)\,dr + C \\ G(s) &= \frac{1}{2}\phi(s) + \frac{1}{2}\int_s^0 \psi(r)\,dr - C \end{aligned}$$

The zero-dimensional sphere $\mathbb{S}^0$ consists of two points, which we identify as "left" (-1) and "right" $(+1)$. The forward radiation field associated to the initial data (ϕ, ψ) is given by

$$\mathcal{R}_+(\phi,\psi)(s,\theta) = \lim_{r\to\infty} \partial_s u(s+r, r\theta) = \begin{cases} F'(s) & \theta = -1 \\ G'(s) & \theta = +1 \end{cases}$$

In other words, the "left" component of the forward radiation field is the left-moving wave form F, while the "right" component is the right-moving wave form G. In particular, we have that

$$\mathcal{R}_+(\phi,\psi)(s,\theta) = \begin{cases} \frac{1}{2}\phi'(s) + \frac{1}{2}\psi(s) & \theta = -1 \\ \frac{1}{2}\phi'(s) - \frac{1}{2}\psi(s) & \theta = +1 \end{cases}$$

We calculate here the L^2 norm of $\mathcal{R}_\pm$ and its relationship to the energy of u:

$$\begin{aligned} \|\mathcal{R}_+(\phi,\psi)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^0)} &= \int_{\mathbb{R}} \left[\frac{1}{4}\left(\phi'(s)+\psi(s)\right)^2 + \frac{1}{4}\left(\phi'(s)-\psi(s)\right)^2\right] \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(|\phi'(s)|^2 + |\psi(s)|^2\right) ds = E(\phi,\psi) \end{aligned}$$

It is well-known that the one-dimensional wave equation obeys equipartition of energy for compactly supported smooth initial data. Indeed, suppose that both ϕ and ψ are smooth and supported in the ball of radius R, so that $F(s)$ and $G(s)$ are both constant for $|s| \geq R$. In particular, $G'(s) = F'(s) = 0$ for $|s| \geq R$. We then compute

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}} \left(|\partial_t u(t,x)|^2 - |\partial_x u(t,x)|^2\right) dx \\ &\quad= \frac{1}{2}\int_{\mathbb{R}} \left(|F'(x+t) - G'(x-t)|^2 - |F'(x+t) + G'(x-t)|^2\right) dx \\ &\quad= -2\int_{\mathbb{R}} F'(x+t)G'(x-t)\,dx \end{aligned}$$

For $t \geq R$ and $x \in \mathbb{R}$, we have that either $|t+x| \geq R$ or $|t-x| \geq R$ and therefore one of the two factors in the integral vanishes, so u obeys equipartition of energy.

In fact, a consequence of the theorem in Section 3 is the asymptotic equipartition of energy for the one-dimensional wave equation. (This can also be seen more directly by taking limits in $\dot{H}^1 \times L^2$ above.)

3. Equipartition of energy

Suppose now that s is a smooth function of t and x and define the kinetic and potential parts of the energy for fixed t:

$$E_K(\lambda, t) = \int_{s(t,x)\leq\lambda} e_K \, dg,$$
$$E_P(\lambda, t) = \int_{s(t,x)\leq\lambda} e_P \, dg.$$

We note that the densities $E_\bullet(t)$ described in the introduction are given by $E_\bullet(t) = E_\bullet(\infty, t)$, for $\bullet = K, P$.

The results of this paper all follow from the following elementary proposition.

Proposition 2. *Suppose that u satisfies equation (1) on $\mathbb{R} \times X$ with initial data (ϕ, ψ) and there is a function $F \in L^2(\mathbb{R})$ satisfying the following two conditions:*

$$\lim_{T\to\infty} E_K(\lambda, T) = \lim_{T\to\infty} E_P(\lambda, T) = \frac{1}{2}\int_{-\infty}^{\lambda} |F|^2 \, ds, \tag{3}$$

for all λ, and

$$\int_X (e_K + e_P) \, dg = \|F\|^2_{L^2(\mathbb{R})} . \tag{4}$$

Then u exhibits asymptotic equipartition of energy, i.e.,

$$\lim_{T\to\infty} (E_K(T) - E_P(T)) = 0.$$

Proof. Fix $\epsilon > 0$ and take λ_0 such that

$$\int_{\infty}^{\lambda_0} |F|^2 \, ds \geq \|F\|^2_{L^2(\mathbb{R})} - \epsilon.$$

Condition (3) allows us to take T_0 so that $E_K(\lambda_0, T)$ and $E_P(\lambda_0, T)$ are within ϵ of $\frac{1}{2}\int_\infty^{\lambda_0} |F|^2$ for all $T \geq T_0$.

Let $E(t) = \int_X (e_K(t) + e_P(t)) \, dg$. By conservation of energy, $E(t) = E(0)$ for all t. Condition (4) implies that $E(t) = \|F\|^2_{L^2(\mathbb{R})}$ for all t. We now estimate

$$\begin{aligned}\int_{s(T,x)>\lambda_0} (e_K(t) + e_P(t)) \, dg &= E(T) - E_K(\lambda_0, T) - E_P(\lambda_0, T) \\ &= \|F\|^2_{L^2(\mathbb{R})} - E_K(\lambda_0, T) - E_P(\lambda_0, T) \\ &\leq 3\epsilon\end{aligned}$$

for all $T \geq T_0$. Our choice of T_0 implies that $|E_K(\lambda_0, T) - E_P(\lambda_0, T)| \leq 2\epsilon$ for all $T \geq T_0$. Putting the two estimates together yields that $|E_K(T) - E_P(T)| \leq 5\epsilon$ for all $T \geq T_0$, proving the claim. □

Remark 1. The proof of Proposition 2 is essentially contained in the work of Friedlander [Fri80], who observed that such solutions exhibit a non-quantitative form of local energy decay. Indeed, if λ and T_0 are large enough, then

$$\int_{s(T,x)>\lambda} (e_K(t) + e_P(t)) \, dg \leq 3\epsilon.$$

In particular, after waiting long enough, solutions decay in a forward light cone. Moreover, the argument shows that quantitative decay rates for the function F (i.e., for the radiation field below) yield quantitative local energy decay statements.

4. Applications

In this section, we establish conditions (3) and (4) for radiation fields in various geometric settings. In what follows, X is always a compact manifold with boundary Y and x is always a boundary defining function, i.e., $x = 0$ at $Y = \partial X$ and $dx \neq 0$ at Y.

4.1. **Scattering manifolds.**

Definition 1. A metric g on X is a scattering metric if it is a Riemannian metric on the interior of X and, in a collar neighborhood $[0, \epsilon)_x \times Y_y$ of the boundary, g has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, \, dy)}{x^2},$$

where $h(x, y, \, dy)$ is a smoothly varying (in x) family of Riemannian metrics on Y.

Scattering metrics are a class of asymptotically conic metrics introduced by Melrose [Mel94]. In the case where $Y = \mathbb{S}^{n-1}$ and $h(0, y, \, dy)$ is the round metric on the sphere, they are a class of asymptotically Euclidean metric (in this case $x = r^{-1}$).

Friedlander [Fri80, Fri01] showed that the radiation fields for solutions of the wave equation on scattering manifolds exist and are unitary on the orthogonal complement of the kernel of the radiation field. Sá Barreto [SB03] later proved a support theorem for the radiation field and showed that Friedlander's unitarity theorem could be seen as a consequence of work of Hassell and Vasy [HV99].

Let u be a solution of equation (1) on a scattering manifold and let $s_{\pm}(t, x) = t \mp \frac{1}{x}$. The rescaling v, given by $v_{\pm}(x, s, y) = x^{-\frac{n-1}{2}} u(s \pm \frac{1}{x}, x, y)$ is smooth down to $x = 0$ and the radiation field is defined by

$$\mathcal{R}_{\pm}(\phi, \psi)(s, y) = \partial_s v_{\pm}(0, s, y).$$

Consider now, for fixed t,

$$E_K(\lambda, t) = \frac{1}{2} \int_{t - \frac{1}{x} \leq \lambda} |\partial_t u|^2 \, \frac{dx \, dh}{x^{n+1}} = \frac{1}{2} \int_{s(t,x) \leq \lambda} |\partial_s v(x, s(t, x), y)|^2 \, ds \, dh.$$

The function v is smooth in x and so the integrand converges uniformly to $\partial_s v(0, s, y)$ on $s \leq \lambda$ as $t \to \infty$. The dominated convergence theorem then shows that the first part of condition (3) holds for smooth compactly supported data with $F(s) = \int_Y |\mathcal{R}_+(\phi, \psi)(s, y)|^2 \, dh$. The continuity of the radiation field on H_E shows that holds for more general data. A similar calculation shows that the condition holds for $E_P(\lambda, t)$ as well.

The fact that the radiation field is an isometry in this setting shows that condition (4) holds on H_E, proving the theorem in the asymptotically Euclidean case.

4.2. **Asymptotically hyperbolic manifolds.**

Definition 2. (X, g) is asymptotically hyperbolic if $\overline{g} = x^2 g$ is a smooth (up to the boundary) Riemannian metric on X and $|\, dx|_{\overline{g}} = 1$ at the boundary.

Graham–Lee [GL91] and Joshi–Sá Barreto [JSB00] showed that such metrics may be put into a norma form so that, in a collar neighborhood of the boundary,

$$g = \frac{dx^2 + h(x, y, dy)}{x^2},$$

where h is a smoothly varying family of Riemannian metrics on Y. The spectrum of the Laplacian for such a metric was studied by Mazzeo [Maz88, Maz91] and by Mazzeo and Melrose [MM87] and consists of an absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\Delta)$ and a finite pure point spectrum $\sigma_{\mathrm{pp}}(\Delta)$. The spectrum satisfies

$$\sigma_{\mathrm{pp}}(\Delta) \subset \left(0, \frac{(n-1)^2}{4}\right), \quad \sigma_{\mathrm{ac}}(\Delta) = \left[\frac{(n-1)^2}{4}, \infty\right),$$

giving a decomposition

$$L^2(X) = L^2_{\mathrm{pp}}(X) \oplus L^2_{\mathrm{ac}}(X).$$

If $\mathcal{P}_{\mathrm{ac}} : L^2(X) \to L^2_{\mathrm{ac}}(X)$ is the orthogonal projector, we let $E_{\mathrm{ac}}(X) = \mathcal{P}_{\mathrm{ac}}(H_E)$.

We define the operator $H = \Delta_g - \frac{(n-1)^2}{4}$ so that the bottom of the continuous spectrum moves to 0. With this choice of H, the conserved energy is given by

$$E(t) = \frac{1}{2}\int_X \left(|\partial_t u|^2 + |\nabla_g u|^2 - \frac{(n-1)^2}{4}|u|^2 \right) dg,$$

and the energy densities are given by

$$\begin{aligned} e_K &= \frac{1}{2}|\partial_t u|^2, \\ e_P &= \frac{1}{2}\left(|\nabla_g u|^2 - \frac{(n-1)^2}{4}|u|^2 \right). \end{aligned}$$

The conserved energy is positive definite if u is in the image of $\mathcal{P}_{\mathrm{ac}}$.

Sá Barreto [SB05] showed the existence and unitarity of the radiation field for initial data in E_{ac}. Indeed, let $s = t \mp \log x$ and, for a solution u of equation (1) (with $H = \Delta_g - \frac{(n-1)^2}{4}$) on an asymptotically hyperbolic manifold, define the functions

$$v_\pm(x, s, y) = x^{-n/2} u(s \mp \log x, x, y).$$

Sá Barreto showed that for compactly supported smooth initial data v is smooth to $x = 0$. As before, the radiation field is given in terms of v:

$$\mathcal{R}_+(\phi, \psi)(s, y) = \partial_s v(0, s, y).$$

A similar computation to the one in section 4.1 shows that condition (3) is satisfied, again with $F(s) = \int_Y |\mathcal{R}_+(s, y)|^2 \, dh$. The unitarity of the radiation field on E_{ac} shows that condition (4) is satisfied as well, showing that equipartition of energy holds for initial data in E_{ac}.

One reason for projecting off of the pure point spectrum is that the eigenfunctions here do not radiate any energy to null infinity. Moreover, they correspond to zero-energy (and in fact exponentially growing/decaying) solutions with respect to the energy form above.

4.3. **Asymptotically complex hyperbolic manifolds.** Guillarmou and Sá Barreto [GSB08] showed the existence and unitarity of the radiation field on asymptotically complex hyperbolic manifolds. We refer the reader to their paper for the relevant definition of asymptotically complex hyperbolic manifolds. As with asymptotically real hyperbolic manifolds, the spectrum splits into an absolutely continuous part and a pure point part consisting of finitely many eigenvalues. The radiation field for equation (1) with $H = \Delta_g - \frac{n^2}{4}$ is well defined and unitary on E_{ac}. The conserved energy is given by

$$E(t) = \frac{1}{2}\int_X \left(|\partial_t u|^2 + |\nabla_g u|^2 - \frac{n^2}{4}|u|^2 \right) dg,$$

and computations identical to those in section 4.2 show that equipartition of energy holds in this setting.

4.4. **The Schwarzschild spacetime.** Recent work by Wang and the author [BW] establishes the existence and unitarity of the radiation fields on the Schwarzschild black hole background. Topologically, the Schwarzschild spacetime is diffeomorphic to $\mathbb{R}_t \times (2M, \infty)_r \times \mathbb{S}^2$ and is endowed with a Lorentzian metric

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\omega^2.$$

The spatial part of the spacetime has two ends corresponding to infinity ($r = \infty$) and to the event horizon of the black hole ($r = 2M$). Wang and the author define the radiation field to consist of two functions: the restriction of the solution u to the event horizon and the rescaled restriction of the solution to null infinity and in the process use the pointwise decay of solutions of the wave equation to show that it is unitary.

One key feature of the Schwarzschild spacetime is that it is *static*, i.e., ∂_t is a Killing vector field that is orthogonal to the constant t hypersurfaces. This implies that there is a conserved energy and that it splits into kinetic and potential parts. As ∂_t is a Killing vector field for $r > 2M$, the conserved energy is given by

$$E(t) = \frac{1}{2}\int_{2M}^{\infty}\int_{\mathbb{S}^2} \left(\left(1 - \frac{2M}{r}\right)^{-1} |\partial_t u|^2 + \left(1 - \frac{2M}{r}\right)|\partial_r u|^2 + \frac{1}{r^2}|\nabla_\omega u|^2 \right) r^2\, d\omega\, dr,$$

and so the kinetic and potential energy densities are given by

$$e_K = \frac{1}{2}\left(1 - \frac{2M}{r}\right)^{-1} |\partial_t u|^2,$$
$$e_P = \frac{1}{2}\left(1 - \frac{2M}{r}\right)|\partial_r u|^2 + \frac{1}{2r^2}|\nabla_\omega|^2.$$

Near the event horizon, in coordinates ($\rho = r - 2M, \tau = t + r + 2M\log(r - 2M)$), the solution is smooth down to $\rho = 0$ and one half of the radiation field is defined to be

$$f_{\mathcal{H}} = 2M \partial_\tau u|_{\rho=0}.$$

Near null infinity, in coordinates ($\rho = r^{-1}, \tau = t - r - 2M\log(r - 2M)$), the solution is again smooth down to $\rho = 0$ and the other half of the radiation field is defined to be

$$f_{\mathcal{I}} = \partial_\tau(\rho^{-1}u)|_{\rho=0}.$$

For finite-energy (with respect to the energy form above) initial data, the radiation field is unitary, i.e.,

$$E(0) = \|f_{\mathcal{H}}\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} + \|f_{\mathcal{I}}\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} .$$

Proposition 2 then implies that finite-energy solutions on the Schwarzschild exterior exhibit equipartition of energy (on the constant t hypersurfaces).

4.5. **The energy-critical semilinear wave equation on $\mathbb{R}^3$.** In recent work [BSB12], Sá Barreto and the author define a nonlinear radiation field for the energy-critical semilinear wave equation on $\mathbb{R}^3$. Indeed, consider the following semilinear wave equation on $\mathbb{R}^3$:

$$\begin{aligned} \partial_t^2 - \Delta u + |u|^4 u = 0, \\ (u, \partial_t u)|_{t=0} = (\phi, \psi). \end{aligned} \tag{5}$$

The energy densities are given by

$$\begin{aligned} e_P &= \frac{1}{2}|\nabla u|^2 + \frac{1}{6}|u|^2 , \\ e_K &= \frac{1}{2}|\partial_t u|^2 , \end{aligned}$$

and the total energy,

$$E = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\partial_t u|^2 + |\nabla u|^2\right)\, dz + \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\, dz,$$

is conserved.

Grillakis [Gri90] proved global existence and regularity for solutions of this equation with compactly supported smooth initial data, while Shatah and Struwe [SS94] later extended the global well-posedness result to initial data in $\dot{H}^1\times L^2$. Bahouri and Shatah [BS98] showed that the nonlinear (u^6) term in the energy tends to 0 as $t\to\infty$. Bahouri and Gérard [BG99] extended this result to show that solutions also exhibit scattering in the energy space.

The nonlinear radiation field $\mathcal{L}_+(\phi,\psi)$ associated to initial data (ϕ,ψ) with finite energy is given just as in the linear case. In particular, if u solves equation (5) then the limit

$$\mathcal{L}_+(\phi,\psi)(s,\theta) = \lim_{r\to\infty}\partial_s u(s+r, r\theta)$$

exists because $u\in L^5([0,\infty), L^{10}(\mathbb{R}^3))$. Moreover, the map $(\phi,\psi)\mapsto\mathcal{L}_+(\phi,\psi)$ is norm-preserving in the sense that

$$\|\mathcal{L}_+(\phi,\psi)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^2)} = \frac{1}{2}\|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|\psi\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{6}\int_{\mathbb{R}^3}|\phi|^6 .$$

The injectivity of the map $(\phi,\psi)\mapsto\mathcal{L}_+(\phi,\psi)$, together with the definition of $\mathcal{L}_+(\phi,\psi)$, shows that the radiation field satisfies properties (3) and (4). We may thus conclude that the semilinear wave equation exhibits asymptotic equipartition of energy. Moreover, together with the Bahouri and Shatah [BS98] result, we conclude that the critical semilinear wave exhibits asymptotic equipartition of energy in a more traditional sense, i.e.,

$$\lim_{T\to\infty}\int_{\mathbb{R}^3}\left(|\partial_t u(T)|^2 - |\nabla u(T)|^2\right)\, dz = 0.$$

This provides an alternative proof of a result similar to the integrated equipartition result of Vega–Visciglia [VV08].

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