

# A PARAMETRIX FOR THE FUNDAMENTAL SOLUTION OF THE KLEIN-GORDON EQUATION ON ASYMPTOTICALLY DE SITTER SPACES

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ABSTRACT. In this paper we construct a parametrix for the forward fundamental solution of the wave and Klein-Gordon equations on asymptotically de Sitter spaces without caustics. We use this parametrix to obtain asymptotic expansions for solutions of  $(\square - \lambda)u = f$  and to obtain a uniform  $L^p$  estimate for a family of bump functions traveling to infinity.

## 1. INTRODUCTION

The de Sitter space is an exact solution of the vacuum Einstein equations with positive cosmological constant. In this paper, we study the forward fundamental solution of the wave and Klein-Gordon equations on asymptotically de Sitter spaces. This is the unique operator  $E_+$  (which we identify with its Schwartz kernel) which satisfies  $(\square - \lambda)E_+ = I$  and is supported in the forward light cones, i.e. for a compactly supported smooth function  $f$ , the function  $u = E_+f$  satisfies

$$\begin{aligned}(\square - \lambda)u &= f \\ u &\equiv 0 \text{ near past infinity.}\end{aligned}\tag{1}$$

We construct a parametrix for this problem and establish asymptotic expansions for solutions. As an application of our parametrix, we prove a uniform  $L^p$  estimate for the operator applied to a family of bump functions tending toward infinity. We postpone to a future paper the consideration of Strichartz estimates and semilinear wave equations on asymptotically de Sitter spaces.

The study of the decay properties of the wave equation on various natural classes of spacetimes is an active area of research. For example, Dafermos and Rodnianski [DR07] and Melrose, Sá Barreto, and Vasy [MSBV08] have obtained decay results for solutions of the wave equation in the context of the de Sitter-Schwarzschild model of a black hole spacetime.

Our definition of asymptotically de Sitter spaces is given in [Vas07] and follows the definition of asymptotically hyperbolic spaces given in [MM87] and [Maz88]. An asymptotically de Sitter space is a compact manifold with boundary equipped with a Lorentzian metric having a prescribed asymptotic form near the boundary. This pushes the boundary off “to infinity”.

The microlocal structure of the fundamental solution for general real principal type operators has been studied extensively. The solution operator for the Cauchy problem for general real principal type operators is a Fourier integral operator associated to a Lagrangian submanifold of phase space given by the flowout of the Hamilton vector field of the principal symbol of the operator. This was first described in this language by Duistermaat and

Hörmander in [DH72]. In [MU79], Melrose and Uhlmann constructed the forward fundamental solution for a real principal type operator as a paired Lagrangian distribution. These are distributions associated to two cleanly intersecting Lagrangian submanifolds in phase space.

Guillemin and Uhlmann [GU81], Joshi [Jos98], Melrose and Zworski [MZ96], Hassell and Vasy [HV99], and others have all generalized the notion of paired Lagrangian distributions. Guillemin and Uhlmann defined a much more general class of paired Lagrangian distributions, which Joshi restricted slightly in order to construct a well-behaved calculus. Joshi then used this calculus to construct complex powers of the wave operator on Riemannian manifolds. Melrose and Zworski defined a class of distributions associated to an intersecting pair of Legendrians, while Hassell and Vasy later expanded this notion to describe the spectral projections on a scattering manifold.

Polarski [Pol89] computed exactly the solution operator for the Cauchy problem on de Sitter space. Yagdjian and Galstian [YG08] considered the related equation

$$((s\partial_s)^2 - (n-1)(s\partial_s) + s^2\Delta_z - \lambda)u = \delta(s-1)\delta(z)$$

on the upper half-space  $\mathbb{R}_s^+ \times \mathbb{R}_z^{n-1}$  and solved this problem explicitly using hypergeometric functions. Vasy [Vas07] generalized and extended Polarski's result to asymptotically de Sitter spaces. He exhibited the well-posedness of the Cauchy problem and showed that on such spaces, the solution  $u$  of  $(\square - \lambda)u = 0$  with smooth Cauchy data has an asymptotic expansion at infinity. Indeed, if  $x$  is a boundary defining function for  $X$  and  $\sqrt{\frac{(n-1)^2}{4} + \lambda}$  is not a half-integer, then  $u$  has an expansion

$$u = u_+ x^{\frac{n-1}{2} + \sqrt{\frac{(n-1)^2}{4} + \lambda}} + u_- x^{\frac{n-1}{2} - \sqrt{\frac{(n-1)^2}{4} + \lambda}},$$

where  $u_{\pm}$  are smooth on  $X$ . (The difference between this expression and the corresponding one in [Vas07] is due to differing sign conventions for Lorentzian metrics.) In the case of an integer coincidence,  $u_-$  is instead in  $C^\infty(X) + x^2 \sqrt{\frac{(n-1)^2}{4} + \lambda} (\log x) C^\infty(X)$ . Vasy also showed that solutions of the wave equation exhibit scattering, i.e., that the data  $u_{\pm}$  may be specified at one of  $Y_{\pm}$ , which fixes the data at  $Y_{\mp}$ . He further showed that the scattering map, sending the data  $(u_+, u_-)|_{Y_-}$  at  $Y_-$  to the data  $(u_+, u_-)|_{Y_+}$  at  $Y_+$  is a Fourier integral operator associated to a canonical relation given by nullbicharacteristic flow from  $Y_-$  to  $Y_+$ .

Our result extends the work of Vasy to the study of the fundamental solution  $(\square - \lambda)E_+ = I$  on asymptotically de Sitter spaces. We require three global assumptions in our study of asymptotically de Sitter spaces:

- (A1)  $Y = Y_+ \cup Y_-$ , with  $Y_+$  and  $Y_-$  a union of connected components of  $Y$ ,
- (A2) each bicharacteristic  $\gamma$  of  $P$  converges to  $Y_+$  as  $t \rightarrow +\infty$  and to  $Y_-$  as  $t \rightarrow -\infty$ , or vice versa, and
- (A3) The projection from  $T^*X$  to  $X$  of the flowout of the forward light cone from any point  $p \in Y_-$  is an embedded submanifold of  $X$  (except at the point  $p$ , where it always has a conic singularity).

The first two of these assumptions are not particularly restrictive. They imply that the manifold is topologically a product  $Y_+ \times \mathbb{R}$ , but are reasonable from a physical viewpoint in that they imply a time orientation on the manifold. Colloquially, assumptions (A1) and (A2) prevent the breakdown of causality on  $X$ .

Assumption (A3) is needed only to obtain sharp global statements about the fundamental solution, but our construction works in some neighborhood of  $Y_+$  (i.e., a neighborhood of future infinity) without this assumption.

Assumption (A3) prohibits the development of caustics, which significantly narrows the class of manifolds considered. The exact de Sitter space just misses being covered by (A3), though a slight modification of our construction still applies here. Indeed, the projection of the flowout of the light cone from a point  $p \in Y_-$  is a smooth embedded submanifold of the interior of the de Sitter space, but intersects itself at  $Y_+$ . Section 16 discusses the minor modifications needed to handle this case.

If we slightly enlarge the spherical cross section of de Sitter space, then this new space satisfies the assumptions above. The assumptions above are stable under perturbation, so the construction applies to perturbations of this enlarged version of the de Sitter space.

In order to remove the assumption (A3), we could combine the Poisson operator construction of [Vas07] with the local description of the forward fundamental solution (given in [MU79]). Phrasing this in a geometric way is left to a future paper.

The main result of this paper is the following (we state it more precisely later):

**Theorem 1.1.** *There is a compactification  $\widetilde{X}_0^2$  of the interior of  $X \times X$  to a compact manifold with corners such that the closures of the diagonal and the light cone both intersect all boundary hypersurfaces transversely.  $\widetilde{X}_0^2$  is constructed by first blowing up the boundary of the diagonal in  $X \times X$  and then blowing up the set where the projection of the flowout of the light cone hits the side face. On this compactification, the forward fundamental solution of  $\square - \lambda$  lifts to be the sum of a paired Lagrangian distribution, smooth down to the front face, and a conormal distribution associated to the light cone with polyhomogeneous expansions at the other faces of  $\widetilde{X}_0^2$ .*

Call a tempered distribution  $f$  on  $X$  *forward-directed* if it is a smooth function on the interior of  $X$  that vanishes to all orders at  $Y_-$ . The work of Vasy [Vas07] implies that if  $f$  is forward-directed, then we may apply  $E_+$  to  $f$ .

A function  $f$  is polyhomogeneous with index set  $E$  on  $X$  if it has an asymptotic expansion of the form

$$\sum_{(r,l) \in E} x^r (\log x)^l a_{rl}(y)$$

near  $Y_{\pm}$ .

Let  $F_1$  and  $F_2$  be the following two index sets:

$$\begin{aligned} F_1 &= \{(j, l) : j, l \in \mathbb{N}_0, l \leq j\}, \\ F_2 &= \{(s_{\pm}(\lambda) + m, 0) : m \in \mathbb{N}_0\}, \end{aligned} \tag{2}$$

where  $s_{\pm}(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} + \lambda}$ .

A corollary of Theorem 1.1 describes the polyhomogeneity of the solutions of  $P(\lambda)u = f$ .

**Theorem 1.2.** *If  $f$  is forward-directed and polyhomogeneous on  $X$  with index set  $E$  at  $Y_+$ , then  $E_+f$  is forward-directed and polyhomogeneous with index set  $F$ , where*

$$F = F_1 \bar{\cup} F_2 \bar{\cup} E.$$

Here  $E_+$  is regarded as an operator and  $\bar{\cup}$  denotes the extended union of two index sets:

$$(z, p) \in E \bar{\cup} F \Leftrightarrow (z, p) \in E \cup F \text{ or } p = p' + p'' + 1 \text{ with } (z, p') \in E \text{ and } (z, p'') \in F.$$

As an application of our parametrix, we obtain a uniform  $L^p$  estimate for a family of bump functions traveling to infinity. We leave more general  $L^p$  mapping properties and Strichartz estimates to a future manuscript.

**Theorem 1.3.** *Suppose  $\phi \in C_c^\infty(\mathbb{R}_s^+ \times \mathbb{R}_z^{n-1})$  is supported near  $(1, 0)$ . For  $(\tilde{x}, \tilde{y}) \in X$ , Let  $f_{(\tilde{x}, \tilde{y})}(x, y) = \phi(\frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}})$ . Suppose that  $2 < p < \infty$ ,  $r > \max\left(\frac{1}{2}, \frac{n-1}{2} + \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}\right)$ , and  $l \geq \max\left(2, 1 - \frac{n-1}{2} + \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}\right)$ . Then*

$$\|E_+ f_{(\tilde{x}, \tilde{y})}\|_{x^{-l+2(l-r)/p} L^p(X; dg)} \leq C \tilde{x}^{-2r/p}. \quad (3)$$

Here the constant  $C$  depends on the family  $f_{(\tilde{x}, \tilde{y})}$  but not on  $(\tilde{x}, \tilde{y})$ .

In particular, for the wave equation,  $\lambda = 0$ , and we may let  $r = l > n - 1$ , and then

$$\|E_+ f_{(\tilde{x}, \tilde{y})}\|_{x^{-l} L^p(X; dg)} \leq C \tilde{x}^{-2r/p}.$$

Theorem 1.1 follows from a parametrix construction that comprises the bulk of this paper. The main ingredient in the proof of Theorem 1.2 is the pushforward theorem of Melrose. Theorem 1.3 follows from Theorem 1.1 and the  $L^2$  estimates proved by Vasy [Vas07].

The paper is broadly divided into three parts. The first part consists of Sections 2 through 9 and develops the tools necessary to construct the parametrix. The second part contains the construction, which begins with an outline in Section 10 and concludes in Section 15. The last part, consisting of Sections 17 and 18, proves Theorems 1.2 and 1.3.

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## 2. ASYMPTOTICALLY DE SITTER SPACES

We start by describing the de Sitter space. Recall that hyperbolic space can be realized as one sheet of the two-sheeted hyperboloid in Minkowski space. It inherits a Riemannian metric from the Lorentzian metric in Minkowski space. De Sitter space, on the other hand, is the one-sheeted hyperboloid  $\{-X_0^2 + \sum_{i=1}^n X_i^2 = 1\}$  in Minkowski space, but now the induced metric is Lorentzian. A good set of coordinates on this space, which is topologically just  $S^n \times \mathbb{R}$ , is

$$\begin{aligned} X_0 &= \sinh \tau \\ X_i &= \omega_i \cosh \tau, \end{aligned}$$

where  $\omega_i$  are coordinates on the unit sphere. The de Sitter metric is then

$$-d\tau^2 + \cosh^2 \tau d\omega^2.$$

If we let  $T = e^{-\tau}$  near  $\tau = +\infty$ , then  $\tau = +\infty$  corresponds to  $T = 0$  and the metric now has the form

$$\frac{-dT^2 + \frac{1}{4}(T^2 + 1)^2 d\omega^2}{T^2}. \quad (4)$$

This resembles the Riemannian metric on hyperbolic space, which in the ball model is

$$\frac{dr^2 + r^2 d\omega^2}{(1 - r^2)^2}.$$

Our definition of an asymptotically de Sitter space is based on the form (4) of the de Sitter metric.

**Definition 2.1.**  $(X, g)$  is asymptotically de Sitter if  $X$  is a compact manifold with boundary  $Y$ ,  $g$  is a Lorentzian metric on the interior of  $X$ , and, for a boundary defining function  $x$ , there is some collar neighborhood  $[0, \epsilon)_x \times Y$  of the boundary on which  $g$  has the form

$$g = -\frac{dx^2}{x^2} + \frac{h(x, y, dy)}{x^2},$$

where  $h$  is a smooth symmetric  $(0, 2)$ -tensor on  $Y$  that is a metric on the boundary  $\{0\} \times Y$ .

A calculation similar to the ones in [Maz88] or [MM87] shows that the sectional curvatures of asymptotically de Sitter spaces approach 1 as  $x \rightarrow 0$ .

Let  $\square$ , the wave operator, be the Laplace-Beltrami operator associated to  $g$ , and, for  $\lambda \in \mathbb{R}$ , let  $P = P(\lambda) = \square - \lambda$  be the Klein-Gordon operator. We are seeking the fundamental solution of the Klein-Gordon equation, i.e. a distribution  $E_{+, \lambda}$  such that  $P(\lambda)E_{+, \lambda} = I$ . Note that the principal symbol  $\sigma_2(P)$  is given by the dual metric function of  $g$ .

**Definition 2.2.** We say that the *bicharacteristics* of  $P$  over  $X^\circ$ , the interior of  $X$ , are the integral curves of the Hamilton vector field of the principal symbol  $\sigma_2(P)$  inside the characteristic set of  $P$  (the set where  $\sigma_2(P)$  vanishes).

In [Vas07], Vasy showed the existence of a solution to the Cauchy problem for  $\square - \lambda$  under the following two global assumptions:

- (A1)  $Y = Y_+ \cup Y_-$ , with  $Y_+$  and  $Y_-$  a union of connected components of  $Y$ , and
- (A2) each bicharacteristic  $\gamma$  of  $P$  converges to  $Y_+$  as  $t \rightarrow +\infty$  and to  $Y_-$  as  $t \rightarrow -\infty$ , or vice versa.

Throughout this paper, we assume that both (A1) and (A2) hold. These assumptions are introduced primarily to ensure that  $X$  exhibits a coherent causal structure.

Because  $g$  is conformal to the incomplete pseudo-Riemannian metric

$$\hat{g} = -dx^2 + h$$

near  $Y$ , the bicharacteristics of  $P$  (near  $Y$ ) are reparametrizations of the bicharacteristics of  $-dx^2 + h$ .  $g$  is complete, so the global assumptions imply that each bicharacteristic  $\gamma(t)$  has a limit in  $S_Y^*X$  as  $t \rightarrow \pm\infty$ .

Physicists (e.g. [Ger70]) have long known that these assumptions imply the existence of a global ‘time’ function  $T \in C^\infty(\overline{X})$  such that  $T|_{Y_\pm} = \pm 1$  and is monotone on the nullbicharacteristics of  $P$ . Note that  $1 - x$  and  $x - 1$  have the desired properties near  $Y_+$  and  $Y_-$ , so the assumptions mean that such functions can be extended to all of  $X$ . Moreover,  $T$  gives a fibration  $\overline{X} \rightarrow [-1, 1]$  and so  $X$  is diffeomorphic to  $X \times S$  for a compact manifold  $S$ . This also shows that  $Y_+$  and  $Y_-$  are both diffeomorphic to  $S$ .

We also impose the additional assumption:

- (A3) The projection of the flowout of the forward light cone from any point  $p \in Y_-$  is an embedded submanifold of  $X$  (except at the point  $p$ , where it always has a conic singularity).

This assumption ensures that the boundary of the flowout of the light cone in the cotangent bundle is actually the cotangent bundle of a submanifold in the base. It allows us to work with an adapted class of conormal distributions rather than a class of Lagrangian distributions.

Note that the projection of the flowout is automatically an embedded submanifold for small times. Moreover, because the nullbicharacteristics agree in the compact and non-compact settings (i.e. for  $\hat{g}$  and  $g$ ), there is a neighborhood of  $Y_+$  such that the projection of the flowout of the forward light cone from any point in this neighborhood is an embedded submanifold. If we then restrict to data supported in this neighborhood, we may use our construction below even in the absence of assumption (A3).

In this paper we adopt the convention that  $\int \delta(x - \tilde{x})\delta(y - \tilde{y})f(\tilde{x}, \tilde{y})d\hat{g} = f(x, y)$ . In particular, this breaks the symmetry of the operator  $P(\lambda)$ . When we seek a right parametrix, then, we must use the transpose of the operator  $P(\lambda)$  with respect to this metric. We may write this out explicitly as

$$P^t(\lambda) = (x\partial_x)^2 + (n+1)x\partial_x + \frac{x\partial_x\sqrt{h}}{\sqrt{h}}x\partial_x + x^2\Delta_y + n\left(1 + \frac{x\partial_x\sqrt{h}}{\sqrt{h}}\right) - \lambda. \quad (5)$$

We also adopt the convention that when we are applying a differential operator to the variables in the right factor, we use a subscript  $R$ .

In our consideration of the forward fundamental solution of  $P(\lambda)$ , it is useful to have the notion of *forward-directed* data, which are simply tempered distributions (or polynomially bounded smooth functions) on  $X$  that vanish to all orders at  $Y_-$ .

### 3. THE CAUCHY PROBLEM AND SCATTERING

In [Vas07], Vasy studied both the Cauchy problem and the scattering problem on asymptotically de Sitter spaces. The Cauchy problem seeks a solution the equation

$$\begin{cases} P(\lambda)u = 0, \\ u|_{\Sigma_0} = \psi_0, \quad Vu|_{\Sigma_0} = \psi_1 \end{cases}, \quad (6)$$

where  $\Sigma_0$  is a Cauchy hypersurface,  $V$  is a vector field transverse to  $\Sigma_0$ , and  $\psi_0, \psi_1$  are smooth functions on  $\Sigma_0$ .

Via a positive commutator argument, Vasy showed global existence and uniqueness for solutions of the Cauchy problem (6) on asymptotically de Sitter spaces. One may use any “time slice”  $\{T = \text{const}\}$  from the diffeomorphism  $X \cong \mathbb{R} \times Y_+$  as the Cauchy hypersurface needed to pose the Cauchy problem.

Vasy further proved the following a global solvability result for the inhomogeneous equation on asymptotically de Sitter spaces under assumptions (A1) and (A2).

**Theorem 3.1** (Theorem 5.4 of [Vas07]). *Suppose that  $\lambda \in \mathbb{R}$ ,  $l_+ > \max(\frac{1}{2}, l(\lambda))$ ,  $l_- < -\max(\frac{1}{2}, l(\lambda))$ . Then for  $f \in H_0^{0, l_+, l_-}(X)$ ,  $Pu = f$  has a unique solution  $u \in H_0^{1, l_+, l_-}(X)$ . Here  $l(\lambda) = \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}$  while the  $H_0^{m, q_+, q_-} = x_+^{q_+} x_-^{q_-} H_0^m$  measure both regularity and decay at  $Y_+, Y_-$  separately.*

Here the  $H_0^m$  are weighted Sobolev spaces measuring regularity with respect to the 0-vector fields defined in the following section.

In particular, for any  $f \in \dot{C}^\infty(X)$  (indeed, for any forward-directed tempered smooth function), there is a unique solution  $u \in x_+^\infty x_-^{-\infty} H_0^1(X)$  to  $Pu = f$ . There is thus a distribution  $E_+$  on  $X \times X$  that can be called the forward fundamental solution of  $P$ . In other words,  $E_+$  is such that for each  $p \in X$ ,  $PE_+(p) = \delta_p$  and  $E_+(p)(q) \equiv 0$  for  $q \in X$  not in the domain of influence of  $p$ . In terms of the identification  $X \cong \mathbb{R}_t \times Y_+$ , this means that  $E_+(p)(q)$  vanishes when  $t(q) < t(p)$ .

Vasy proved also that solutions of the Cauchy problem (6) have unique asymptotic expansions near the boundary  $Y$  of  $X$  of the form

$$u = u_+ x^{\frac{n-1}{2} + \sqrt{\frac{(n-1)^2}{4} + \lambda}} + u_- x^{\frac{n-1}{2} - \sqrt{\frac{(n-1)^2}{4} + \lambda}},$$

where  $u|_{\pm}$  are smooth up to the boundary of  $X$ . When the difference between the two exponents is an integer, then  $u_-$  is instead only in  $C^\infty(X) + x^2 \sqrt{\frac{(n-1)^2}{4} + \lambda} (\log x) C^\infty(X)$ . Moreover, one may specify  $u_{\pm}|_{Y_{\pm}}$ , uniquely determining  $u_{\pm}|_{Y_{\pm}}$ . He showed that when there are no integer coincidences the map sending the data

$$(u_+|_{Y_-}, u_-|_{Y_-}) \mapsto (u_+|_{Y_+}, u_-|_{Y_+})$$

is a Fourier integral operator associated to the canonical relation given by bicharacteristic flow from  $Y_-$  to  $Y_+$ . This is via an explicit parametrix construction on an appropriate blow-up of  $X \times Y$ , and the static model of de Sitter space appears in the consideration of the rescaled normal operator at the front face of this blow-up.

Observe now that for  $(\tilde{x}_0, \tilde{y}_0) \in X$  away from the boundary of  $X$ , the restriction of the kernel of the fundamental solution to the slice  $\tilde{x} = \tilde{x}_0$  agrees with a multiple of the solution of the Cauchy problem

$$\begin{aligned} P(\lambda)u &= 0, \\ u(\tilde{x}_0, \cdot) &= 0, \\ Vu(\tilde{x}_0, \cdot) &= \delta(y - \tilde{y}), \end{aligned}$$

at least in the region to the future of  $(\tilde{x}_0, \tilde{y}_0)$ . In particular, we can understand the fundamental solution in this region by understanding the solution operator for the Cauchy problem, which Vasy studied.

The solution operator for the Cauchy problem is the composition of the Poisson operator and the Cauchy-to-scattering operator. Vasy showed that these are Fourier integral operators, and their canonical relations intersect transversely, so we can compose them. The result is an operator with canonical relation given by the restriction of the conormal bundle of the flowout of the light cone, restricted to  $\tilde{x} = \tilde{x}_0$  but lifted to a blown-up space. This blown-up space agrees with a slice of the space we define later. Vasy's construction blows up  $[X \times Y_+, \text{diag} Y_+]$ , which turns into what we call  $\text{lcf}_+$  under this composition. Applying this operator to a delta distribution gives a conormal distribution on our space, and the log terms in Vasy's construction become the log terms in our construction, though this requires careful bookkeeping. Viewing this as a distribution parametrized by  $\tilde{x}$  gives the fundamental solution in a neighborhood of the interior of what we call  $\text{lf}_+$  away from the other boundary hypersurfaces.

#### 4. 0-GEOMETRY

Recall from [MM87] the Lie algebra of 0-vector fields,

$$\mathcal{V}_0 = \{\text{vector fields vanishing at } \partial X\}.$$

The universal enveloping algebra of this Lie algebra is the algebra of 0-differential operators,  $\text{Diff}_0^*(X)$ .

We may use the Lie algebra  $\mathcal{V}_0(X)$  to define the natural tensor bundle  ${}^0T X$ , whose smooth sections are elements of  $\mathcal{V}_0(X)$ . Its dual,  ${}^0T^* X$ , has smooth sections spanned (over  $C^\infty(X)$ ) by  $\frac{dx}{x}$  and  $\frac{dy_j}{x}$ .

Recall that  $T^*X$  is endowed with a canonical 1-form given by

$$\alpha = \xi dx + \mu \cdot dy = x\xi \frac{dx}{x} + x\mu \cdot \frac{dy}{x}.$$

In particular,  ${}^0T^*X$  is endowed with the canonical 1-form

$${}^0\alpha = \tau \frac{dx}{x} + \eta \cdot \frac{dy}{x}. \quad (7)$$

Just as  $T^*X$  is endowed with a symplectic form given by  $\omega = d\alpha$ ,  ${}^0T^*X$  is endowed with the symplectic form  ${}^0\omega = d{}^0\alpha$ .

As in [MM87] or [Maz88], we define the 0-double space  $X \times_0 X = X_0^2$  as the blown-up space  $[X \times X, \partial\Delta]$ .  $X \times_0 X$  is a manifold with corners agreeing with  $X \times X$  on the interior. It has three boundary hypersurfaces – the left face  $\text{lf} = \{x = 0\}$ , which is the lift of the boundary hypersurface  $\{0\} \times X$  in  $X \times X$ ; the right face  $\text{rf} = \{\tilde{x} = 0\}$ , which is the lift of  $X \times \{0\}$ ; and the front face  $\text{ff}$ , which is the boundary hypersurface introduced by the blow up construction. Recall also that  $X_0^2$  is equipped with a blow-down map  $\beta : X_0^2 \rightarrow X \times X$ .

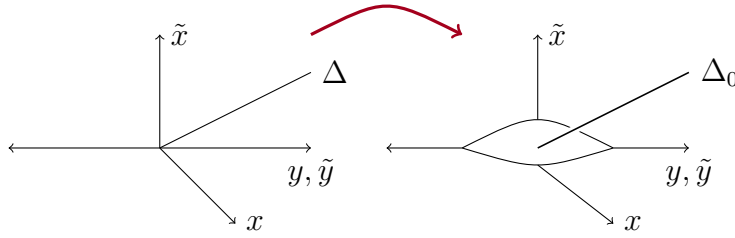


FIGURE 1. Passing from  $X \times X$  to the 0-double space  $X_0^2 = [X^2, \partial\Delta]$ .

This construction is perhaps best thought of as an invariant way of introducing spherical coordinates near the boundary of the diagonal in  $X \times X$ . Indeed, a valid coordinate system near the front face  $\text{ff}$  is given by spherical coordinates:

$$\rho_{\text{ff}} = \sqrt{x^2 + \tilde{x}^2 + |y - \tilde{y}|^2}$$

$$\theta_0 = \frac{x}{\rho_{\text{ff}}}, \quad \theta_n = \frac{\tilde{x}}{\rho_{\text{ff}}}, \quad \theta' = \frac{y - \tilde{y}'}{\rho_{\text{ff}}},$$

where  $\theta \in S_{++}^n$ , a quarter sphere. It is often more convenient to work with projective coordinates near the front face:

$$s = \frac{x}{\tilde{x}}, z = \frac{y - \tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y}.$$

In the projective coordinates  $(s, z, \tilde{x}, \tilde{y})$ , we may compute the lifts of the vector fields  $x\partial_x, x\partial_y, \tilde{x}\partial_{\tilde{x}}$ , and  $\tilde{x}\partial_{\tilde{y}}$ . Indeed, we have

$$x\partial_x \rightsquigarrow s\partial_s \quad x\partial_y \rightsquigarrow s\partial_z$$

$$\tilde{x}\partial_{\tilde{x}} \rightsquigarrow \tilde{x}\partial_{\tilde{x}} - s\partial_s - z \cdot \partial_z \quad \tilde{x}\partial_{\tilde{y}} \rightsquigarrow \tilde{x}\partial_{\tilde{y}} - \partial_z.$$

Recall that the front face is the total space of a fibration. Indeed, it is the total space of a bundle over  $Y$  with quarter-sphere fibers. In local projective coordinates  $(s, z, \tilde{x}, \tilde{y})$  near the front face, the fibers are given by  $\{\tilde{x} = 0, \tilde{y} = \text{const}\}$ . An asymptotically de Sitter metric on  $X$  induces a Lorentzian metric  $-\frac{ds^2}{s^2} + \frac{h(0, \tilde{y}, dz)}{s^2}$  on the interior of the fibers. By an affine



change of coordinates in the fiber, this metric may be written as  $-\frac{ds^2}{s^2} + \frac{dz^2}{s^2}$ , which Wick rotates to the negative of the hyperbolic metric on an upper half-space.

Due to our assumptions (A1) and (A2), the space  $X$  has two boundary components  $Y_+$  and  $Y_-$ , so it is useful at this point to give names to the components of the left face, right face, and front face. We call  $\text{ff}_+$  the component of the front face that comes from blowing up the diagonal near  $Y_+ \times Y_+$ , and  $\text{ff}_-$  the other component. Similarly, we give the name  $\text{lf}_+$  to the lift of the hypersurface  $Y_+ \times X$  and  $\text{lf}_-$  to the lift of  $Y_- \times X$  (and  $\text{rf}_+$ ,  $\text{rf}_-$  denote the lifts of  $X \times Y_+$  and  $X \times Y_-$ , respectively).

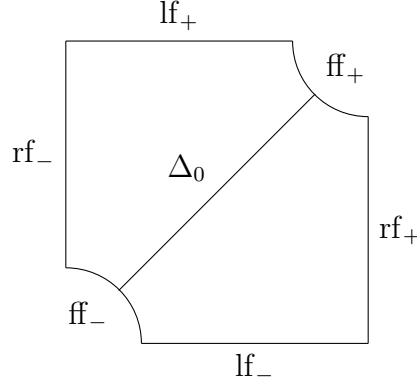


FIGURE 2. A two-dimensional view of  $X_0^2$ .

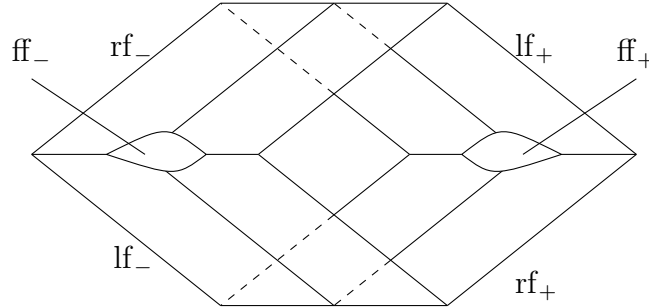


FIGURE 3. Another view of  $X_0^2$ . The horizontal lines represent the  $y, \tilde{y}$  axes.

We should also recall the trivial half-density bundles  $\Omega^{\frac{1}{2}}(X)$  and  $\Omega^{\frac{1}{2}}(X_0^2)$ , which are trivial bundles. In local coordinates  $(x, y)$ ,  $\Omega^{\frac{1}{2}}(X)$  is trivialized by the global section  $\gamma = |d\hat{g}|^{1/2}$ . The bundle  $\Omega^{\frac{1}{2}}(X \times X)$  is trivialized by  $v = |d\hat{g}_L d\hat{g}_R|^{1/2}$ , where  $d\hat{g}_L$  and  $d\hat{g}_R$  are the lifts of the densities from the left and right factors of  $X \times X$ . Up to a nonvanishing factor, the Jacobian determinant of the blow-down map  $X_0^2 \rightarrow X \times X$  is  $r^n$ , so  $v$  lifts to  $r^{n/2}\mu$ , where  $\mu$  is a nonvanishing section of the standard half-density bundle  $\Omega^{\frac{1}{2}}(X_0^2)$ . For example, we may take  $\mu = |dr d\theta d\tilde{y}|^{1/2}$  in local polar coordinates. In projective coordinates, we may take  $\mu = |ds dz d\tilde{x} d\tilde{y}|^{1/2}$ , and then  $v$  lifts to  $\tilde{x}^{n/2}\mu$  (times a nonvanishing factor). In this paper, we adopt the convention that  $v$  and  $\gamma$  are flat, i.e. for a differential operator  $L$ ,  $L(u\gamma) = (Lu)\gamma$  and  $L(uv) = (Lu)v$ . In particular,

$$L(u\mu) = L(\tilde{x}^{-n/2}uv) = (L(\tilde{x}^{-n/2}u))v = (\tilde{x}^{n/2}L(\tilde{x}^{-n/2}u))\mu.$$

Our operator  $P(\lambda)$  commutes with  $\tilde{x}$ , so  $P(\lambda)(u\mu) = (P(\lambda)u)\mu$ .

On the fibers of the front face, elements of  $\text{Diff}_0^*(X)$  lift to differential operators on  $X_0^2$  that restrict to differential operators on the fibers of the front face by setting  $\tilde{x} = 0$ . We call this restriction to the fiber over  $p \in Y$  the normal operator and denote it as  $N_p(A)$  for  $A \in \text{Diff}_0^*(X)$ . As an example, the normal operator of  $P(\lambda)$  is given by

$$N_p(P(\lambda)) = (s\partial_s)^2 - (n-1)(s\partial_s) + s^2\Delta_z - \lambda.$$

The inclusions  $\text{ff} \subset X_0^2$  and  $\text{lf} \subset X_0^2$  induce inclusions on the tangent bundles  $T\text{ff} \subset T_{\text{ff}}X_0^2$  and  $T\text{lf} \subset T_{\text{lf}}X_0^2$ . These inclusions induce natural projections on the cotangent bundle:

$$\pi_{\text{ff}} : T_{\text{ff}}^*X_0^2 \rightarrow T^*\text{ff} \quad \text{and} \quad \pi_{\text{lf}} : T_{\text{lf}}^*X_0^2 \rightarrow T^*\text{lf}.$$

Let  $2X_0^2$  denote the doubling of  $X_0^2$  across  $\text{ff}$ , and let  $T_{\text{ff}}^*X_0^2$  denote the restriction of the cotangent space to the front face. As in [JS01], we say that a smooth closed conic Lagrangian submanifold  $\Lambda \subset T^*X_0^2$  is extendible if it intersects  $T_{\text{ff}}^*X_0^2$  transversely. Recall that in this case there is a smooth closed conic Lagrangian submanifold  $\Lambda^e \subset T^*2X_0^2$  such that

$$\Lambda = \Lambda^e \cap T^*X_0^2, \quad \Lambda^0 = \pi_{\text{ff}}(\Lambda \cap T_{\text{ff}}^*X_0^2).$$

Note that there are many choices for the extension  $\Lambda^e$ .

We recall a result [JS01]:

**Lemma 4.1** (Lemma 2.1 of [JS01]). *Let  $\Lambda \subset T^*X_0^2$  be an extendible Lagrangian. Then  $\Lambda_0 = \pi_{\text{ff}}(\Lambda \cap T_{\text{ff}}^*X_0^2)$  is a Lagrangian submanifold of  $T^*\text{ff}$ .*

For future use, we collect a bit more information about the symplectic structure of  $T^*X_0^2$ . The space  ${}^0T^*X \times {}^0T^*X$  is endowed with the symplectic form

$$\omega = \pi_1^*\omega_X + \pi_2^*\omega_X, \tag{8}$$

where  $\omega_X$  is the symplectic form on  ${}^0T^*X$  (coming from the canonical 1-form (7)) and  $\pi_j$  the projection of the  $j$ th copy of  ${}^0T^*X$ . Moreover, the blow-down map  $\beta$  induces a smooth map

$${}^0T^*X \times {}^0T^*X \rightarrow T^*X_0^2$$

which is an isomorphism on the interior.

The identification of  $T^*X$  and  ${}^0T^*X$  over the interior of  $X$  then induces a smooth map  ${}^0T^*X \times {}^0T^*X \rightarrow T^*X_0^2$  over the interior of  $X \times X$ . This map identifies the 1-forms

$$\tau \frac{dx}{x} + \tilde{\tau} \frac{d\tilde{x}}{\tilde{x}} + \eta \cdot \frac{dy}{x} + \tilde{\eta} \cdot \frac{d\tilde{y}}{\tilde{x}} \quad \text{and} \quad \sigma ds + \xi d\tilde{x} + \zeta \cdot dz + \mu \cdot d\tilde{y},$$

and so we must have

$$\begin{aligned} \sigma &= \frac{\tau\tilde{x}}{x}, & \xi &= \frac{\tau}{\tilde{x}} + \frac{\tilde{\tau}}{\tilde{x}} + \frac{\eta}{x} \cdot \frac{y-\tilde{y}}{\tilde{x}} \\ \zeta &= \frac{\tilde{x}\eta}{x}, & \mu &= \frac{\eta}{x} + \frac{\tilde{\eta}}{\tilde{x}}. \end{aligned} \tag{9}$$

Observe that in these coordinates, the symplectic form  $\omega$  takes a familiar form:

$$d\sigma \wedge ds + d\xi \wedge d\tilde{x} + d\zeta \wedge dz + d\mu \wedge d\tilde{y}.$$

## 5. POLYHOMOGENEITY AND CONORMAL DISTRIBUTIONS

In order to consider the pushforward of a distribution, we need the notion of a b-fibration. We recall from [Mel92] that an interior b-map  $f : X \rightarrow Y$  of manifolds with corners is a function mapping  $X \rightarrow Y$  and  $X \setminus \partial X \rightarrow Y \setminus \partial Y$  such that each boundary defining function for  $Y$  pulls back to a sum of products of boundary defining functions for  $X$ . More precisely, suppose that  $M_1(X)$  is the set of boundary hypersurfaces of  $X$ , and, for each  $G \in M_1(X)$ , let  $I(G)$  be the ideal of functions vanishing at  $G$ . Suppose that  $H$  is a boundary hypersurface for  $Y$  and  $I(H)$  is the ideal of functions vanishing on  $H$ . We say that  $f : X \rightarrow Y$  is an interior b-map if

$$f^*I(H) = \prod_{G \in M_1(X)} I(G)^{e_f(G,H)}.$$

Here  $e_f(G, H)$  is a collection of nonnegative integers that we call the exponent matrix of  $f$ . We set  $(e_f) = \{H \in M_1(X) : e_f(H, G) = 0 \forall G \in M_1(Y)\}$ .

For an interior b-map  $f$ , the differential  $f_* : T_p X \rightarrow T_{f(p)} Y$  extends by continuity to the b-differential

$$f_* : {}^bT_p X \rightarrow {}^bT_{f(p)} Y.$$

Recall that a b-fibration between two manifolds with corners is one that is both b-normal and a b-submersion. A b-submersion is a b-map with surjective b-differential. A b-normal map is one such that the b-differential is surjective as a map  ${}^bN_x H \rightarrow {}^bN_{f(x)} G$  on the interior of each boundary hypersurface  $H$ . Here  $G$  is the face that contains the image of  $H$ .

We say that a discrete subset  $E \subset \mathbb{C} \times \mathbb{N}_0$  is an index set if  $(z_j, k_j) \in E$  with  $|(z_j, k_j)| \rightarrow \infty$  implies that  $\Re z_j \rightarrow \infty$ ,  $(z, k) \in E$  implies that  $(z + p, k) \in E$  for all  $p \in \mathbb{N}_0$ , and if  $(z, k) \in E$ , then  $(z, p) \in E$  for all  $p \in \mathbb{N}_0$  with  $0 \leq p < k$ . We say that  $\mathcal{E} = \{E_H : H \text{ is a boundary hypersurface of } M\}$  is an index family if each  $E_H$  is an index set.

Our construction below shows that certain distributions have asymptotic expansions. We make this notion more precise by recalling the definition of a polyhomogeneous distribution. A polyhomogeneous distribution is one that is smooth on the interior of  $M$ , and, near each boundary hypersurface  $H$  of  $M$  has an asymptotic expansion of the form

$$\sum_{(s,l) \in E_H} x_H^s (\log x_H)^l a_{sl}(x, y),$$

where  $x_H$  is a defining function for  $H$ ,  $E_H$  is an index set for  $H$ , and  $a_{sl}$  are smooth functions independent of  $x_H$ . Near the corners of  $M$ , we require that polyhomogeneous distributions have an appropriate product-type expansion. For an extended discussion of polyhomogeneous distributions, we refer the reader to [Mel92], [Mel93], or [Maz91]. We adopt the convention that  $a_{sl}$  should take values in the half-density bundles.

Recall that if  $f$  and  $g$  are polyhomogeneous with index families  $\mathcal{F}$  and  $\mathcal{G}$ , then  $fg$  is polyhomogeneous with index set  $\mathcal{F} + \mathcal{G}$ .

We also require the following lemma, which can be found in [Mel92] or [Mel93], and allows us to understand the polyhomogeneity of the pushforward of a distribution:

**Lemma 5.1.** *If  $f : X \rightarrow Y$  is a b-fibration between compact manifolds with corners, then for any index family  $\mathcal{K}$  for  $X$  with*

$$\Re \mathcal{K}(H) > -1 \quad \text{if } H \in (e_f),$$

the pushforward gives

$$\begin{aligned} f_* : \mathcal{A}_{phg}^{\mathcal{K}}(X; \Omega \otimes f^*E) &\rightarrow \mathcal{A}_{phg}^{\mathcal{J}}(Y; \Omega \otimes E), \\ \mathcal{J} &= f_{\#}\mathcal{K}. \end{aligned}$$

Here  $\mathcal{J} = f_{\#}\mathcal{K}$  is the pushforward of the index family  $\mathcal{K}$ . In particular, for a boundary hypersurface  $H$  of  $Y$ ,

$$\mathcal{J}(H) = \bar{\cup}_{G \in M_1(X), e_f(G, H) \neq 0} \{(z/e_f(G, H), p) : (z, p) \in \mathcal{K}(G)\},$$

where  $\bar{\cup}$  is the extended union defined by  $(z, p) \in K \bar{\cup} J$  if and only if  $(z, p) \in K \cup J$  or  $p = p' + p'' + 1$  with  $(z, p') \in K$  and  $(z, p'') \in J$ .

We recall the definition of the space of conormal distributions (see, for example, [Hör07] or [Mel92]). If  $M$  is a manifold with corners and  $Z$  is an interior  $p$ -submanifold, then  $I^p(M; Z)$  is the space of conormal distributions of order  $p$  associated to  $Z$ . Here we only need the case when  $Z$  has codimension one and meets the boundary hypersurfaces transversely. In local coordinates  $(x, y)$  where  $M$  is given by  $x_i \geq 0$  for  $i = 1, \dots, k$  and  $Z$  is given by  $y_1 = 0$ , then  $u \in I^p(M; Z)$  has the form

$$\int_{\mathbb{R}} e^{iy_1\eta} a(x, y, \eta) d\eta,$$

with  $a$  a half-density valued classical symbol of order  $p + n/4$  and  $n = \dim M$  (note that we are using the order convention consistent with the orders of Lagrangian distributions in [MU79]). Because  $x$  behaves as a parameter, the boundaries cause no problems here.

We also need several refinements of this notion. The first is when we allow the symbol  $a$  to have an asymptotic expansion (in  $x$ ) near the boundary of  $M$ . We need the space  $\mathcal{A}_{phg}^{\mathcal{E}} I^p(M; Z)$  of conormal distributions associated to  $Z$  with polyhomogeneous expansion at the boundary of  $M$ . These are merely conormal distributions in the sense above, but where we allow the symbol  $a$  to have a polyhomogeneous expansion in the  $x$  variables.

The second generalization we require is to allow the symbol to have a mild type of singularity at  $\eta = 0$ . In particular, we allow symbols of the form  $\sum_j (\eta + i0)^{r-j} a_j$ . The singularity at 0 only affects the growth of the distribution as  $y \rightarrow \pm\infty$ , and so we may avoid complications by multiplying our distribution by a cutoff function supported near  $y = 0$ . We need this generalization later in order to prove a lemma about the support of our distributions.

## 6. THE HAMILTON VECTOR FIELDS

By analogy with the definition of the Hamilton vector field, we recall from [JS01] the definition of the 0-Hamilton vector field.

**Definition 6.1.** Given  $p \in C^\infty(T^*X)$ , the 0-Hamilton vector field of  $p$ , denoted  ${}^0H_p$ , is defined by

$${}^0\omega(\cdot, {}^0H_p) = dp.$$

In local coordinates where  ${}^0\alpha$  is given by (7),  ${}^0H_p$  is given by

$${}^0H_p = x \frac{\partial p}{\partial \tau} \frac{\partial}{\partial x} + x \frac{\partial p}{\partial \eta} \cdot \frac{\partial}{\partial y} - \left( x \frac{\partial p}{\partial x} + \frac{\partial p}{\partial \eta} \cdot \eta \right) \frac{\partial}{\partial \tau} - \left( x \frac{\partial p}{\partial y} - \frac{\partial p}{\partial \tau} \eta \right) \cdot \frac{\partial}{\partial \eta}.$$

We are interested in the operator  $P(\lambda)$ , whose principal symbol is the length function

$$p = -x^2\xi^2 + x^2(h_0(y, \mu) + xh_1(x, y, \mu)) = -\tau^2 + h_0(y, \eta) + xh_1(x, y, \eta),$$

where we have written  $h(x, y, \mu) = h_0(y, \mu) + \bar{x}h_1(x, y, \mu)$ . This is also the principal symbol of  $P$  when  $P$  is treated as an operator acting on the left factor of the product space  $X \times X$ .

In particular, in the coordinates given by (9),  $p$  pulls back to

$$\tilde{p} = -s^2\sigma^2 + s^2h_0(\tilde{y} + \tilde{x}z, \zeta) + s^3\tilde{x}h_1(\tilde{x}s, \tilde{y} + \tilde{x}z, \zeta).$$

Thus if  $\tilde{\omega}$  is the lift of the symplectic form (8) (as in [JS01]) to  $T^*X_0^2$ ,  ${}^0H_p$  lifts to  $H_{\tilde{p}}$ , which is given by  $\tilde{\omega}(\cdot, H_{\tilde{p}}) = d\tilde{p}$ . In these coordinates, we may write

$$\begin{aligned} H_{\tilde{p}} = & \frac{\partial \tilde{p}}{\partial \sigma} \frac{\partial}{\partial s} - \frac{\partial \tilde{p}}{\partial s} \frac{\partial}{\partial \sigma} + \frac{\partial \tilde{p}}{\partial \zeta} \cdot \frac{\partial}{\partial z} - \frac{\partial \tilde{p}}{\partial z} \cdot \frac{\partial}{\partial \zeta} \\ & + \frac{\partial \tilde{p}}{\partial \xi} \frac{\partial}{\partial \tilde{x}} - \frac{\partial \tilde{p}}{\partial \tilde{x}} \frac{\partial}{\partial \xi} + \frac{\partial \tilde{p}}{\partial \mu} \cdot \frac{\partial}{\partial y} - \frac{\partial \tilde{p}}{\partial y} \cdot \frac{\partial}{\partial \mu} \end{aligned}$$

An elementary computation then yields

$$\begin{aligned} H_{\tilde{p}} = & -2s^2\sigma \frac{\partial}{\partial s} - s \left( -2\sigma^2 + 2h_0 + 3s\tilde{x}h_1 + s^2\tilde{x}^2 \frac{\partial h_1}{\partial x} \right) \frac{\partial}{\partial \sigma} \\ & + s^2 \left( \frac{\partial h_0}{\partial \eta} + s\tilde{x} \frac{\partial h_1}{\partial \eta} \right) \cdot \frac{\partial}{\partial z} - s^2 \left( \tilde{x} \frac{\partial h_0}{\partial y} + s\tilde{x}^2 \frac{\partial h_1}{\partial y} \right) \cdot \frac{\partial}{\partial \zeta} \\ & - s^2 \left( z \cdot \frac{\partial h_0}{\partial y} + sh_1 + s^2\tilde{x} \frac{\partial h_1}{\partial x} + s\tilde{x}z \cdot \frac{\partial h_1}{\partial y} \right) \frac{\partial}{\partial \xi} - s^2 \left( \frac{\partial h_0}{\partial y} + s\tilde{x} \frac{pdh_1}{\partial y} \right) \cdot \frac{\partial}{\partial \mu}, \end{aligned}$$

where  $h_0$  and  $h_1$  are evaluated at  $(\tilde{x}s, \tilde{y} + \tilde{x}z, \zeta)$ .

Note that the characteristic set of  $P$  on  $X_0^2$  is the set where  $\tilde{p} = 0$ , i.e. the set where  $s^2\sigma^2 = s^2(h(\tilde{y} + \tilde{x}z, \zeta) + s\tilde{x}h_1(s\tilde{x}, \tilde{y} + \tilde{x}z, \zeta))$ . In this set,  $H_{\tilde{p}}$  has the form

$$\begin{aligned} H_{\tilde{p}} = & s^2 \left( -2\sigma \frac{\partial}{\partial s} - \left( \tilde{x}h_1 + s\tilde{x}^2 \frac{\partial h_1}{\partial x} \right) \frac{\partial}{\partial \sigma} + \left( \frac{\partial h_0}{\partial \eta} + s\tilde{x} \frac{\partial h_1}{\partial \eta} \right) \cdot \frac{\partial}{\partial z} \right. \\ & - \left( \tilde{x} \frac{\partial h_0}{\partial y} + s\tilde{x}^2 \frac{\partial h_1}{\partial y} \right) \cdot \frac{\partial}{\partial \zeta} - \left( \frac{\partial h_0}{\partial y} + s\tilde{x} \frac{pdh_1}{\partial y} \right) \cdot \frac{\partial}{\partial \mu} \\ & \left. - \left( z \cdot \frac{\partial h_0}{\partial y} + sh_1 + s^2\tilde{x} \frac{\partial h_1}{\partial x} + s\tilde{x}z \cdot \frac{\partial h_1}{\partial y} \right) \frac{\partial}{\partial \xi} \right), \end{aligned} \quad (10)$$

In particular, we have proved the following proposition:

**Proposition 6.2.** *Inside the characteristic set of  $P$  on  $X_0^2$ ,  $H_L = H_{\tilde{p}} = s^2\tilde{H}_L$ , where  $\tilde{H}_L$  is a smooth vector field tangent to the front face that is nondegenerate at  $s = 0$ . Similarly, in coordinates  $(x, y, \tilde{s} = s^{-1}, \tilde{z} = s^{-1}z)$ , the Hamilton vector field for the lift from the right factor can be written  $H_R = \tilde{s}^2\tilde{H}_R$ , where  $\tilde{H}_R$  is nondegenerate at  $\tilde{s} = 0$ .*

The fact that we may factor a power of  $s^2$  from the Hamilton vector field is useful in the next section.

## 7. THE LAGRANGIANS

The two Lagrangians that interest us are the lift of the diagonal and the flowout from the characteristic set of  $P$  within this conormal bundle by the Hamilton vector field  $H_L$ .

The coordinates on phase space given by (9) imply that the conormal bundle of the diagonal,

$$\{x = \tilde{x}, y = \tilde{y}, \tau = -\tilde{\tau}, \eta = -\tilde{\eta}\}$$

lifts to

$$\Lambda_0 = \{s = 1, z = 0, \xi = 0, \mu = 0\}.$$

$\Lambda_0$  intersects  $T_{\text{ff}}^* X_0^2$  transversely at

$$\Lambda_0^0 = \{s = 1, z = 0, \xi = 0, \mu = 0, \tilde{x} = 0\} = T_{\Delta_0 \cap \text{ff}}^* \text{ff},$$

and so  $\Lambda_0$  is extendible.

We now set  $\Lambda_1$  to be the forward flowout of  $H_{\tilde{p}}$  from  $\Lambda_0 \cap \Sigma(P)$ . Because we may write  $H_{\tilde{p}} = s^2 \tilde{H}_{\tilde{p}}$ , with  $\tilde{H}_{\tilde{p}}$  a smooth nondegenerate vector field tangent to  $\text{ff}_+$ ,  $\Lambda_1$  is a smooth submanifold of  $T^* X_0^2$ , intersecting  $T_{\text{ff}_+}^* X_0^2$  transversely.  $\Lambda_1$  is Lagrangian by general theory (see, for example, [DS99]). Thus  $\Lambda_1$  is an extendible Lagrangian near  $\text{ff}_+$ .

To see that  $\Lambda_1$  is extendible near  $\text{ff}_-$ , we require the following proposition:

**Proposition 7.1.** *The Lagrangian  $\Lambda_0$  is invariant under the flow of  $H_L - H_R$ . Moreover,  $\Lambda_1$  is also the flowout by  $H_R$  of  $\Lambda_0 \cap \Sigma(P)$ .*

*Proof.* A straightforward calculation shows that  $H_L - H_R$  preserves  $\Lambda_0$ . Indeed, though this is a nonzero vector field, the coefficients of  $\partial_s$ ,  $\partial_z$ ,  $\partial_\xi$ , and  $\partial_\mu$  all vanish at  $\Lambda_0$ . Note further that  $\Sigma(P_L) \cap \Lambda_0 = \Sigma(P_R) \cap \Lambda_0$ , and so  $H_L - H_R$  preserves this set.

In order to see that  $\Lambda_1$  is also the flowout of  $H_R$ , we observe that  $H_L$  and  $H_R$  must commute because they are lifts (from  $X \times X$ ) of commuting vector fields. In particular, because  $H_R - H_L$  is tangent to  $\Lambda_0 \cap \Sigma(P_L)$ , it is also tangent to  $\Lambda_1$ . In particular,  $H_R$  is also tangent to  $\Lambda_1$ .  $H_R$  is not tangent to  $\Lambda_0$ , so its flowout must also be  $\Lambda_1$ .  $\square$

An argument similar to the one above then shows that  $\Lambda_1$  is extendible near  $\text{ff}_-$  (with  $H_L$  replaced by  $H_R$ ). (For what we use below, the argument used above suffices, as we only need that  $\Lambda_1$  is extendible in a neighborhood of the diagonal.)

The projection of  $H_L$  in  $\Sigma(P)$  to  $T^* \text{ff}_+$  is given by

$$H_L = s^2 \left( -2\sigma \frac{\partial}{\partial s} + 2 \frac{\partial h}{\partial \zeta} \cdot \frac{\partial}{\partial z} - \frac{\partial h}{\partial y} \cdot \frac{\partial}{\partial \mu} \right),$$

so that  $\Lambda_1^0$  is given by the flowout of this vector field from  $\Lambda_0^0$  in  $T^* \text{ff}_+$ .

Note that the assumption (A3) implies that the projection of  $\Lambda_1$  to the base  $X_0^2$  is a smooth submanifold. We call this submanifold  $\text{LC} = \pi \Lambda_1$ . This is a smooth codimension one submanifold. Note further that  $\text{LC}$  is the boundary of an open subset of  $X_0^2$ , which we call  $\text{LC}^{\text{int}}$ .

## 8. PAIRED LAGRANGIAN DISTRIBUTIONS

We are interested in a calculus of paired Lagrangian distributions adapted to the 0-geometry. These are distributions with a model form

$$\begin{aligned} u &= \int_0^\infty \int_{\mathbb{R}^n} e^{i(x-\tilde{x}-t\tilde{x})\frac{\xi}{x} + i(y-\tilde{y})\frac{\mu}{\tilde{x}}} a \left( t, \frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y}, \xi, \mu \right) dt d\xi d\mu \\ &= \int_0^\infty \int_{\mathbb{R}^n} e^{i(s-1-t)\xi + iY \cdot \mu} a(t, s, Y, \tilde{x}, \tilde{y}, \xi, \mu) dt d\xi d\mu, \end{aligned} \quad (11)$$

where  $a$  is a symbol.

We briefly recall the definition of a paired Lagrangian distribution from [MU79]. Suppose  $M$  is a manifold and  $L_0, L_1$  are conic Lagrangian submanifolds of  $T^*M$  intersecting cleanly

in codimension 1. We say that  $u \in I^m(M; L_0, L_1; \Omega^{\frac{1}{2}}(M))$  if it is a distributional  $\frac{1}{2}$ -density modeled microlocally on

$$\int_0^\infty \int_{\mathbb{R}^n} e^{i(x_1-s)\xi_1 + ix' \cdot \xi'} a(s, x, \xi) d\xi ds,$$

where  $a$  is a symbol of order  $m + \frac{1}{2} - \frac{n}{4}$ . In the general case, we require the phase function to parametrize the Lagrangian pair  $(L_0, L_1)$  and  $a$  to be a symbol of order  $m + \frac{1}{2} + \frac{n-2N}{4}$ , where  $N$  is the number of variables required to parametrize  $L_1$ . In this model, the phase function parametrizes  $L_0 = N^*\{0\}$ , while  $L_1 = \{x_1 \geq 0, x' = 0, \xi_1 = 0\}$ .

We now recall what it means to parametrize a cleanly intersecting Lagrangian pair. Suppose that  $L_0, L_1 \subset T^*M$  are two closed conic Lagrangian submanifolds intersecting cleanly in codimension 1 and that  $\partial L_1 = L_1 \cap L_0$ . Suppose  $\phi : M_x \times \mathbb{R}_\xi^N \setminus \{0\} \rightarrow \mathbb{R}$  is homogeneous of degree 1 in  $\xi$ , and let

$$C = \{(x, \xi) : x \in M, \xi \in \mathbb{R}^N \setminus \{0\}, d_\xi \phi(x, \xi) = 0\}.$$

We say  $\phi$  is nondegenerate if  $d \frac{\partial \phi}{\partial \xi_j}$ ,  $j = 1, \dots, N$ , are linearly independent at any point in  $C$ . We say that  $\phi$  parametrizes a single conic Lagrangian  $L$  if  $L$  is locally the image of the map  $C \rightarrow T^*M$  given by  $(x, \xi) \mapsto (x, d_x \phi(x, \xi))$ . A simple example is the case when  $L = N^*\{0\}$ , in which case we may take  $\phi(x, \xi) = x \cdot \xi$ .

In the case of an intersecting Lagrangian pair, we say that  $\phi$  is a nondegenerate parametrization of  $(L_0, L_1)$  near  $q \in \partial L_1$  if

$$\phi(x, s, \xi) = \phi_0(x, \xi) + s\phi_1(x, s, \xi), \quad \xi \in \mathbb{R}^n,$$

where  $q = (x_0, d_x \phi(x, 0, \xi))$ ,  $d_{(s, \xi)} \phi(x, 0, \xi) = 0$ , and such that  $\phi_0$  is a nondegenerate parametrization of  $L_0$  near  $q$ , and  $\phi$  parametrizes  $L_1$  near  $q$  for  $s > 0$  with  $ds$ ,  $d \frac{\partial \phi}{\partial \xi_j}$ , and  $d \frac{\partial \phi}{\partial s}$  all linearly independent at  $q$ . In particular, the phase function  $(s-1-t)\xi + Y \cdot \mu$  parametrizes  $(\Lambda_0, \Lambda_1)$  near the front face of  $X_0^2$ .

**Definition 8.1.** We say that  $u \in I^m(X_0^2; \Lambda_0, \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$  is a paired Lagrangian distribution of order  $m$  associated to  $\Lambda_0$  and  $\Lambda_1$  on  $X_0^2$  if it may be written as the restriction to  $X_0^2$  of some distribution in  $I^m(2X_0^2; \Lambda_0^e, \Lambda_1^e; \Omega^{\frac{1}{2}}(2X_0^2))$ .

**Definition 8.2.** The class  $I_0^m(X_0^2; \Lambda_0, \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$  consists of the paired Lagrangian distributions that are supported away from the side faces of  $X_0^2$ .

We similarly define the classes  $I_0^m(X_0^2; \Lambda_0; \Omega^{\frac{1}{2}}(X_0^2))$  and  $I_0^m(X_0^2; \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$  as restrictions of distributions on  $2X_0^2$  that are supported away from the side faces.

We typically shorten  $I^m(X_0^2; \Lambda_0, \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$  and  $I_0^m(X_0^2; \Lambda; \Omega^{\frac{1}{2}}(X_0^2))$  by suppressing the half-density factor.

The following proposition follows immediately from Proposition 4.1 of [MU79].

**Proposition 8.3.** *If  $u \in I_0^m(X_0^2; \Lambda_0, \Lambda_1)$ , then  $\text{WF } u \subset \Lambda_0 \cup \Lambda_1$ . If  $B$  is a properly supported  $b$ -pseudodifferential operator of order 0 on  $X_0^2$ , then*

$$\begin{aligned} \text{WF}'(B) \cap \Lambda_0 = \emptyset &\implies Bu \in I_0^m(X_0^2; \Lambda_1 \setminus \partial \Lambda_1). \\ \text{WF}'(B) \cap \Lambda_1 = \emptyset &\implies Bu \in I_0^{m-1/2}(X_0^2; \Lambda_0 \setminus \partial \Lambda_0). \end{aligned}$$

Note that the orders on  $\Lambda_0$  and  $\Lambda_1$  differ by  $\frac{1}{2}$ . This can be seen by integrating by parts once in  $s$ .

Just as in [MU79], this gives us two symbol maps. For  $u \in I_0^m(X_0^2; \Lambda_0, \Lambda_1)$ ,

$$\begin{aligned}\sigma^{(1)}(u) &\in C_{m+n/4}^\infty(\Lambda_1 \setminus \partial\Lambda_1; \Omega^{\frac{1}{2}} \otimes L_1) \\ \sigma^{(0)}(u) &\in C_{m-1/2+n/4}^\infty(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{\frac{1}{2}} \otimes L_0),\end{aligned}$$

where the subscript indicates the degree of homogeneity and  $L_i$  is the Maslov bundle over  $\Lambda_i$ . Admissible symbols are subject to a compatibility condition at  $\partial\Lambda_1$  as in [MU79], but we do not need the explicit form of this condition here. We do, however, use that if the principal symbol of  $u$  vanishes then  $u$  is one order better.

For these classes to be well-defined, we must show that these classes are independent of the choice of extensions  $\Lambda_0^e$  and  $\Lambda_1^e$ . (It was already shown in [JS01] that the classes  $I_0^m(X_0^2; \Lambda)$  are independent of the choice of extension.)

**Lemma 8.4.** *The class  $I_0^m(X_0^2; \Lambda_0, \Lambda_1)$  is independent of the choice of extensions  $\Lambda_0^e$  and  $\Lambda_1^e$ .*

*Proof.* Suppose that  $\Lambda_0^e$  and  $\tilde{\Lambda}_0^e$  both extend  $\Lambda_0$ , while  $\Lambda_1^e$  and  $\tilde{\Lambda}_1^e$  extend  $\Lambda_1$ . Let  $u \in I_0^m(2X_0^2; \tilde{\Lambda}_0^e, \tilde{\Lambda}_1^e)$ . We may find  $v \in I_0^m(2X_0^2; \Lambda_0^e, \Lambda_1^e)$  with the same symbol on  $\Lambda_0 \cup \Lambda_1$ .

$u - v$  is then order  $m - 1$  on  $\Lambda_0 \cup \Lambda_1$ . We now iteratively solve away principal symbols to find  $w$  with  $u - w$  of order  $-\infty$  on  $\Lambda_0 \cup \Lambda_1$ , i.e. such that  $u - w$  is a smooth function on  $X_0^2$  up to the front face, but supported away from the side faces.  $u - w$  is then clearly the restriction of an element of  $I_0^{-\infty}(2X_0^2; \Lambda_0^e, \Lambda_1^e)$ , proving the claim.  $\square$

The following proposition follows easily from Proposition 5.4 of [MU79]:

**Proposition 8.5.** *Suppose  $A$  is a properly supported  $b$ -differential operator of order  $m$  on  $X_0^2$  and is such that  $\sigma(A)$  vanishes on  $\Lambda_1$ . If  $u \in \tilde{x}^p I_0^k(X_0^2; \Lambda_0, \Lambda_1)$ , then  $Au = \tilde{x}^p h + \tilde{x}^p g$ , with  $h \in I_0^{m+k-1/2}(X_0^2; \Lambda_0)$  and  $g \in I_0^{k+m-1}(X_0^2; \Lambda_0, \Lambda_1)$ .*

*Proof.* We start by noting that the action of  $\text{Diff}_b^m(X_0^2)$  (extended to  $2X_0^2$ ) commutes with restriction to  $X_0^2$ . This follows from the observation that  $x\partial_x\chi_{\{x>0\}} = 0$ , where  $\chi_{\{x>0\}}$  is the characteristic function of a half-plane.

We start by writing  $u = \tilde{x}^p \tilde{u}|_{X_0^2}$ , where  $\tilde{u} \in I_0^k(2X_0^2; \Lambda_0^e, \Lambda_1^e)$ . We apply Proposition 5.4 of [MU79] to  $\tilde{u}$ , giving us  $A\tilde{u} = \tilde{h} + \tilde{g}$ .

In particular, we have that  $Au = \tilde{x}^p h + \tilde{x}^p g + [A, \tilde{x}^p]u$ .  $A$  is a  $b$ -differential operator, so we know that  $[A, \tilde{x}^p] = \tilde{x}^p B$ , where  $B$  is a  $b$ -differential operator of order  $m - 1$ . We thus have that  $Au = \tilde{x}^p h + \tilde{x}^p(g + Bu)$ , where  $h$  and  $g$  are the restrictions of  $\tilde{h}$  and  $\tilde{g}$  to  $X_0^2$ .  $g, Bu \in I_0^{k+m-1}(X_0^2; \Lambda_0, \Lambda_1)$ , proving the claim.  $\square$

We also need another statement from Proposition 5.4 of [MU79], namely that the principal symbol of the distribution  $g$  above is given by the action of the Hamilton vector field of the principal symbol of  $A$  on the principal symbol of  $u$ . We only require this on the interior of the double space and so we do not prove it here.

**Definition 8.6.** We define the normal operator of a paired Lagrangian distribution (or a Lagrangian distribution, as in [JS01]) to be the restriction of its Schwartz kernel to the front face ff.



Note that this definition makes sense by wavefront considerations and the transversality of  $\Lambda_0$  and  $\Lambda_1$  to  $\text{ff}$ . Moreover, by considering the model form (11), we find that the restriction is a paired Lagrangian distribution in the class  $I^m(\text{ff}; \Lambda_0^0, \Lambda_1^0)$ .

Just as in [MM87], we have a short exact sequence.

**Proposition 8.7.** *The normal operator induces exact sequences*

$$0 \rightarrow \tilde{x}I_0^m(X_0^2; \Lambda_0) \rightarrow I_0^m(X_0^2; \Lambda_0) \rightarrow I_0^m(\text{ff}; \Lambda_0^0) \rightarrow 0 \quad (12)$$

and

$$0 \rightarrow \tilde{x}I_0^m(X_0^2; \Lambda_0, \Lambda_1) \rightarrow I_0^m(X_0^2; \Lambda_0, \Lambda_1) \rightarrow I_0^m(\text{ff}; \Lambda_0^0, \Lambda_1^0) \rightarrow 0 \quad (13)$$

such that for any differential operator  $P \in \text{Diff}_0^m(X)$  and any paired Lagrangian distribution  $u$ ,

$$N_p(Pu) = N_p(P)N_p(u) \quad (14)$$

*Proof.* The exactness of the sequences (12) and (13) follows from Taylor expansion in  $\tilde{x}$  near  $\text{ff}$ .

The proof of (14) is identical to the one in [MM87].  $\square$

Also note that because LC is an embedded submanifold, elements of  $I_0^{k-1/2}(X_0^2; \Lambda_1)$  may be identified with distributions of order  $k - 1/2$  conormal to LC.

## 9. ANOTHER BLOW-UP

In order to understand the solutions of the transport equations, we must introduce another blow-up. Because our solutions eventually have differing asymptotic behaviors along the light cone and on the interior of the light cone, we blow up the boundary of the light cone (i.e., at the left and right faces). This blow-up always makes sense locally near the front face, but to make sense of it globally we use assumption (A3).

Because the boundary of the light cone meets the corner  $\text{lf}_+ \cap \text{rf}_-$ , the order in which we blow up the two submanifolds  $\text{LC} \cap \text{lf}_+$  and  $\text{LC} \cap \text{rf}_-$  matters. We deal with this situation by blowing up the submanifold  $\text{LC} \cap \text{lf}_+ \cap \text{rf}_-$  of the corner. We include this discussion primarily for completeness, as this piece of the construction is unnecessary as long as we restrict to forward-directed data. Indeed, the product of the pullback of a forward-directed function and any tempered distribution on  $\tilde{X}_0^2$  will vanish to all orders at this new face.

We first define what we refer to as the intermediate double space. This is the space on which we solve the transport equations for the conormal singularity. If we restrict to data supported away from  $Y_-$ , this space suffices for our entire construction.

**Definition 9.1.** We define the intermediate double space  $X_{0,t}^2 = [X_0^2, \text{LC} \cap \text{rf}, \text{LC} \cap \text{lf}]$ .

This is a new manifold with corners. We will call  $\text{lcf}_+$  the lift of  $\text{LC} \cap \text{lf}$  and  $\text{lcf}_-$  the lift of  $\text{LC} \cap \text{rf}$ .

Though we may think of this new manifold as an invariant way of introducing polar coordinates near  $\text{LC} \cap \text{lf}$ , projective coordinates are more convenient for our applications. Near  $\text{lcf}_+$  and  $\text{ff}_+$  but away from  $\text{lf}_+$ , we may use coordinates  $\rho/s$ ,  $s$ , and  $\tilde{x}$ . Similarly, near  $\text{lcf}_-$  and  $\text{ff}_-$  and away from  $\text{rf}_-$ , we may use  $\rho/\tilde{s}$ ,  $\tilde{s}$ , and  $x$  (and the remaining  $\tilde{z}$  and  $y$  variables), where  $\tilde{s} = \tilde{x}/x$ .

Away from the front face and the corner  $\text{lf}_+ \cap \text{rf}_-$ , this is just the blow-up of an intersection of two hypersurfaces  $\{\rho = 0\}$  and  $\{x = 0\}$  (or  $\{\tilde{x} = 0\}$ ). Near  $\text{lcf}_-$  but away from  $\text{lcf}_+$  and  $\text{ff}_+$ ,  $\rho/\tilde{x}$  and  $\tilde{x}$  are valid coordinates, while near  $\text{lcf}_+$  away from  $\text{lcf}_-$  and  $\text{ff}_-$ ,  $\rho/x$  and  $x$  are

valid coordinates. Near their intersection but away from  $\text{lf}_+$  and  $\text{rf}_-$ ,  $\rho/(\tilde{x}x)$ ,  $\tilde{x}$ , and  $x$  are valid.

Because  $\text{lcf}_+$  and  $\text{lcf}_-$  intersect, the order in which we performed the blow-up matters. Near the interiors of the new faces the two spaces are locally diffeomorphic, but they are not globally diffeomorphic. This is relevant when understanding the behavior of our parametrix near intersections of these faces, so we instead work on what we call the full double space. In this space we first blow up  $\text{LC} \cap \text{lf}_+ \cap \text{rf}_-$  and then perform the other two blow-ups. Performing this first blow-up separates the lifts of  $\text{LC} \cap \text{lf}_+$  and  $\text{LC} \cap \text{rf}_-$ , so the order in which we perform the other two blow-ups is now irrelevant.

**Definition 9.2.** The full double space is  $\tilde{X}_0^2 = [X_0^2, \text{LC} \cap \text{lf}_+ \cap \text{rf}_-, \text{LC} \cap \text{lf}_+, \text{LC} \cap \text{rf}_-]$ .

This is another new manifold with corners and is the space on which the final parametrix lives. We call the additional corner  $\text{scf}$ , for scattering face. We give it this name because this face is related to Vasy's construction of the scattering operator in [Vas07]. Near the interior of  $\text{scf} \cap \text{lcf}_+$ , we may use coordinates given by  $\rho/(x\tilde{x})$ ,  $x/xt$ , and  $\tilde{x}$ , while near the interior of  $\text{scf} \cap \text{lcf}_-$ , we may use  $\rho/(x\tilde{x})$ ,  $\tilde{x}/x$ , and  $x$ . We again emphasize that this blow-up is unnecessary if we restrict to data supported away from  $Y_-$ .

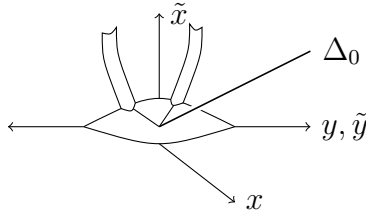


FIGURE 4. The double space  $\tilde{X}_0^2$  near  $\text{ff}_+$  (or  $\text{ff}_-$ ).

Up to a smooth nonvanishing factor, the Jacobian determinant of the blow-down map  $X_{0,t}^2 \rightarrow X_0^2$  is given by  $r_{\text{lcf}_+} r_{\text{lcf}_-}$ , so sections of the bundle  $\Omega^{\frac{1}{2}}(X_0^2)$  lift to sections of  $(r_{\text{lcf}_+} r_{\text{lcf}_-})^{1/2} \Omega^{\frac{1}{2}}(X_{0,t}^2)$ . Similarly, sections of  $\Omega^{\frac{1}{2}}(X_0^2)$  lift to sections of  $r_{\text{scf}}^{1/2} r_{\text{lcf}_+}^{1/2} r_{\text{lcf}_-}^{1/2} \Omega^{\frac{1}{2}}(\tilde{X}_0^2)$ .

## 10. OUTLINE OF THE CONSTRUCTION

We present now an outline of the parametrix construction, which consists of five steps. At each step we construct an approximation of the fundamental solution that captures the “worst” remaining singularity and yields an error term that is less singular.

We simultaneously construct a distribution that is both a left and a right parametrix for our operator. In other words, we construct a distribution  $K$  such that  $P(\lambda)K = I + R_1$  and  $P(\lambda)_R^t K = I + R_2$ . Here the subscript  $R$  denotes the operator acting on the right variables and  $P(\lambda)^t$  is as in equation (5), i.e.,

$$P^t(\lambda) = (x\partial_x)^2 + (n+1)x\partial_x + \frac{x\partial_x\sqrt{h}}{\sqrt{h}}x\partial_x + x^2\Delta_y + n\left(1 + \frac{x\partial_x\sqrt{h}}{\sqrt{h}}\right) - \lambda.$$

The distributions  $R_1$  and  $R_2$  are smoothing operators that vanish to infinite order at relevant boundary hypersurfaces and should be considered “negligible”.

First we show that the construction in [MU79] is valid in a neighborhood of the diagonal, smooth down to the front face of  $X_0^2$ . The Schwartz kernel of the identity is a distribution

conormal to the lift of the diagonal to  $X_0^2$ , and so the fundamental solution of  $P(\lambda)$  is a paired Lagrangian distribution associated to the lift of the diagonal and its flowout via the Hamilton vector field, uniformly down to the front face. This is the most singular piece of the distribution but is the most standard portion of the construction. After cutting off this distribution outside a neighborhood of the lifted diagonal, this step yields a remainder term that is Lagrangian on the interior, uniformly down to the front face, and supported away from the side faces. Our assumption (A3) (which is unnecessary if we are only considering the construction in a neighborhood of  $Y_+$ ) guarantees that the Lagrangian remainder term is in fact a conormal distribution associated to the projection of the flowout of the light cone.

In the second step of the construction, we solve away the conormal error term to obtain a remainder that is smooth on the interior of the double space. Because the light cone is characteristic for  $P(\lambda)$ , this reduces to solving a sequence of transport ordinary differential equations. These may be solved on the interior, but solving each successive transport equation on  $X_0^2$  causes a loss of an order of decay at the boundary. In other words, with each successive improvement of regularity at the interior, the behavior of the solution at the boundary becomes worse.

We instead solve these equations on the intermediate double space  $X_{0,t}^2$ . We solve these equations on  $X_{0,t}^2$  rather than the full double space  $\tilde{X}_0^2$  because transport equations have a slightly simpler form here. Indeed, the main modification needed to handle the exact de Sitter space is to solve the equations on  $\tilde{X}_0^2$  rather than  $X_{0,t}^2$ . This step then yields a remainder term that is smooth on the interior and polyhomogeneous at the light cone faces  $\text{lcf}_+$  and  $\text{lcf}_-$  and the scattering face  $\text{scf}$ .

The third step solves away part of the polyhomogeneous remainder at the light cone faces. We write down a formal series expansion for the solution and solve term by term, which yields a sequence of ordinary differential equations. The numbers  $s_{\pm}(\lambda)$  are the indicial roots of these equations near the side faces of  $\tilde{X}_0^2$ , so  $x^{s_{\pm}(\lambda)}$  solve the equations to first order. In other words, the fundamental solution has an expansion at the side faces starting with  $x^{s_{\pm}(\lambda)}$ . The remainder term from this step vanishes to all orders at the light cone face and has an expansion at the side faces.

In the fourth step we solve away the remaining error term at the front face and side faces. This reduces to another formal power series calculation, and was carried out in [Vas07]. The distribution we obtain by adding the distributions found in the four steps then solves the left (right) equation up to an error term that is smooth and vanishing to all orders at the “plus” (“minus”) boundary faces.

We finally remove this last error term via an argument showing that the fundamental solution itself must be in our class of distributions.

## 11. THE DIAGONAL SINGULARITY

The aim of this section is to solve away the diagonal singularity of the fundamental solution, leaving us with a Lagrangian error. The method here is similar to the one found in [JS01]. We use the normal operator to solve away the paired Lagrangian singularity at the front face up to  $O(\tilde{x}^\infty)$ , which allows us to invoke the construction in [MU79].

**Proposition 11.1.** *Given  $f \in \tilde{x}^p I_0^k(X_0^2; \Lambda_0)$ , there is a paired Lagrangian distribution  $u \in \tilde{x}^p I_0^{-3/2+k}(X_0^2; \Lambda_0, \Lambda_1)$  such that*

$$P(\lambda)u - f \in \tilde{x}^p I_0^{-1/2+k}(X_0^2; \Lambda_1).$$

In particular, if  $f$  is the Schwartz kernel of the identity operator, then

$$f = \delta(s-1)\delta(z)\tilde{x}^{-n/2}\mu \in \tilde{x}^{-n/2}I_0^0(X_0^2; \Lambda_0),$$

and there is a paired Lagrangian distribution  $u \in \tilde{x}^{-n/2}I_0^{-3/2}(X_0^2; \Lambda_0, \Lambda_1)$  such that  $P(\lambda)u - f \in \tilde{x}^{-n/2}I_0^{-1/2}(X_0^2; \Lambda_1)$ .

*Proof.* On the interior of the manifold we may use the construction in [MU79], so we localize near the front face. Because we are only considering distributions supported away from the side faces for now, we do not need separate arguments for dealing with  $\text{ff}_+$  and  $\text{ff}_-$ .

We start by fixing a cutoff function  $\chi \in C^\infty(X_0^2)$  such that  $\chi \equiv 1$  in a neighborhood of the diagonal, but supported away from the side faces of  $X_0^2$ . We also note now that  $P(\lambda)$  commutes with multiplication by  $\tilde{x}$ .

Consider now the normal operator  $N_p(P(\lambda))$  acting on the fiber  $\text{ff}_p$  over the front face. In the projective coordinates  $(s, z, \tilde{x}, \tilde{y})$ , this has the form

$$N(P(\lambda)) = (s\partial_s)^2 - (n-1)(s\partial_s) + s^2\Delta_z - \lambda.$$

It is easy to verify that  $N_p(P(\lambda))$  satisfies the assumptions required to apply Proposition 6.6 of [MU79]. We may thus find  $\kappa \in I^{k-3/2}(\text{ff}; \Lambda_0^0, \Lambda_1^0)$  such that  $N_p(P(\lambda))\kappa(p) - N_p(\tilde{x}^{-p}f) \in C^\infty(\text{ff})$ . Multiplying  $\kappa$  by the cutoff function  $\chi$  gives

$$N_p(P(\lambda))(\chi\kappa(p)) - N_p(x^{-r}f) \in I_0^{k-1/2}(\text{ff}; \Lambda_1^0).$$

By appealing to the short exact sequence (13), we may find some paired Lagrangian distribution  $u_0 \in \tilde{x}^p I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$  such that  $N(\tilde{x}^{-p}u_0) = \tilde{x}^{-p}\chi\kappa$ .  $P(\lambda)$  is characteristic on  $\Lambda_1$ , so we may appeal to Propositions 8.5 and 8.7 to write

$$P(\lambda)u_0 - f = v_0 + w_0,$$

where  $v_0 \in \tilde{x}^{p+1}I_0^k(X_0^2; \Lambda_0)$  and  $w_0 \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$ .

The distribution  $v_0$  is supported away from the side face, so  $\tilde{x}^{-1}v_0 \in \tilde{x}^p I_0^k(X_0^2; \Lambda_0)$ . We may now again use equation (12), Proposition 6.6 of [MU79], and equation (13) to find  $u_1 \in \tilde{x}^p I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$  such that

$$N(P(\lambda))N(\tilde{x}^{-p}u_1) = N(\chi\tilde{x}^{-1}v_0).$$

In particular, we may then again appeal to Propositions 8.5 and 8.7 to write

$$P(\lambda)(\chi(u_0 + \tilde{x}u_1)) = v_1 + w_0 + \tilde{x}w_1,$$

with  $v_1 \in \tilde{x}^{p+2}I_0^k(X_0^2; \Lambda_0)$  and  $w_i \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$ .

Iterating this process gives us distributions  $u_j$  such that

$$P(\lambda) \left( \chi \sum_{j=0}^N \tilde{x}^j u_j \right) = v_N + \sum_{j=0}^N \tilde{x}^j w_j$$

with  $v_N \in \tilde{x}^{p+N+1}I_0^k(X_0^2; \Lambda_0)$  and  $w_j \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$ .

We may now asymptotically sum the  $\tilde{x}^j u_j$  to find  $u \in \tilde{x}^p I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$  such that

$$P(\lambda)(\chi u) - f = v + w, \quad v \in \tilde{x}^\infty I_0^k(X_0^2; \Lambda_0), w \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1).$$

We now extend  $v$  by zero to  $2X_0^2$ , and use proposition 6.6 of [MU79] to find  $\tilde{u} \in I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$  such that  $P(\lambda)\tilde{u} + v \in C^\infty(2X_0^2)$ . In particular,  $P(\lambda)(\chi\tilde{u}) - v = \tilde{w} \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$ . The distribution  $\chi u + \chi\tilde{u}$  then satisfies

$$P(\lambda)(\chi u + \chi\tilde{u}) - f \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1). \quad (15)$$

□

We should note here that solving the transport equations for the symbols on  $\Lambda_1$  (i.e., when invoking Proposition 6.6 of [MU79]) actually fixes the behavior of the solution in both components of  $\Lambda_1 \setminus 0$ . In particular, we may arrange it so that the symbol on  $\Lambda_1$  has an expansion in decreasing powers of  $(\eta + i0)$  (here  $\eta$  is the fiber variable in  $\Lambda_1 = N^* \text{LC}$ ). This allows us to guarantee that our parametrix  $u$  and the error term in (15) are supported on the interior of the light cone. This is not surprising because the exact solution must also be supported on the interior of the light cone due to the finite speed of propagation for the wave equation.

The following lemma is also useful:

**Lemma 11.2.** *When  $f = \kappa_I$  is the Schwartz kernel of the identity operator, then the distribution  $u$  constructed in this section is also a left parametrix for  $P$ , i.e.  $P_R^t u - f = \tilde{x}^{-n/2} r$ , where  $r$  is smooth in a neighborhood of the diagonal. Here  $P_R^t$  denotes the transpose operator for  $P$  acting on the right factor.*

*Proof.* This is an application of the symbol calculus for paired Lagrangian distributions.

We start by observing that  $(P_L - P_R^t)\kappa_I = 0$  because the identity operator commutes with any operator. In other words, we must have that

$$\int (P_L \kappa_I) v = P v = \int \kappa_I (P v) = \int (P_R^t \kappa_I) v.$$

Now let  $v = (P_L - P_R^t)u \in \tilde{x}^{-n/2} I_0^k(X_0^2; \Lambda_0, \Lambda_1)$ . We know that  $P_L v$  is smooth in a neighborhood of the diagonal, down to the front face, because  $P_L$  and  $P_R^t$  commute. We now invoke the symbol calculus:

$$0 = \sigma(P_L v)|_{\Lambda_0 \setminus \partial \Lambda_1} = \sigma(P_L) \sigma(v)|_{\Lambda_0 \setminus \partial \Lambda_1},$$

and  $\sigma(P_L) \neq 0$  on  $\Lambda_0 \setminus \partial \Lambda_1$ , so  $\sigma(v) = 0$  on this set. This fixes an initial condition for  $\sigma(v)$ , i.e.  $\sigma(v)|_{\partial \Lambda_1} = 0$ . We now use the form of the transport equation

$$0 = \sigma(P_L v)|_{\Lambda_1} = (i\mathcal{L}_{H_L} + c)\sigma(v)|_{\Lambda_1},$$

with initial condition 0, to conclude that  $\sigma(v)|_{\Lambda_1} = 0$ . We may use this argument for any  $k$ , so we must have that  $v \in \tilde{x}^{-n/2} I_0^{-\infty}(X_0^2; \Lambda_0, \Lambda_1)$ , proving the claim. □

## 12. THE TRANSPORT EQUATION

We now wish to solve away the error from the previous section. We call this error  $r$  and note that  $r \in r_{\text{ff}}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$ . By solving a transport equation for some finite time and multiplying by a cutoff function, we may assume that this error is supported in a neighborhood of the side faces. Viewed as a conormal distribution near  $\text{ff}_+$ ,  $r$  may be written as

$$\int e^{i\frac{\rho}{s}\eta} a(s, \frac{\rho}{s}, \theta, \tilde{x}, \tilde{y}, \eta) d\eta. \quad (16)$$

where  $\rho$  is a defining function for the light cone and  $a$  is a classical symbol of order  $(k - \frac{1}{2}) + \frac{n-1}{2}$ . In fact, as noted in Section 11, we may assume that the symbol has an expansion in decreasing powers of  $(\eta + i0)$ . Near  $\text{lcf}_+ \cap \text{lcf}_-$ , we must replace  $\frac{\rho}{s}$  with  $\frac{\rho}{x\bar{x}}$ .

We first fix a defining function  $\rho$  for the submanifold LC. Assumption (A3) guarantees that LC is an embedded submanifold of  $X_0^2$ . Because  $N^* \text{LC}$  is characteristic for  $P(\lambda)$  and  $d\rho$  spans  $N^* \text{LC}$ , we must have that  $\hat{g}(d\rho, d\rho) = 0$  at LC, i.e.

$$\hat{g}(d\rho, d\rho) = \rho b, \quad (17)$$

where  $b$  is a smooth function.

In coordinates near  $\partial X = Y$ , we may write

$$P(\lambda) = (s\partial_s)^2 - (n-1)s\partial_s + \frac{\tilde{x}s\partial_s\sqrt{h}}{\sqrt{h}}s\partial_s + s^2\Delta_h - \lambda.$$

Our ansatz is that  $u$  is polyhomogeneous at  $\text{lcf}_+$  and  $\text{lcf}_-$  and conormal to LC, and so we seek an expression of the form (16) with  $a$  a classical symbol of order  $k - \frac{3}{2} + \frac{n-1}{2} = k + \frac{n}{2} - 2$  with an expansion in powers of  $(\eta + i0)$ , i.e., of the form

$$u \sim \sum_{j \geq 0} \int e^{i\frac{\rho}{s}\eta} (\eta + i0)^{k + \frac{n}{2} - 2 - j} a_j d\eta. \quad (18)$$

This is because our error  $r$  from Section 11 is a Lagrangian distribution of order  $k - \frac{1}{2}$  associated to the conormal bundle of LC. LC is characteristic for  $P(\lambda)$ , so we expect the solution of  $P(\lambda)u = r$  to be conormal of one order better, i.e. of order  $k - \frac{3}{2}$ . Moreover, at each step, we multiply the symbols  $a_j$  by a compactly supported smooth function in  $\frac{\rho}{s}$  that is identically 1 near 0, which makes the singularity of  $(\eta + i0)^{k + \frac{n}{2} - 2 - j}$  at 0 superfluous. Note that we could equivalently insist on expressing our ansatz in powers of  $(\frac{\rho}{s})_+$ . If  $k = 0$ , then the top power seen here would be  $-1 + 2 - \frac{n}{2} = 1 - \frac{n}{2}$ , the same powers seen in the construction of the fundamental solution of the wave equation on Minkowski space (in our convention,  $n$  is the total dimension of the spacetime).

**Lemma 12.1.** *Suppose  $u$  is of the form (18), and  $a_j = \tilde{a}_j\nu$ , where  $\nu$  is a fixed nonvanishing section of  $\Omega^{\frac{1}{2}}(X_{0,t}^2)$ . If we write  $\gamma_j = k + \frac{n}{2} - 2 - j$ , then near  $\text{lcf}_+$  away from  $\text{lcf}_+ \cap \text{lcf}_-$  we have*

$$\begin{aligned} P(\lambda)u &\sim \sum_{j=0}^{\infty} \int e^{i\frac{\rho}{s}\eta} (\eta + i0)^{\gamma_j+1} (-2i\hat{g}(d\rho, sd\tilde{a}_j) - i(\partial_s\rho)(n-3-2\gamma_j+O(s))\tilde{a}_j) d\eta\nu \\ &\quad + \sum_{j=0}^{\infty} \int e^{i\frac{\rho}{s}\eta} (\eta + i0)^{\gamma_j} \left( P(\lambda) + (\gamma_j + O(s))s\partial_s - (\gamma_j + 1)(n-3-2\gamma_j) \right. \\ &\quad \left. + \frac{1}{2} \left( n - \frac{1}{2} \right) + O(s)s\partial_z + O(s) \right) \tilde{a}_j d\eta\nu \end{aligned} \quad (19)$$

where  $O(s)$  is taken to mean an element of  $sC^\infty$ .

*Proof.* We first show the result near  $\text{ff}_+ \cap \text{lf}_+$ .

Write  $u_j = \tilde{u}_j\nu$ , where  $\nu$  is a fixed trivialization of  $\Omega^{\frac{1}{2}}(X_{0,t}^2)$ . Say, for concreteness, that  $\nu = r_{\text{lcf}_+}^{-1/2} \tilde{x}^{-n/2} v$ . We then have that

$$P(\lambda)(\tilde{u}_j\nu) = P(\lambda)(\tilde{u}_j r_{\text{lcf}_+}^{-1/2} \tilde{x}^{-n/2} v) = (P(\lambda)\tilde{u}_j) v + r_{\text{lcf}_+}^{1/2} \tilde{x}^{n/2} \left( [P(\lambda), r_{\text{lcf}_+}^{-1/2} \tilde{x}^{-n/2}] \right) \tilde{u}_j.$$

We note also that near  $\text{lcf}_+$  but away from  $\text{lf}_+$ ,  $r_{\text{lcf}_+} = s(1 + \alpha s)$ , where  $\alpha$  is smooth, and so we may easily calculate this commutator:

$$(r_{\text{lcf}_+}^{1/2} \tilde{x}^{n/2})[P(\lambda), r_{\text{lcf}_+}^{-1/2} \tilde{x}^{-n/2}] = (-1 + O(s)) s \partial_s + O(s) s \partial_z + \frac{1}{2} \left( n - \frac{1}{2} + O(s) \right).$$

We now use this calculation to drop the density factor. We apply  $P(\lambda)$  to our ansatz and use equation (17). Integration by parts allows us to exchange powers of  $\frac{\rho}{s}$  for decreasing powers of  $(\eta + i0)$ , as  $\frac{\rho}{s} e^{i \frac{\rho}{s} \eta} = \frac{1}{i} \partial_\eta e^{i \frac{\rho}{s} \eta}$ . If we write  $\gamma_j = k + \frac{n}{2} - 2 - j$ , this yields

$$\begin{aligned} P(\lambda) \tilde{u} &= \left( P(\lambda) + (-1 + O(s)) s \partial_s + \frac{1}{2} \left( n - \frac{1}{2} + O(s) \right) \right) \tilde{u} \\ &\sim \sum_{j=0}^{\infty} \int e^{i \frac{\rho}{s} \eta} (\eta + i0)^{\gamma_j + 2} \hat{g}(d\rho, d\rho) \tilde{a}_j d\eta \\ &\quad + \sum_{j=0}^{\infty} \int e^{i \frac{\rho}{s} \eta} (\eta + i0)^{\gamma_j + 1} \left( -2i \hat{g}(d\rho, s d\tilde{a}_j) \right. \\ &\quad \quad \left. - i(\partial_s \rho)(n - 1 - 2\gamma_j - 3) \tilde{a}_j - i(\partial_s \rho) \tilde{a}_j + O(s) \tilde{a}_j \right) d\eta \\ &\quad + \sum_{j=0}^{\infty} \int e^{i \frac{\rho}{s} \eta} (\eta + i0)^{\gamma_j} \left( P(\lambda) + 2(\gamma_j + 1) s \partial_s - (\gamma_j + 1)(n - 1 - 2\gamma_j - 1) \right. \\ &\quad \quad \left. + (\gamma_j + 1) - s \partial_s + \frac{1}{2} \left( n - \frac{1}{2} \right) + O(s) s \partial_s + O(s) s \partial_z + O(s) \right) \tilde{a}_j d\eta, \end{aligned}$$

where  $O(s)$  is taken to mean an element of  $sC^\infty$ . We now use that  $\hat{g}(d\rho, d\rho) = \rho b$  and integrate by parts to prove the first part of the claim.

We finish the proof with a similar calculation, where we change our operator to  $P_R^t$  and our ansatz to be one of the following two forms:

$$\begin{aligned} u &\sim \sum_{j=0}^{\infty} \int e^{i \frac{\rho}{s} \eta} a_j (\eta + i0)^{\gamma_j} d\eta, \\ u &\sim \sum_{j=0}^{\infty} \int e^{i \frac{\rho}{\tilde{x}} \eta} a_j (\eta + i0)^{\gamma_j} d\eta, \end{aligned}$$

where  $\tilde{s} = \frac{\tilde{x}}{x}$ , and  $P_R^t$  is given by equation (5). We use here that  $r_{\text{lcf}_-} = \tilde{s}(1 + \alpha \tilde{s})$  in the first case and that  $r_{\text{lcf}_-} = \tilde{x}(1 + \alpha \tilde{x})$  and  $r_{\text{lcf}_+} = x(1 + \alpha x)$  in the second case.  $\square$

**Lemma 12.2.** *Similarly, we may compute the right operator near  $\text{lcf}_-$  away from the corner:*

$$\begin{aligned} P(\lambda) {}^t_R u &= \sum_{j=0}^{\infty} \int e^{i \frac{\rho}{s} \eta} (\eta + i0)^{\gamma_j + 1} (-2i \hat{g}(d_R \rho, \tilde{s} d_R \tilde{a}_j) + i(\partial_{\tilde{s}} \rho)(n + 3 + 2\gamma_j + O(\tilde{s})) \tilde{a}_j) d\eta \nu \\ &\quad + \sum_{j=0}^{\infty} e^{i \frac{\rho}{s} \eta} (\eta + i0)^{\gamma_j} \left( P(\lambda) {}^t_R + (\gamma_j + O(\tilde{s})) \tilde{s} \partial_{\tilde{s}} + (\gamma_j + 1)(n + 3 + 2\gamma_j) \right. \\ &\quad \quad \left. + \frac{1}{2} \left( n - \frac{1}{2} \right) + O(\tilde{s}) \tilde{s} \partial_z + O(\tilde{s}) \right) \tilde{a}_j d\eta \nu \end{aligned}$$

We may also compute the behavior of the left and right operators near  $\text{lcf}_+ \cap \text{lcf}_-$ :

$$\begin{aligned}
P(\lambda)u &\sim \sum_{j=0}^{\infty} \int e^{i\frac{\rho}{x\tilde{x}}\eta} (\eta + i0)^{\gamma_j+1} \frac{1}{\tilde{x}} (-2i\hat{g}(d\rho, xd\tilde{a}_j) - i(\partial_x\rho)(n-3-2\gamma_j + O(x))\tilde{a}_j) d\eta \nu \\
&\quad + \sum_{j=0}^{\infty} \int e^{i\frac{\rho}{x\tilde{x}}\eta} (\eta + i0)^{\gamma_j} \left( P(\lambda) + (\gamma_j + O(x))x\partial_x - (\gamma_j + 1)(n-3-2\gamma_j) \right. \\
&\quad \left. + \frac{1}{2} \left( n - \frac{1}{2} \right) + O(x)x\partial_y + O(x) \right) \tilde{a}_j d\eta \nu \\
P(\lambda)_R^t u &= \sum_{j=0}^{\infty} \int e^{i\frac{\rho}{x\tilde{x}}\eta} (\eta + i0)^{\gamma_j+1} \frac{1}{x} (-2i\hat{g}(d_R\rho, \tilde{x}d_R\tilde{a}_j) + i(\partial_{\tilde{x}}\rho)(n+3+2\gamma_j + O(\tilde{x}))\tilde{a}_j) d\eta \nu \\
&\quad + \sum_{j=0}^{\infty} e^{i\frac{\rho}{x\tilde{x}}\eta} (\eta + i0)^{\gamma_j} \left( P(\lambda)_R^t + (\gamma_j + O(\tilde{x}))\tilde{x}\partial_{\tilde{x}} + (\gamma_j + 1)(n+3+2\gamma_j) \right. \\
&\quad \left. + \frac{1}{2} \left( n - \frac{1}{2} \right) + O(\tilde{x})\tilde{x}\partial_{\tilde{y}} + O(\tilde{x}) \right) \tilde{a}_j d\eta \nu
\end{aligned}$$

Because we wish to solve  $P(\lambda)u = r$  up to smooth terms, we wish to iteratively solve away the terms in the above expansions. The first transport equation we must solve is then

$$-2\hat{g}(d\rho, sd\tilde{a}_0) - (\partial_s\rho)(n-3-2\gamma_0 + O(s))\tilde{a}_0 = r_0,$$

where  $r_0$  is compactly supported and comes from the inhomogeneous term.

Because  $\partial_s\rho$  is nonzero, we may divide by it to obtain the transport equation

$$-\frac{h^{kl}(\partial_{z_k}\rho)s\partial_{z_l}\tilde{a}_0}{\partial_s\rho} + s\partial_s\tilde{a}_0 - \left(\frac{n}{2} - \frac{3}{2} - \gamma_0 + O(s)\right)\tilde{a}_0 = \frac{r_0}{2\partial_s\rho}$$

Note that  $\gamma_0 = k + \frac{n}{2} - 2$ , so the coefficient of  $\tilde{a}_0$  is just  $k - \frac{1}{2} + O(s)$ .

Near the face  $\text{lcf}_+$ , given by  $s = 0$ , we may use a parameter  $t$  along the light cone LC.  $t$  is then equivalent to  $s$ , so we may change coordinates to

$$t\partial_t\tilde{a}_0 + \left(k - \frac{1}{2}\right)\tilde{a}_0 = O(t)\tilde{a}_0 + \tilde{r}.$$

We note that the solution  $\tilde{a}_0$  of this equation must have a polyhomogeneous expansion in  $t$ . In fact, we prove a more precise version of this statement.

**Lemma 12.3.** *Suppose that  $v$  solves the differential equation*

$$t\partial_tv - \left(j - k + \frac{1}{2}\right)v = t \cdot c(t)v + b,$$

where  $r$  is smooth in  $t$ , and  $b$  is polyhomogeneous in  $t$  with index set

$$\mathcal{E}_{j-1} = \left\{ \left(-k + \frac{1}{2} + l, i\right) : l \in \mathbb{N}_0, i \leq l \text{ if } l \leq j-1, i = j-1 \text{ if } l \geq j \right\}$$

when  $j \neq 0$  and  $b$  is supported away from  $t = 0$  for  $j = 0$ . Then  $v$  has a polyhomogeneous expansion in  $t$  with index set

$$\mathcal{E}_j = \left\{ \left(-k + \frac{1}{2} + l, i\right) : l \in \mathbb{N}_0, i \leq l \text{ if } l \leq j, i = j \text{ if } l > j \right\}.$$



*Remark 12.4.* We may prove a similar lemma for solutions of the transport equations on the right, with appropriate modifications for the index sets.

*Proof.* We show this by constructing a formal power series solution.

We start with the case  $j = 0$  so that near  $t = 0$ ,  $b \equiv 0$ . We seek a formal power series solution of the form

$$t^{-k+\frac{1}{2}} \sum_{l=0}^{\infty} v_l t^l.$$

Indeed, with this ansatz, the equation becomes

$$t^{-k+\frac{1}{2}} \sum_{l=0}^{\infty} l v_l = t^{-k+\frac{1}{2}} \sum_{l=0}^{\infty} q_{l-1},$$

where  $q_{l-1}$  depends only on  $c(t)$  and the first  $l-1$  coefficients  $v_i$  (so  $q_{-1} = 0$ ). The coefficient  $v_0$  is fixed by the initial condition of the differential equation, and then the remaining coefficients may be found iteratively.

We then sum this series with Borel summation to find a function with this power series at  $t = 0$  and the difference between this function and the solution of the differential equation vanishes to all orders at  $t = 0$ .

For general  $j$ , we write

$$b = \sum_{l=0}^{j-1} \sum_{i=0}^{\min(l, j-1)} b_{li} t^l (\log t)^i$$

with a similar expression for  $v$ . A similar calculation then reduces the equation to

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{i=0}^{\min(l, j)} t^{-k+\frac{1}{2}+l} (\log t)^i ((l-j)v_{li} + (i+1)v_{l, i+1}) \\ &= \sum_{l=1}^{\infty} \sum_{i=0}^l t^{-k+\frac{1}{2}+l} (\log t)^i (q_{l-1, i} + b_{li}), \end{aligned}$$

where  $q_{l-1, i}$  depends on the function  $c(t)$  and the coefficients  $v_{l' i'}$  where  $l' \leq l-1$  or  $l' = l$  and  $i' > i$ . In particular, we may again iteratively solve for each coefficient and then Borel sum the result. □

We may now apply Lemma 12.3 to the first transport equation

$$t \partial_t \tilde{a}_0 - (j - k + \frac{1}{2}) \tilde{a}_0 + O(t) \tilde{a}_j = b(t)$$

and find that  $a_0$  is polyhomogeneous in  $t$  with index set  $\mathcal{E}_0$ . Changing coordinates back to  $s$  tells us that  $a_0$  is polyhomogeneous in  $s$  with index set  $\mathcal{E}_0$ .

Letting  $Q_j$  be the operator acting on  $\tilde{a}_j$  in the coefficient of  $(\eta + i0)^{\gamma_0 - j}$  in equation (19), the  $j$ th transport equation is then

$$-\frac{h^{kl} (\partial_{z_k} \rho) s \partial_{z_l} \tilde{a}_j}{\partial_s \rho} + s \partial_s \tilde{a}_j - (j - k + \frac{1}{2}) \tilde{a}_j + O(s) \tilde{a}_j = -Q_j \tilde{a}_{j-1}.$$

By applying Lemma 12.3 again, we may conclude that  $a_j$  is polyhomogeneous in  $s$  with index set  $\mathcal{E}_j$ .

Now let  $\mathcal{E} = \cup_j \mathcal{E}_j$ . The process above gives us conormal distributions  $u_j \in I_{\mathcal{E}}^{k-3/2-j}(X_{0,t}^2; \text{LC})$  with symbols  $a_j$  such that

$$P(\lambda) \sum_{j=0}^N u_j - r \in I_{\mathcal{E}}^{k-1/2-N-1}(X_{0,t}^2; \text{LC}).$$

We now wish to asymptotically sum this expression to find  $u$  such that  $P(\lambda)u - f$  is smooth on the interior of  $X_{0,t}^2$ .

**Lemma 12.5.** *There is a distribution  $u \in I_{\mathcal{E}}^{k-3/2}(X_{0,t}^2; \text{LC})$ , supported on the interior of the light cone, such that*

$$u \sim \sum_{j=0}^{\infty} u_j.$$

*Proof.* Each symbol  $(\eta + i0)^{\gamma_0 - j} a_j$  is analytic (in  $\eta$ ) and exponentially decreasing in the upper half plane  $\Im \eta > 0$ , so the Paley-Wiener theorem tells us that each  $u_j$  is supported in the region  $\{\frac{\rho}{s} \geq 0\}$ . A standard Borel summation argument completes the proof.  $\square$

Putting our factors of  $r_{\text{ff}}$  from Section 11 back in, we have thus proved:

**Proposition 12.6.** *Given  $r \in r_{\text{ff}}^p I_0^{k-1/2}(X_0^2; \text{LC})$ , there is a conormal distribution,*

$$u \in r_{\text{ff}}^p \mathcal{A}_{phg}^{\mathcal{F}_1} I^{k-3/2}(X_{0,t}^2; \text{LC}),$$

*such that*

$$P(\lambda)u - r \in r_{\text{ff}}^p \mathcal{A}_{phg}^{\mathcal{F}_1}(X_{0,t}^2; \text{lcf}_+),$$

*where  $u$  is supported away from the corner  $\text{ff}_+ \cap \text{lf}_+$  and  $\mathcal{F}_1$  is as in equation (2), i.e.*

$$\mathcal{F}_1 = \left\{ \left( j - k + \frac{1}{2}, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\}.$$

*We may further arrange that the distribution is supported in the interior of the light cone.*

We observe here that an argument similar to the one given in the previous section shows that  $P_R^t u - r$  is again smooth on the interior when we are constructing a parametrix for the fundamental solution. In particular, the symbols of  $u$  satisfy the transport equations for the actions on the right factor, and so the following proposition holds:

**Proposition 12.7.** *There is a conormal distribution  $u \in r_{\text{ff}}^{-n/2} \mathcal{A}^{\mathcal{F}} I^{-3/2}(X_{0,t}^2; \text{LC})$  such that*

$$\begin{aligned} P(\lambda)u - r &\in r_{\text{ff}}^{-n/2} \mathcal{A}_{phg}^{\mathcal{E}_L}(X_{0,t}^2), \\ P(\lambda)_R^t u - r &\in r_{\text{ff}}^{-n/2} \mathcal{A}_{phg}^{\mathcal{E}_R}(X_{0,t}^2). \end{aligned}$$

Here the index families  $\mathcal{F}$  and  $\mathcal{E}$  are given by

$$\begin{aligned} F_{\text{lcf}_+} &= \left\{ \left( j + \frac{1}{2}, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\} \\ F_{\text{lcf}_-} &= \left\{ \left( j + \frac{1}{2} - n, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\} \\ E_{\text{lcf}_+, L} &= \left\{ \left( j + \frac{1}{2}, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\} \\ E_{\text{lcf}_-, L} &= \left\{ \left( j + \frac{1}{2} - n - 1, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\} \\ E_{\text{lcf}_+, R} &= \left\{ \left( j + \frac{1}{2} - 1, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\} \\ E_{\text{lcf}_-, R} &= \left\{ \left( j + \frac{1}{2} - n, l \right) : j, l \in \mathbb{N}_0, l \leq j \right\}. \end{aligned}$$

Note that the decrease of  $-1$  for two of the error terms come from the factors of  $\frac{1}{x}$  in the expression given in Lemma 12.2. The  $-n$  on the “minus” faces comes from taking the transpose of our operator. This is because we are using sections of the standard half-density bundle rather than the 0-half-density bundle.

### 13. THE LIGHT CONE FACE

We now wish to solve away the error from Proposition 12.7. We aim to solve away the left error at  $\text{lcf}_+$  and the right error at  $\text{lcf}_-$ . We show only the left calculation here and observe that the right calculation is nearly identical.

We again call the error  $r$ , which is in  $r_{\text{ff}}^p \mathcal{A}_{\text{phg}}^{\mathcal{E}L}(X_{0,t}^2)$ , i.e. near  $\text{ff}_+$ ,  $r$  has an expansion of the form

$$r \sim \tilde{x}^p \sum_j^\infty \sum_{l=0}^j s^{\gamma_j} (\log s)^l r_{jl} \nu,$$

where  $\gamma_j = j - k - \frac{1}{4}$ . We drop the power of  $\tilde{x}$  for now because  $P(\lambda)$  commutes with  $\tilde{x}$ .

We first claim that this error lifts to be polyhomogeneous on the full double space  $\tilde{X}_0^2$ .

**Lemma 13.1.** *Suppose that  $r$  is a polyhomogeneous distribution on  $X_{0,t}^2$  supported in a small neighborhood of LC with index family  $\mathcal{F}$ . Then  $r$  lifts to a polyhomogeneous distribution on  $\tilde{X}_0^2$  with index family  $\mathcal{G}$ , where  $\mathcal{G}$  is given by*

$$\begin{aligned} G_{\text{lcf}_+} &= F_{\text{lcf}_+}, \\ G_{\text{lcf}_-} &= F_{\text{lcf}_-}, \\ G_{\text{scf}} &= F_{\text{lcf}_+} + F_{\text{lcf}_-} + 1. \end{aligned}$$

*Proof.* The result follows because the two possible orders of the blow-up are locally diffeomorphic near  $\text{LC} \cap \text{lcf}_+ \cap \text{lcf}_-$ . The extra 1 in  $G_{\text{scf}}$  is because sections of  $\Omega^{\frac{1}{2}}(X_{0,t}^2)$  lift to sections of  $r_{\text{scf}} \Omega^{\frac{1}{2}}(\tilde{X}_0^2)$ .  $\square$

We may thus consider  $r$  as a polyhomogeneous function on  $\tilde{X}_0^2$ .

We now proceed in two steps. The first is to solve away near  $\text{lcf}_+$  and the second is to show that it has the desired form at  $\text{lcf}_+$ . This only away from the scattering face  $\text{scf}$ , though

the computation near scf is nearly identical. In this section there are many terms that come from differentiating our ansatz. We attempt to indicate the origin of the important terms.

Note that the statement about the support in Proposition 12.7 means that  $r_{jl}$  is supported on the interior of the light cone and vanishes to infinite order at LC.

Because we are working near  $\text{lcf}_+$ , we first use projective coordinates  $(s, w = \frac{\rho}{s}, \theta)$ , where  $\rho$  is a defining function for LC (as above) and  $\theta$  are the remaining variables. In these coordinates, derivatives of the function  $\rho$  appear as coefficients of  $\partial_w$  in our operator  $P(\lambda)$ . In particular,  $s\partial_s$  lifts to  $s\partial_s + (\partial_s\rho)\partial_w - w\partial_w$ .

We again expect a polyhomogeneous expansion in  $s$ . In other words, we expect an expansion of the form

$$u \sim \sum_{j=0}^{\infty} \sum_{l=0}^j s^{\gamma_j} (\log s)^l u_{jl}, \quad (20)$$

where  $u_{jl}$  is regarded as a function of  $w$ ,  $\theta$ ,  $\tilde{x}$ , and  $\tilde{y}$ .

Write  $u_{jl} = \tilde{u}_{jl}\nu$ . We again note that we may again take  $r_{\text{lcf}_+}$  equivalent to  $s$  in this region, giving us an extra  $-\frac{1}{2}s\partial_s - \frac{1}{2}(n - \frac{1}{2})$  in our operator. Applying  $P(\lambda)$  to our ansatz yields

$$\begin{aligned} P(\lambda)u &\sim \sum_j \sum_{l=0}^j s^{\gamma_j} (\log s)^l \left[ -\bar{g}(d\rho, d\rho) - 2w(\partial_s\rho) + w^2 \right] (\partial_w^2 \tilde{u}_{jl}) \\ &\quad - \left( (n + \frac{1}{2} - 2\gamma_j)(\partial_s\rho) - (n + \frac{1}{2})w \right) \partial_w \tilde{u}_{jl} \\ &\quad - \left( (n - 1 - \gamma_j)\gamma_j + \frac{1}{2}(n - \frac{1}{2}) - \lambda \right) \tilde{u}_{jl} \\ &\quad + A_{j-2}\tilde{u}_{j-2,l} + B_{j-1}\tilde{u}_{j-1,l} + B'_{l+1}\tilde{u}_{j-1,l+1} + X_{jl}\tilde{u}_{j,l+1} + Y_l\tilde{u}_{j,l+2} \Big] \nu, \end{aligned}$$

where

$$\begin{aligned} A_{j+2} &= - \sum_q (\Delta_h \theta_q) \partial_{\theta_q} - \sum_{q,r,i,k} \left( \frac{\partial \theta_r}{\partial z_i} \right) \left( \frac{\partial \theta_q}{\partial z_k} \right) \partial_{\theta_r} \partial_{\theta_q}, \quad B'_{l+1} = \frac{\tilde{x} \partial_s \sqrt{h}}{\sqrt{h}} (l+1), \\ B_{j+1} &= - \sum_{i,k,r} 2h^{ik} \left( \frac{\partial \theta_r}{\partial z_i} \right) \left( \frac{\partial \rho}{\partial z_k} \right) \partial_{\theta_r} \partial_w + (\Delta_h \rho) \partial_w + (\partial_s^2 \rho) \partial_w, \\ X_{jl} &= 2(l+1)(\partial_s \rho - w) \partial_w - (l+1)(n - \frac{1}{2} - 2\gamma_j), \quad Y_l = (l+1)(l+2). \end{aligned}$$

The constants above come from the operator and from  $s\partial_s$  landing on the powers of  $s$ . The leading  $\partial_w$  terms come from the  $(\partial_s \rho) \partial_w$  terms when we lift  $s\partial_s$  and the  $h^{kl}(\partial_{z_k} \rho)(\partial_{z_l} \rho) \partial_w^2$  term in the lift of the Laplacian in  $z$ .

Note that because LC is characteristic for  $\square$ , we must have that  $\bar{g}(d\rho, d\rho) = \rho b$ .  $\rho = sw$ , so by replacing  $B_{j+1}$  with  $\tilde{B}_{j+1} = B_{j+1} - wb\partial_w^2$ , we may write

$$\begin{aligned} P(\lambda)u &\sim \sum_j \sum_{l=0}^j s^{\gamma_j} (\log s)^l \left[ Q_j \tilde{u}_{jl} + A_{j-2} \tilde{u}_{j-2,l} + \tilde{B}_{j-1} \tilde{u}_{j-1,l} \right. \\ &\quad \left. + B'_{l+1} \tilde{u}_{j-1,l+1} + X_{jl} \tilde{u}_{j,l+1} + Y_l \tilde{u}_{j,l+2} \right] \nu, \end{aligned}$$

where

$$Q_j = -(\partial_s \rho) + w) w \partial_w^2 - \left( (\partial_s \rho) \left( n + \frac{1}{2} - 2\gamma_j \right) - \left( n + \frac{1}{2} \right) w \right) \partial_w \\ - (n - 1 - \gamma_j) \gamma_j - \frac{1}{2} \left( n - \frac{1}{2} \right) - \lambda.$$

We thus wish to solve a sequence of inhomogeneous ordinary differential equations  $Q_j \tilde{u}_{jl} = \tilde{r}_{jl}$ , where

$$\tilde{r}_{jl} = r_{jl} - X_{jl} \tilde{u}_{j,l+1} - Y_l \tilde{u}_{j,l+2} - A_{j-2} \tilde{u}_{j-2,l} - \tilde{B}_{j-1} \tilde{u}_{j-1,l} - B'_{l+1} \tilde{u}_{j-1,l+1}.$$

where all terms are supported in  $\{w \geq 0\}$ , vanishing to infinite order at  $w = 0$ . (We know already that  $r_{jl}$  has this property, and we show at each step that  $\tilde{u}_{jl}$  does as well.)

It is clear that 0 is a regular singular point of the differential operator  $Q_j$ , so the solutions of  $Q_j v = 0$  have formal power series expansions at  $w = 0$  with first term given by  $w^{\mu_i}$ . Here  $\mu_i$  are the roots of the indicial equation  $-\mu(\mu - 1) - (n + \frac{1}{2} - 2\gamma_j)\mu = 0$  (see, for example, [BR89]), i.e.

$$\mu_1 = 0, \quad \mu_2 = 2j - 2k - n$$

Here we have used that  $\gamma_j = j - k + \frac{1}{2}$ . Standard ODE techniques (i.e., variation of parameters) then give us solutions  $\tilde{u}_{jl}$  to  $Q_j \tilde{u}_{jl} = \tilde{r}_{jl}$  in terms of a basis of solutions for  $Q_j v = 0$ . Moreover, because  $\tilde{r}_{jl}$  vanishes to all orders at  $w = 0$ , we may also guarantee that  $\tilde{u}_{jl}$  is supported in  $\{w \geq 0\}$ , vanishing to all orders at  $w = 0$ . Indeed, if  $v_1$  and  $v_2$  are a basis for the solutions of  $Q_j v = 0$ , then

$$\tilde{u}_{jl}(w) = -v_1(w) \int_0^w \frac{v_2(w') \tilde{r}_{jl}(w')}{W(v_1, v_2)(w')} dw' + v_2(w) \int_0^w \frac{v_1(w') \tilde{r}_{jl}(w')}{W(v_1, v_2)(w')} dw'.$$

Because  $\tilde{r}_{jl}$  vanishes to all orders at  $w = 0$ , (and  $v_i$  are bounded by  $w^{-N}$  for some  $N$ ), the integrals make sense and vanish to all orders at  $w = 0$ . Multiplication by  $v_i$  then preserves this property.

We may thus solve these equations and now wish to consider their asymptotics near the corner  $\text{lcf}_+ \cap \text{lf}_+$ . Near this corner, the coordinates  $(s, w, \theta)$  are invalid and so we must use the other set of projective coordinates  $(\rho = sw, W = w^{-1}, \theta)$ . In these coordinates,  $s \partial_s$  lifts to  $(\partial_s \rho) \partial_\rho + W \partial_W - (\partial_s \rho) W^2 \partial_W$ . The top order terms below then come from the  $W \partial_W$  and the constants come only from the operator.

Because polyhomogeneous distributions are independent of our choice of coordinate systems, we may also express our ansatz (20) in terms of these coordinates  $(\rho, W, \theta)$ . In this case, expansions in  $s$  are equivalent to expansions in  $\rho$ , so our ansatz here has the form

$$u \sim \sum_j \sum_{l=0}^j \rho^{\gamma_j} (\log \rho)^\ell v_{jl}.$$

In the computation that follows, the important point is that  $s_\pm(\lambda)$  are the indicial roots of our operator  $P(\lambda)$ , and that this behavior is dominant away from LC because the fundamental solution is a smooth solution of the homogeneous equation here.

Apply  $P(\lambda)$  to such an ansatz to see

$$\begin{aligned} P(\lambda)u \sim \sum_{(j,l) \in \mathcal{F}_1} \rho^j (\log \rho)^l & [(1 - q_2 W) W^2 \partial_W^2 \tilde{v}_{jl} \\ & - (n - 2 + q_1 W) W \partial_W \tilde{v}_{jl} - (\lambda + q_0 W) \tilde{v}_{jl} \\ & + W A_{j-1,l} \tilde{v}_{j-1,l} + W A'_{j-1,l+1} \tilde{v}_{j-1,l+1} \\ & + W B_{j-2,l} \tilde{v}_{j-2,l} + W B'_{j-2,l+1} \tilde{v}_{j-2,l+1} + W B''_{j-2,l+2} \tilde{v}_{j-2,l+2}], \end{aligned}$$

where we have (via a similar, but more tedious, calculation)

$$\begin{aligned} q_2 &= 2(\partial_s \rho) - (\partial_s \rho)^2 W + |d_z \rho|_h^2 W, \\ q_1 &= -(n + 2\gamma_j - 4)(\partial_s \rho) + (2\gamma_j - 2)\bar{g}(d\rho, d\rho)W, \\ q_0 &= (n - 2)\gamma_j(\partial_s \rho) - \gamma_j(\gamma_j - 1)\bar{g}(d\rho, d\rho)W, \end{aligned}$$

while

$$\begin{aligned} A_{j-1,l} &= 2h^{ik}(\partial_{z_i} \theta_p)(\partial_{z_k} \rho) W^2 \partial_{\theta_p} \partial_W - 2\gamma_j h^{ik}(\partial_{z_i} \rho)(\partial_{z_k} \theta_p) W \partial_{\theta_p} \\ &\quad - (\square_{\bar{g}} \rho) W^2 \partial_W - \gamma_j (\square_{\bar{g}} \rho) W, \\ A'_{j-1,l+1} &= (l+1) [(-2(\partial_s \rho) + 2\bar{g}(d\rho, d\rho)) W \partial_W \\ &\quad - (\partial_s \rho)(n-2) - (2\gamma_j - 2)\bar{g}(d\rho, d\rho)W], \end{aligned}$$

and

$$\begin{aligned} B_{j-2,l} &= (\Delta_h \theta_p) W \partial_{\theta_p} - h^{ik}(\partial_{z_i} \theta_p)(\partial_{z_k} \theta_q) W \partial_{\theta_p} \partial_{\theta_q} \\ B'_{j-2,l+1} &= -2(l+1)h^{ik}(\partial_{z_k} \rho)(\partial_{z_i} \theta_p) W \partial_{\theta_p} + (l+1)(\square_{\bar{g}} \rho) W \\ B''_{j-2,l+2} &= -(l+1)(l+2)\bar{g}(d\rho, d\rho)W. \end{aligned}$$

We must still take into account the density factor here. In this region,  $r_{\text{lf}_+}$  is equivalent to  $\rho$ . A similar computation to the one above shows that this leads to two types of terms. One of these is  $O(W)W\partial_W$ , while the other is  $O(W)$ . This means that they may be absorbed into  $q_1$  and  $q_0$ . Let us now write

$$Q_j = (1 - q_2 W) W^2 \partial_W^2 - (n - 2 + q_1 W) W \partial_W - (\lambda + q_0 W).$$

We wish to solve a sequence of transport equations (which are really the same equations as above written in these coordinates) given by

$$\begin{aligned} Q_j \tilde{v}_{jl} &= \tilde{r}'_{jl} - W A_{j-1,l} \tilde{v}_{j-1,l} - W A'_{j-1,l+1} \tilde{v}_{j-1,l+1} - W B_{j-2,l} \tilde{v}_{j-2,l} \\ &\quad - W B'_{j-2,l+1} \tilde{v}_{j-2,l+1} - W B''_{j-2,l+2} \tilde{v}_{j-2,l+2} = \tilde{r}'_{jl}. \end{aligned}$$

We may again construct formal power series solutions (or apply Lemma 5.44 of Melrose ([Mel93])) to understand the solutions of these transport equations as polyhomogeneous conormal functions. Here the indicial roots of  $Q_j$  are  $s_{\pm}(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} + \lambda}$ . More precisely, we have

**Lemma 13.2.** *Suppose that  $u_{jl}$  solves  $Q_j u_{jl} = \tilde{r}'_{jl}$  as above. Suppose first that  $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$ . Suppose that  $\tilde{r}'_{jl}$  is polyhomogeneous conormal at  $\text{lf}_+$  with index set  $F + 1 = (F^+ + 1) \cup (F^- + 1)$ , where*

$$F^{\pm} = \{(s_{\pm}(\lambda) + m, 0) : m \in \mathbb{N}_0, \},$$

then  $u_{jl}$  is polyhomogeneous with index set  $F = F^+ \cup F^-$ . If  $s_+(\lambda) - s_-(\lambda) = N \in \mathbb{Z}$ , then if  $\tilde{r}'_{jl}$  is polyhomogeneous with index set  $\tilde{F} + 1$ , where

$$\tilde{F} = \{(s_-(\lambda) + m, l) : m \in \mathbb{N}_0, l = 0 \text{ for } l < N, l = 1 \text{ for } l \geq N\},$$

then  $u_{jl}$  must be polyhomogeneous with index set  $\tilde{F}$ .

*Remark 13.3.* Here the notation  $F + 1$  is shorthand for

$$(\alpha, l) \in F + 1 \text{ if and only if } (\alpha - 1, l) \in F.$$

*Remark 13.4.* Note that the index set above means that if  $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$ , then  $u_{jl} \in W^{s_+(\lambda)}C^\infty + W^{s_-(\lambda)}C^\infty$ . If  $s_+(\lambda) - s_-(\lambda) \in \mathbb{Z}$ , then

$$u_{jl} - \sum_{m=0}^{s_+(\lambda)-s_-(\lambda)-1} W^{s_-(\lambda)+m} u_{jl}^{(m)} \in (1 + \log W)C^\infty.$$

*Proof.* We may again construct a formal power series solution (or an apply of Lemma 5.44 of Melrose ([Mel93])). We omit this here because it has been described in detail already. The key point is that the operators  $A_{j-1,l}$ ,  $A'_{j-1,l+1}$ ,  $B_{j-2,l}$ ,  $B'_{j-2,l+1}$ , and  $B''_{j-2,l+2}$  are all elements of  $W \text{Diff}_b^*$  (products of  $W$  and  $b$ -differential operators). This ensures that the terms we are solving away vanish to one order better at  $W = 0$ .  $\square$

Asymptotically summing  $\sum \sum \rho^{\nu_j} (\log \rho)^l u_{jl}$  then solves away the error  $r$  at  $\text{lcf}_+$ . Note that because  $s_\pm(\lambda)$  are the indicial roots of  $N(P)$ , we in fact have that the error term vanishes to one order better.

When we are constructing a parametrix for the fundamental solution, we may perform the same construction on the right and the left. We may also add a distribution solving away the right error at  $\text{lcf}_-$ .

Remembering our factors of  $r_{\text{ff}}$  now, we have now proved the following proposition.

**Proposition 13.5.** *Given  $r_1 \in r_{\text{ff}}^{-n/2} \mathcal{A}_{\text{phg}}^{\mathcal{E}_L}(\tilde{X}_0^2)$  and  $r_2 \in r_{\text{ff}}^{-n/2} \mathcal{A}_{\text{phg}}^{E_R}(\tilde{X}_0^2)$  as above, we may find a smooth function  $u \in r_{\text{ff}}^{-n/2} \mathcal{A}_{\text{phg}}^{\mathcal{F}}(\tilde{X}_0^2)$  vanishing outside the light cone such that  $P(\lambda)u - r_1$  vanishes to all orders at  $\text{lcf}_+$  and is polyhomogeneous with index family  $\mathcal{G}_1$ , while  $P(\lambda)_R^t u - r_2$  vanishes to all orders at  $\text{lcf}_-$  and is polyhomogeneous with index set  $\mathcal{G}_2$ . Here we have that*

$$\begin{aligned} F_{\text{lcf}_+} &= G_{\text{lcf}_+,2} = \{(s_\pm(\lambda) + m, 0) : m \in \mathbb{N}_0\}, & F_{\text{lcf}_+} &= E_{\text{lcf}_+,L}, & (21) \\ F_{\text{rf}_+} &= G_{\text{rf}_+,1} = \{(-n + s_\pm(\lambda) + m, 0) : m \in \mathbb{N}_0\}, & F_{\text{lcf}_-} &= E_{\text{lcf}_-,R}, \\ G_{\text{lcf}_-,1} &= F_{\text{lcf}_-} - 1, & G_{\text{lcf}_+,1} &= F_{\text{lcf}_+} + 1, \\ G_{\text{lcf}_+,2} &= F_{\text{lcf}_+} - 1, & G_{\text{rf}_+,2} &= F_{\text{rf}_+} + 1, \\ G_{\text{lcf}_+,1} &= G_{\text{lcf}_-,2} = E_{\text{lcf}_+,L} = E_{\text{rf}_-,L} = E_{\text{lcf}_+,R} = E_{\text{rf}_+,R} = \emptyset, \\ F_{\text{sfc}} &= E_{\text{sfc},L} = E_{\text{sfc},R} = G_{\text{sfc},1} = G_{\text{sfc},2} = \{(-n + j, l) : l \leq j\}, \end{aligned}$$

if  $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$ . If  $s_+ - s_-(\lambda) = N \in \mathbb{Z}$ , then the index sets become

$$\begin{aligned} \tilde{F}_{\text{lcf}_+} &= \{(s_-(\lambda) + m, l) : m \in \mathbb{N}_0, l = 0 \text{ for } m < N, l = 1 \text{ for } m \geq N\}, \\ \tilde{F}_{\text{rf}_+} &= \{(-n + s_-(\lambda) + m, l) : m \in \mathbb{N}_0, l = 0 \text{ for } m < N, l = 1 \text{ for } m \geq N\}, \end{aligned}$$

with corresponding changes for  $G_{\text{lcf}_+}$  and  $G_{\text{rf}_+}$ .

## 14. THE FRONT FACE

We now wish to solve away the errors on the front face, which we again call  $r$ , from the previous step. We show how to solve away the error term at  $\text{ff}_+$  for the operator acting on the left, and the corresponding calculation at  $\text{ff}_-$  for the operator acting on the right is nearly identical. We now suppose that  $r$  is the error term from Proposition 13.5 for the operator acting on the left.

Because  $r$  vanishes to all orders at  $\text{lcf}_+$ , we may blow down  $\text{lcf}_+$  to solve away  $r$ . In this view,  $\tilde{x}^{-p}r$  is smooth on  $\text{ff}_+$ , supported inside the light cone, and has an expansion at  $\text{lf}_+$  of the form

$$r \sim \tilde{x}^p \sum_{(\alpha_j, l) \in \mathcal{G}} s^{\alpha_j} (\log s)^l r_{jl} \mu.$$

The generic case here is that  $l = 0$  when  $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$ . We again drop the powers of  $\tilde{x}$  from our notation.

We wish to solve this error away with a function of the same form:

$$u \sim \sum_{(\alpha_j, l) \in \mathcal{G}} s^{\alpha_j} (\log s)^l u_{jl} \mu.$$

Applying  $N(P)$  to such an ansatz yields

$$\begin{aligned} N(P)u &\sim \sum_j \sum_l s^{\alpha_j} (\log s)^l [\alpha_j^2 u_{jl} + 2\alpha_j(l+1)u_{j,l+1} + (l+1)(l+2)u_{j,l+2} \\ &\quad - (n-1)\alpha_j u_{jl} - (n-1)(l+1)u_{j,l+1} + \Delta_z u_{j-2,l} - \lambda u_{jl}] \nu \\ &= \sum_j \sum_l s^{\alpha_j} (\log s)^l [ -((n-1-\alpha_j)\alpha_j + \lambda) u_{jl} - (n-1-2\alpha_j)(l+1)u_{j,l+1} \\ &\quad + (l+1)(l+2)u_{j,l+2} + \Delta_z u_{j-2,l}] \mu. \end{aligned} \tag{22}$$

The coefficient of  $u_{jl}$  in this expression vanishes precisely when  $\alpha_j = s_{\pm}(\lambda)$ , so we may solve away the error to all orders at  $\text{lf}_+$  because the expansions of  $r$  begin at  $s_{\pm}(\lambda) + 1$ .

**Proposition 14.1.** *For  $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$ , there is a smooth function  $u$  on  $\text{ff}_+$ , polyhomogeneous at  $\text{lf}_+$  with index set  $F_{\text{lf}_+}$ , where  $F_{\text{lf}_+}$  is defined in equation (21), such that  $N(P)u - \tilde{x}^{-p}r$  vanishes to all orders at  $\text{lf}_+ \cap \text{ff}_+$ . An identical statement holds when  $s_+(\lambda) - s_-(\lambda) \in \mathbb{Z}$ , with  $F_{\text{lf}_+}$  replaced by  $\tilde{F}_{\text{lf}_+}$ .*

*Proof.* This is clear from the expression (22) for  $N(P)u$ . □

On the front face, we are then left with an error term  $r_0$  vanishing to all orders at  $\text{lf}_+ \cap \text{ff}_+$  and supported inside the light cone. Because  $\text{ff}_+$  is an asymptotically de Sitter space, we may now use Corollary 3.6 of Vasy ([Vas07]) to find a smooth function on  $\text{ff}_+$ , vanishing to all orders at  $\text{lf}_+$ , such that  $N(P)u = \tilde{x}^{-p}r_0$ .

By iterating this construction (and extending it to the interior of  $\text{lf}_+$  as well), we may find  $u \in \tilde{x}^p \mathcal{A}_{\text{phg}}^{\mathcal{F}_2}(X_0^2; \text{lf}_+)$  such that  $P(\lambda)u - r$  vanishes to all orders at  $\text{ff}_+$  and  $\text{lf}_+$ . By gluing two of these functions together, we may simultaneously solve the left error near the “plus” faces and the right error near the “minus” faces. This proves the following proposition.

**Proposition 14.2.** *We may solve away the left error at  $\text{ff}_+$  and  $\text{lf}_+$ , and the right error at  $\text{rf}_-$  and  $\text{ff}_-$  with a function  $u \in \mathcal{A}_{\text{phg}}^{\mathcal{H}}(\tilde{X}_0^2)$ , where  $\mathcal{H}$  is given by*

$$H_{\text{lcf}_+} = H_{\text{lcf}_-} = \emptyset, H_* = F_* \text{ for the other index sets.}$$



The remaining error terms are in  $\mathcal{A}_{phg}^{\mathcal{K}_L, \mathcal{K}_R}(\tilde{X}_0^2)$ , where  $\mathcal{K}_L$  and  $\mathcal{K}_R$  are given by

$$\begin{aligned} K_{\text{lf}_+, L} &= K_{\text{rf}_-, R} = K_{\text{lcf}_+} = K_{\text{lcf}_-} = \emptyset, & K_{\text{lf}_+, R} &= F_{\text{lf}_+}, \\ K_{\text{rf}_-, L} &= F_{\text{rf}_-}, & K_{\text{scf}, L} &= K_{\text{scf}, R} = H_{\text{scf}}. \end{aligned}$$

## 15. THE FULL PARAMETRIX

We now take the various pieces of the parametrix constructed in sections 11, 12, 13, and 14 to construct a parametrix for the fundamental solution of  $P$ .

Putting together the results of Propositions 11.1, 12.7, 13.5, and 14.2, we have proved the following theorem:

**Theorem 15.1.** *Suppose that  $X$  is an asymptotically de Sitter space, satisfying assumptions (A1), (A2), and (A3). We may find a left parametrix  $K$  such that  $P(\lambda)K = I + R_1$  and  $KP(\lambda) = I + R_2$ , where the Schwartz kernels of  $R_1$  and  $R_2$  are smooth on the interior of  $\tilde{X}_0^2$  and are polyhomogeneous with index families  $\mathcal{E}_L$  and  $\mathcal{E}_R$  on  $\tilde{X}_0^2$ . We may write  $K = K_1 + K_2 + K_3$ , where  $K_1$  is supported in a neighborhood of the diagonal,  $K_2$  is supported in a small neighborhood of the light cone LC away from the diagonal, and all three pieces are supported on the interior of the light cone. Moreover,*

$$\begin{aligned} K_1 &\in r_{\text{ff}}^{-n/2} I_0^{-3/2}(X_0^2; \Lambda_0, \Lambda_1), \\ K_2 &\in r_{\text{ff}}^{-n/2} \mathcal{A}_{phg}^{\mathcal{F}} I^{-3/2}(X_{0,t}^2; \text{LC}), \\ K_3 &\in r_{\text{ff}}^{-n/2} \mathcal{A}_{phg}^{\mathcal{F}}(\tilde{X}_0^2). \end{aligned} \tag{23}$$

The index families  $\mathcal{E}_L$ ,  $\mathcal{E}_R$ , and  $\mathcal{F}$  are given by

$$\begin{aligned} F_{\text{lcf}_+} &= \{(j + \frac{1}{2}, l) : l \leq j, j \in \mathbb{N}_0\}, & F_{\text{lcf}_-} &= F_{\text{lcf}_+} - n, \\ F_{\text{lf}_+} &= E_{\text{lf}_+, R} = \{(s_{\pm}(\lambda) + m, 0) : m \in \mathbb{N}_0\}, & E_{\text{rf}_-, L} &= F_{\text{rf}_-} = F_{\text{lf}_+} - n, \\ E_{\text{lcf}_-, L} &= F_{\text{lcf}_-} - 1, & E_{\text{lcf}_+, R} &= F_{\text{lcf}_+} - 1, \\ F_{\text{scf}} &= E_{\text{scf}, L} = E_{\text{scf}, R} = \{(-n + j, l) : l \leq j\}, \\ E_{\text{ff}_-, L} &= E_{\text{ff}_+, R} = \left\{ \left( -\frac{n}{2} + m, 0 \right) : m \in \mathbb{N}_0 \right\}, \\ E_{\text{lcf}_+, L} &= E_{\text{lcf}_-, R} = E_{\text{lf}_+, L} = E_{\text{rf}_-, R} = E_{\text{ff}_+, L} = E_{\text{ff}_-, R} = \emptyset. \end{aligned}$$

If  $s_+(\lambda) - s_-(\lambda) \in \mathbb{Z}$ , then we must modify  $F_{\text{lf}_+}$  and  $F_{\text{rf}_+}$  (and the index sets depending on them) as described earlier.

*Remark 15.2.* As observed earlier, this theorem holds without the assumption (A3) as long as we are willing to multiply our distribution by a cutoff function supported in a neighborhood of the front face. In this case, the extra blow-up to obtain the scattering face scf is unnecessary.

Because our remainder terms lose one order of decay at the light cone faces, we lose an order of decay there when we pass to the exact fundamental solution.

**Theorem 15.3** (Theorem 1.1). *The exact forward fundamental solution  $E_+$  is in this class of distributions, but with index sets  $F'_{\text{lcf}_+} = F_{\text{lcf}_+} - 1$  and  $F'_{\text{lcf}_-} = F_{\text{lcf}_-} - 1$ . In other words,*

we may write  $E_+ = K_1 + K_2 + K_3$  with

$$\begin{aligned} K_1 &\in r_{\text{ff}}^{-n/2} I_0^{-3/2}(X_0^2; \Lambda_0, \Lambda_1), \\ K_2 &\in r_{\text{ff}}^{-n/2} \mathcal{A}_{\text{phg}}^{\mathcal{F}} I^{-3/2}(X_{0,t}^2; \text{LC}), \\ K_3 &\in r_{\text{ff}}^{-n/2} \mathcal{A}_{\text{phg}}^{\mathcal{F}}(\tilde{X}_0^2). \end{aligned} \tag{24}$$

Here the index family  $\mathcal{F}$  is given by

$$\begin{aligned} F_{\text{lcf}_+} &= \{(j - \frac{1}{2}, l) : l \leq j, j \in \mathbb{N}_0\}, & F_{\text{lcf}_-} &= F_{\text{lcf}_+} - n, \\ F_{\text{lf}_+} &= \{(s_{\pm}(\lambda) + m, 0) : m \in \mathbb{N}_0\}, & F_{\text{rf}_-} &= F_{\text{lf}_+} - n, \\ F_{\text{scf}} &= \{(-n + j, l) : l \leq j\}, \end{aligned}$$

with modifications to  $F_{\text{lf}_+}$  and  $F_{\text{rf}_-}$  when  $s_+(\lambda) - s_-(\lambda)$  is an integer.

*Proof.*  $E_+$  is the forward fundamental solution, so if  $f$  is a compactly supported smooth function on  $X$ , then  $P(\lambda)E_+f = f$  and  $E_+P(\lambda)f = f$ . Moreover, continuity allows us to extend this to any forward-oriented distribution. In particular, if  $f$  is any smooth function on the interior of  $X$  vanishing to all orders at  $Y_-$  that is also a tempered distribution on  $X$ , then

$$\begin{aligned} P(\lambda)E_+f &= f \\ E_+P(\lambda)f &= f. \end{aligned}$$

Let  $K$  be the parametrrix for  $E_+$  constructed in Theorem 15.1. If  $f$  is a smooth function on  $X$ , vanishing to all orders at  $Y$ , then  $Kf$  vanishes to all orders at  $Y_-$  because  $K$  is identically zero in a neighborhood of  $\text{lf}_-$  and the lift of  $f$  vanishes to all orders at  $\text{lcf}_-$ ,  $\text{ff}_-$ ,  $\text{scf}$ , and  $\text{rf}_-$ . A similar argument applies to  $R_1f$  and  $R_2f$ . We may then write

$$\begin{aligned} (Kf) &= E_+P(\lambda)Kf = E_+f + E_+R_1f \\ (Kf) &= KP(\lambda)E_+f = E_+f + R_2E_+f. \end{aligned}$$

In particular,

$$E_+ = K - KR_1 + R_2E_+R_1.$$

We then observe that the error terms  $KR_1$  and  $R_2E_+R_1$  have the desired properties, finishing the proof.  $\square$

## 16. THE EXACT DE SITTER SPACE

We now briefly discuss the modifications to this construction for the exact de Sitter space. The most important modification is to solve the transport equations on  $\tilde{X}_0^2$  rather than  $X_{0,t}^2$ . (Indeed, we could have done this from the outset, but chose not to because solutions of the transport equations are easier to understand on  $X_{0,t}^2$ .)

Observe that the construction detailed above works without difficulty away from the corner  $\text{lf}_+ \cap \text{rf}_-$ . As mentioned earlier, LC intersects itself in this corner. This intersection is given by  $\{(0, y, 0, -y) : y \in Y\}$ . Near this intersection, we may write  $\rho = x + \tilde{x} - |y + \tilde{y}|$ , plus terms vanishing to higher order at the boundary. Note that the function  $\rho$  is no longer smooth at this intersection because  $|y + \tilde{y}| = 0$  there.

We resolve this singularity by blowing up the submanifold  $\text{LC} \cap \text{lfc}_+ \cap \text{rf}_-$  as in Definition 9.2. Although this submanifold had codimension 3 when  $X$  satisfied (A3), it has codimension  $n + 1$  here. Indeed, this blow-up is almost the same as the one in Section 4 that defined  $X_0^2$ . The function  $\rho$  now lifts to be smooth on this new space  $[X_0^2, \text{LC} \cap \text{lfc}_+ \cap \text{rf}_-]$ . Indeed, after the blow-up we may write

$$\rho = (1 + s) - |z| + O(\tilde{x}) \quad (25)$$

near  $\text{scf} \cap \text{lfc}_+$ . This is now smooth near  $\rho = 0$ , and so we may blow up  $\text{LC} \cap \text{lfc}_+$  and  $\text{LC} \cap \text{rf}_-$  to obtain  $\tilde{X}_0^2$ .

We must also modify the manner in which we solve the transport equations. The behavior at  $\text{lfc}_+$  and  $\text{lfc}_-$  may be obtained in the same way as in Section 12, but the behavior at  $\text{scf}$  requires a slightly different approach. In Section 9, we showed that the symbol of the conormal distribution was polyhomogeneous at  $\text{scf}$  by constructing it on  $X_{0,t}^2$  and lifting it to  $\tilde{X}_0^2$ . Because  $\rho$  is not smooth on  $X_0^2$  when  $X$  is the de Sitter space, we cannot solve the transport equations on the intermediate double space  $X_{0,t}^2$  up to the corner and instead we must solve the transport equation along  $\text{lfc}_-$  and  $\text{lfc}_+$ .

Solving the transport equation on these faces requires using the semi-explicit form of  $\rho$ . The terms where the operator lands entirely on  $\rho$  in equation (19) can no longer be ignored. Using the form (25) for  $\rho$ , we observe that these now contribute a constant term to the equation. Because  $\square\rho = s - (n - 1)s + s^2(n - 1) + O(\tilde{x})$  and  $\partial_s\rho = 1 + O(\tilde{x})$ ,

A computation in the same spirit as those in Section 12 shows that the symbol of  $K_2$  is also polyhomogeneous at  $\text{scf}$  with index set

$$F_{\text{scf}} = \{(-n + m, 0) : m \in \mathbb{N}_0\}.$$

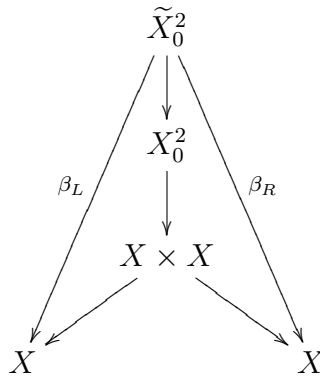
Note that this is the same index set we found before.

The rest of the construction proceeds without change.

## 17. POLYHOMOGENEITY

The aim of this section is to prove Theorem 1.2.

We begin by considering the maps  $\beta_L$  and  $\beta_R$ , where  $\beta_{L,R}$  are given by composing the blow-down maps with projections onto each factor, as in the diagram.



We require the following four lemmas.

**Lemma 17.1.** *The maps  $\beta_L$  and  $\beta_L|_{\text{LC}^{\text{int}}}$  are  $b$ -fibrations.*

**Lemma 17.2.** *The fibers of  $\beta_L$  and  $\beta_L : \text{LC}^{\text{int}} \rightarrow X$  are transverse to  $\text{LC}$ .*

**Lemma 17.3.** *For  $V$  a b-vector field on  $X$ , there is a b-vector field  $\tilde{V}$  on  $\tilde{X}_0^2$  such that  $(\beta_L)_*\tilde{V} = V$  and  $\tilde{V}$  is tangent to LC.*

**Lemma 17.4.** *Suppose that  $M$  and  $N$  are manifolds with corners,  $F : M \rightarrow N$  is a b-fibration,  $H$  is a boundary hypersurface of  $M$ , and  $F(H) = N$ . Suppose also that  $K \in I^0(M; H)$  is polyhomogeneous at the other boundary hypersurfaces of  $M$  (satisfying the hypotheses of Lemma 5.1). Then  $F_*K$  is a polyhomogeneous distribution on  $N$ .*

*Proof of Lemma 17.1.* Here  $\text{LC}^{int}$  is taken as an open manifold away from the cone edge near the diagonal.  $K_2$  is supported away from the cone point, so we may restrict our attention to this region.

That  $\beta_L$  and  $\beta_L|_{\text{LC}^{int}}$  are b-fibrations (defined in Section 5) follows from a more general statement: if  $F : M \rightarrow N$  is a b-fibration, and  $Z \subset \partial M$  is a p-submanifold, then  $F \circ \beta : [M; Z] \rightarrow N$  is a b-fibration. In particular, the blow-down map is a composition of b-fibrations.  $\square$

*Proof of Lemma 17.2.* Because  $\Lambda_1$  is the flowout Lagrangian in the right factor, its intersection with the fibers  $\tilde{x}s = x_0$ ,  $\tilde{y} + \tilde{x}z = y_0$  is the flowout of the light cone with cone point at  $(x_0, y_0)$ . In particular, this is an embedded submanifold of  $X$ , and so the intersection is transverse.  $\square$

*Proof of Lemma 17.3.* This is really a consequence of Lemma 17.2. Indeed, we may choose a local basis of vector fields given by  $\partial_\nu$  and  $V_j$ , where  $\partial_\nu$  is tangent to the fibers of  $\beta_L$  and  $V_j$  are tangent to LC. Then  $(\beta_L)_*\partial_\nu = 0$  because  $\partial_\nu$  is tangent to the fibers of  $\beta_L$ , and so we may choose a lift of the vector fields so that the  $\partial_\nu$  component vanishes at LC.  $\square$

*Proof of Lemma 17.4.* We choose  $K^\epsilon$  polyhomogeneous on  $M$ , supported away from  $H$  such that  $K^\epsilon$  are uniformly bounded in  $I^0(M; H)$  and converge to  $K$  in  $I^\delta(M; H)$  for any  $\delta > 0$ . The pushforward theorem of Melrose (Lemma 5.1) tells us that  $F_*K^\epsilon$  are polyhomogeneous with fixed index set.

We claim now that  $F_*K^\epsilon$  are Cauchy and so converge to  $F_*K$ . This guarantees that  $F_*K$  is polyhomogeneous. The key observation here is that if  $\tilde{V}$  is a lift of  $V$ , then

$$V(F_*(K^{\epsilon_1} - K^{\epsilon_2})) = F_*\left(\tilde{V}(K^{\epsilon_1} - K^{\epsilon_2})\right).$$

The expression  $\tilde{V}(K^{\epsilon_1} - K^{\epsilon_2})$  tends to 0 because  $K^\epsilon \rightarrow K$  as distributions conormal to  $H$ .  $\square$

*Proof of Theorem 1.2.* A simple wavefront set argument gives us that the  $K_2f$  piece is smooth.

Thinking functorially, we write

$$K(v\gamma) = (\beta_L)_*(K \cdot \beta_R^*(v\gamma)) = u\gamma.$$

In particular, if we write  $K = \tilde{K}\nu$ , then

$$u\gamma^2 = u |dx dy| = (\beta_L)_*\left(\tilde{K}(\beta_R^*v)r_{\text{ff}}^{n/2}(r_{\text{lcf}_+}r_{\text{lcf}_-})^{1/2}r_{\text{sct}}\nu^2\right).$$

We now use the decomposition  $K = K_1 + K_2 + K_3$ , where  $K_i$  are as in Theorem 15.3. The lemma above shows that  $\beta_L$  is a b-fibration, and so we may use the pushforward theorem of Melrose in Lemma 5.1 to treat the contribution from  $K_3$ . In particular, we note that if  $v$  has index set  $E$  at  $Y_+$  and vanishes to all orders at  $Y_-$ , then  $\beta_R^*v$  is polyhomogeneous on

$\tilde{X}_0^2$ , smooth at  $\text{lef}_+$  and  $\text{lf}$ , with index set  $E$  at  $\text{ff}_+$ , and vanishing to all orders at  $\text{ff}_-$ ,  $\text{lef}_-$ , and  $\text{scf}$  because it is forward-directed. Index sets add when functions are multiplied, and so we know that  $r_{\text{ff}}^{n/2}(r_{\text{lef}_+}r_{\text{lef}_-})^{1/2}r_{\text{scf}}\tilde{K}_3 \cdot \beta_R^*f$  has index family  $\mathcal{G}$  given by

$$\begin{aligned} G_{\text{ff}_+} &= E, & G_{\text{ff}_-} &= \emptyset \\ G_{\text{lf}_+} &= \{(s_{\pm}(\lambda) + m, 0) : m \in \mathbb{N}_0\}, & G_{\text{lf}_-} &= \emptyset \\ G_{\text{rf}_-} &= \emptyset, & G_{\text{rf}_+} &= \emptyset \\ G_{\text{lef}_+} &= \{(j, l) : l \leq j, j \in \mathbb{N}_0\}, & G_{\text{lef}_-} &= \emptyset \\ G_{\text{scf}} &= \emptyset. \end{aligned}$$

$G_{\text{lf}_+}$  must be modified when  $s_+(\lambda) - s_-(\lambda)$  is an integer. We may now use Lemma 5.1 to conclude that  $K_3f$  is polyhomogeneous on  $X$  with index set

$$E \bar{\cup} G_{\text{lf}_+} \bar{\cup} G_{\text{lef}_+}.$$

To handle the contribution from  $K_2$ , we use the lemmas above. Indeed, Lemmas 17.1, 17.3, and 17.4 show that  $K_2f$  is polyhomogeneous with index set  $G_{\text{lef}_+} \bar{\cup} E$ .

This leaves only the contribution from  $K_1$ . This is just a consequence of the local theory of paired Lagrangian distributions. Indeed, the work of Joshi in [Jos98] implies that the pushforward of  $K_1\beta_R^*f$  exists as a smooth function. The uniformity of  $K_1$  down to the front face  $\text{ff}$  then tells us that  $K_1f$  is polyhomogeneous with index set  $E$ .  $\square$

## 18. AN $L^p$ ESTIMATE

As an application of our parametrix, we consider the behavior of the  $L^p$  norms of a family of smooth compactly supported functions with support tending towards  $Y_+$ .

Suppose first that  $R \in \Psi_0^{-\infty}(X)$  is a smoothing 0-pseudodifferential operator in the small calculus of [MM87] or [Maz88], supported near  $\text{ff}_+$ . In other words, the Schwartz kernel of  $R$  is a smooth function on  $X_0^2$ , supported away from  $\text{lf}$  and  $\text{rf}$  and near  $\text{ff}_+$ . Concretely, let  $\phi$  is a smooth, compactly supported function on  $\mathbb{R}_+^n$ , supported near  $(1, 0)$ , and with  $\int_0^\infty \int_{\mathbb{R}^{n-1}} \phi \frac{ds dz}{s^n} = 1$ . Let  $\chi$  be a smooth function on  $X$  that is identically zero near  $Y_-$ . If  $R(s, z, \tilde{x}, \tilde{y}) = \phi(s, z)\chi(\tilde{x}, \tilde{y})$ , then  $R \in \Psi_0^{-\infty}(X)$  is such an operator.

Define now a family of compactly supported functions  $f$  given by

$$f_{(\tilde{x}, \tilde{y})}(x, y) = R \left( \frac{x}{\tilde{x}}, \frac{y - \tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y} \right) = Rv,$$

where  $v = \delta(x' - \tilde{x})\delta(y' - \tilde{y})$ . Each  $f_{(\tilde{x}, \tilde{y})}$  is a smooth function on  $X$  supported in a compact neighborhood of  $(\tilde{x}, \tilde{y})$  with unit  $L^1(X; dg)$ -norm.

Because  $f_{(\tilde{x}, \tilde{y})}$  is given by applying  $R$  to a  $\delta$  function, pointwise bounds for  $E_+f_{(\tilde{x}, \tilde{y})}$  are equivalent to pointwise bounds on the Schwartz kernel of  $E_+R$ .

The following lemma is useful for obtaining pointwise bounds.

**Lemma 18.1.** *Suppose that  $K = \hat{K}\nu$ , where  $K$  is a polyhomogeneous function on  $\tilde{X}_0^2$  with index family  $\mathcal{F}$  and supported near  $\text{ff}_+$ . Suppose that  $\mathcal{F}$  satisfies*

$$\begin{aligned} F_{\text{ff}_+} &= \{(s_{\text{ff}} + j, p) : j \in \mathbb{N}_0, p \leq p_j\}, \\ F_{\text{lcf}_+} &= \{(s_{\text{lcf}} + j, p) : j \in \mathbb{N}_0, p \leq p_j\}, \\ F_{\text{lrf}_+} &= \{(s_{\text{lrf}} + j, p) : j \in \mathbb{N}_0, p \leq p_j\}, \\ F_{\text{rf}_+} &= \{(s_{\text{rf}} + j, p) : j \in \mathbb{N}_0, p \leq p_j\}, \end{aligned}$$

and  $p_0 = 0$  for each index set (i.e., no log terms appear in the top order part of the expansion). Suppose further that

$$s_{\text{ff}} \geq \frac{n}{2}, \quad s_{\text{lcf}} \geq \frac{1}{2}, \quad s_{\text{lrf}} \geq 0, \quad s_{\text{rf}} \geq 0.$$

Then  $K$  (considered as an operator) satisfies

$$\|Ku\|_{L^\infty(X)} \leq C \|u\|_{L^1(X; d\hat{g})}.$$

If  $s_{\text{ff}} \geq -\frac{n}{2}$  instead, then  $K$  is bounded  $L^1(X; dg) \rightarrow L^\infty(X)$ .

*Proof.* The half-density  $|d\hat{g}_L d\hat{g}_R|^{1/2}$  lifts to a nonvanishing smooth multiple of  $r_{\text{lcf}_+}^{1/2} r_{\text{ff}_+}^{n/2} \nu$  near  $\text{ff}_+$ , so we may write

$$\hat{K}\nu = \hat{K} r_{\text{lcf}_+}^{-1/2} r_{\text{ff}_+}^{-n/2} \beta^* \left( |d\hat{g}_L d\hat{g}_R|^{1/2} \right),$$

where  $\beta^*$  is the pullback by the blow-down map  $\tilde{X}_0^2 \rightarrow X \times X$ . The assumptions above mean that  $\hat{K} r_{\text{lcf}_+}^{-1/2} r_{\text{ff}_+}^{-n/2}$  is a bounded function on  $\tilde{X}_0^2$  and so is the pullback of a bounded function on  $X \times X$ .

The second statement of the lemma follows from the observation that  $d\hat{g} = x^n dg$ .  $\square$

Let  $K_1, K_2$ , and  $K_3$  be the decomposition of  $E_+$  given in Theorem 15.1. The Schwartz kernel of  $K_j R$  is given by

$$\int K_j \left( \frac{x}{s'\tilde{x}}, \frac{y - (\tilde{y} + \tilde{x}z')}{s'\tilde{x}}, s'\tilde{x}, \tilde{y} + \tilde{x}z' \right) R(s', z', \tilde{x}, \tilde{y}) ds' dz' \quad (26)$$

where  $s' = \frac{x'}{\tilde{x}}$  and  $z' = \frac{y' - \tilde{y}}{\tilde{x}}$ . This corresponds to writing the composition of operators  $A$  and  $B$  on  $X \times X$  as

$$\kappa_{AB}(x, y, \tilde{x}, \tilde{y}) = \int \kappa_A(x, y, x', y') \kappa_B(x', y', \tilde{x}, \tilde{y}) dx' dy',$$

where  $\kappa_A$  here denotes the Schwartz kernel of  $A$ . If  $\kappa_A$  and  $\kappa_B$  are instead functions of  $s, z, \tilde{x}, \tilde{y}$ , where  $s = \frac{x}{\tilde{x}}$  and  $z = \frac{y - \tilde{y}}{\tilde{x}}$ , this becomes

$$\begin{aligned} & \int \kappa_A \left( \frac{x}{x'}, \frac{y - y'}{x'}, x', y' \right) \kappa_B \left( \frac{x'}{\tilde{x}}, \frac{y' - \tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y} \right) dx' dy' \\ &= \int \kappa_A \left( \frac{x}{s'\tilde{x}}, \frac{y - (\tilde{y} + \tilde{x}z')}{s'\tilde{x}}, s'\tilde{x}, \tilde{y} + \tilde{x}z' \right) \kappa_B(s', z', \tilde{x}, \tilde{y}) ds' dz', \end{aligned}$$

which yields equation (26).

Note that  $K_1, K_2$ , and  $K_3$  all vanish identically near  $\text{rf}_+$ . Because the family  $f_{(\tilde{x}, \tilde{y})}$  is supported away from  $Y_-$ , we may assume that  $K_i$  are supported away from the ‘‘minus’’ faces  $\text{ff}_-, \text{lcf}_-, \text{rf}_-$ , and  $\text{scf}_-$ .

Consider first  $K_1$ , the piece corresponding to the paired Lagrangian singularity of  $E_+$ . By the theorem, the Schwartz kernel of  $K_1$  is given by  $(x')^{-n/2}\tilde{K}_1$ , where  $\tilde{K}_1$  is a paired Lagrangian distribution on  $X_0^2$  supported away from lf and rf.

Because the fibers of integration in equation (26) are transverse to the diagonal and to the light cone,  $K_1 R = \tilde{x}^{-n/2}u_1$ , where  $u_1$  is a smooth function on  $X_0^2$  supported away from lf and rf. In particular,  $u_1$  is bounded, and Lemma 18.1 implies that

$$\|K_1 f(\tilde{x}, \tilde{y})\|_{L^\infty} \leq C.$$

Suppose now that  $r'_{\text{lcf}_+} = r_{\text{lcf}_+}(x, y, x', y')$ , i.e.,  $r'_{\text{lcf}_+}$  is the defining function of  $\text{lcf}_+$  expressed in terms of  $x, y$  and the intermediate variables  $x', y'$ . Call  $r_{\text{lcf}_+}$  and  $r_{\text{lf}_+}$  the defining functions in terms of  $x, y, \tilde{x}, \tilde{y}$ , i.e., after the composition. The key observation in bounding  $K_2$  is that  $r_{\text{lcf}_+} r_{\text{lf}_+} \leq r'_{\text{lcf}_+}$ . In particular,  $(r'_{\text{lcf}_+})^{1/2} r_{\text{ff}_+}^{n/2} K(x, y, x', y')$  is a conormal distribution with bounded symbol on  $\tilde{X}_0^2$ , and so the Schwartz kernel of  $(r_{\text{lcf}_+} r_{\text{lf}_+})^{1/2} r_{\text{ff}_+}^{n/2} K_2 R$  is bounded on  $\tilde{X}_0^2$ . The Schwartz kernel of  $r_{\text{lcf}_+} r_{\text{lf}_+} K_2 R$  thus satisfies the conditions of the lemma and so

$$\|K_2 f(\tilde{x}, \tilde{y})\|_{x^{-2}L^\infty(X)} \leq C.$$

Consider finally the polyhomogeneous term  $K_3$ . Define  $l_\pm(\lambda)$  by

$$l_\pm(\lambda) = \frac{n-1}{2} \pm \Re \sqrt{\frac{(n-1)^2}{4} + \lambda}.$$

The Schwartz kernel of  $r_{\text{lf}_+}^{-l_-(\lambda)} r_{\text{lcf}_+}$  satisfies the conditions of the lemma, and so

$$\|K_3 f(\tilde{x}, \tilde{y})\|_{x^{l_-(\lambda)-1}L^\infty(X)} \leq C.$$

Now let  $l = \max(1 - l_-(\lambda), 2)$ . Putting the estimates for  $K_1, K_2$ , and  $K_3$  together yields

$$\|E_+ f(\tilde{x}, \tilde{y})\|_{x^{-l}L^\infty(X)} \leq C.$$

An  $L^2$  estimate for this family follows from Theorem 3.1 (from [Vas07]). By reversing the roles of  $Y_-$  and  $Y_+$  in this theorem, we conclude that for any forward-directed  $f \in x^{-r}L^2(X)$  and sufficiently large  $r$ , there is a unique  $u \in x^{-r}H_0^1(X)$  such that  $P(\lambda)u = f$ , and

$$\|u\|_{x^{-r}L^2(X; dg)} \leq \|u\|_{x^{-r}H_0^1(X)} \leq C \|f\|_{x^{-r}L^2(X; dg)}.$$

Here  $H_0^1$  is the 0-Sobolev space of order one, i.e., it measures regularity with respect to the  $x\partial_x$  and  $x\partial_y$  vector fields.

In order to apply this estimate to the family  $f(\tilde{x}, \tilde{y})$ , it is important to understand how the  $L^2$  norms of the functions vary. Indeed, a simple calculation shows that

$$\|f(\tilde{x}, \tilde{y})\|_{x^{-r}L^2(X)}^2 = \int_X |f(\tilde{x}, \tilde{y})(x, y)|^2 \frac{dx dy}{x^{n+r}} = \int |\phi(s, z)|^2 \frac{ds dz}{s^{n+r}\tilde{x}^r} = \tilde{x}^{-r} C_r^2,$$

where  $C_r$  depends on  $r$ , but not on  $\tilde{x}$  or  $\tilde{y}$ . In particular,  $\|f\|_{x^{-r}L^2(X; dg)} = \tilde{x}^{-r/2} C_r$ .

We may now prove Theorem 1.3.

*Proof.* Interpolating between the  $L^\infty$  and  $L^2$  estimates provides an  $L^p$  estimate for  $p \in (2, \infty)$ . Indeed, if  $r > \max(\frac{1}{2}, l_+(\lambda))$ ,  $l = \max(1 - l_-(\lambda), 2)$ , and  $\frac{1}{p} = \frac{\theta}{2}$ ,  $\theta \in [0, 1]$ , then

$$\|E_+ f(\tilde{x}, \tilde{y})\|_{x^{-r\theta-l(1-\theta)}L^{2/\theta}(X; dg)} \leq C \tilde{x}^{-r\theta}, \quad (27)$$

which finishes the proof.  $\square$

*Remark 18.2.* The proof of Theorem 1.3 uses the inclusion  $H_0^1 \subset L^2$ . In particular, we ignore one derivative of  $E_+f$  and so we could modify equation (27) to include a fractional derivative.

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