PRICE'S LAW FOR THE MASSLESS DIRAC–COULOMB SYSTEM

DEAN BASKIN, JESSE GELL-REDMAN, AND JEREMY L. MARZUOLA

Abstract. We consider the pointwise decay of solutions to wave-type equations in two model singular settings. Our main result is a form of Price's law for solutions of the massless Dirac–Coulomb system in $(3+1)$-dimensions. Using identical techniques, we prove a similar theorem for the wave equation on Minkowski space with an inverse square potential. One novel feature of these singular models is that solutions exhibit two different leading decay rates at timelike infinity in two regimes, distinguished by whether the spatial momentum along a curve which approaches timelike infinity is zero or non-zero. An important feature of our analysis is that it yields a precise description of solutions at the interface of these two regions which comprise the whole of timelike infinity.

1. Introduction

We consider the long-time asymptotics of solutions of the wave equation on backgrounds with scale invariant singular potentials and provide pointwise decay estimates in all asymptotic regimes. The models we consider here are those of the massless Dirac–Coulomb equation, and of the wave equation with an inverse square potential, the former equation was considered by Baskin and Gell-Redman (with Booth) [BBGR21] and the latter by Baskin and Marzuola [BM22]. In both instances, in a departure from standard formulations of Price's law, we find that solutions exhibit different leading rates of decay depending on how timelike infinity is approached.

For the Dirac–Coulomb equation, the microlocal framework in prior work [BBGR21] produces regions of timelike infinity at which solutions exhibit distinct decay rates. Indeed, the regions of timelike infinity arise as boundary hypersurfaces of an appropriate spacetime compactification; that work [BBGR21] provides a global description of solutions on a closely related compactification. This requires a clear understanding of the nature of solutions (namely in terms of polyhomogeneity, which they possess in certain regions of spacetime) extracted from [BBGR21]. We show, moreover, that the closely related description of solutions to wave equations in conic setting [BM22] fits into the same framework and thus the solutions to the wave equation with inverse square potentials have the same feature.

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We consider first the massless Dirac–Coulomb equation. Thus, let $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ a solution to

$$\left(\gamma^0 \left(\partial_t + \frac{iZ}{r}\right) + \sum_{j=1}^{3} \gamma^j \partial_j\right) \psi = 0, \tag{1}$$
$$\psi(0) = \psi_0 \in (C_c^\infty(\mathbb{R}^3 \setminus \{0\}))^4,$$

for $\gamma^{0,1,2,3}$ the gamma (or Dirac) matrices. In other words, $\gamma^\alpha \in \mathrm{Mat}(4; \mathbb{C})$ satisfy

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2\eta^{\alpha\beta} \,\mathrm{Id}_4,$$

where $\eta^{\alpha\beta}$ are the components of the Minkowski metric, i.e.,

$$\eta^{\alpha\beta} = \begin{cases} -1 & \alpha = \beta = 0, \\ 1 & \alpha = \beta \in \{1, 2, 3\}, \\ 0 & \alpha \neq \beta. \end{cases}$$

Here
charge is a real constant with $|\mathbf{Z}| < \frac{1}{2}$; this range allows the use of the Hardy inequality and ensures the essential self-adjointness of the underlying Hamiltonian.

We also consider the wave equation on $\mathbb{R} \times \mathbb{R}^n$, $n \geq 3$, with an inverse square potential. That is, we take u a solution to

$$\partial_t^2 u - \Delta u + \frac{\mathsf{F}}{r^2} u = 0, \tag{2}$$
$$(u, \partial_t u)|_{t=0} = (\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n \setminus \{0\}) \times C_c^\infty(\mathbb{R}^n \setminus \{0\}),$$

where Δ is the negative definite Laplacian on $\mathbb{R}^n$ and

$$\mathsf{F} > -\frac{(n-2)^2}{4}.$$

Due to the assumption on F, the Hardy inequality implies that the form $\langle (-\Delta + \frac{\mathsf{F}}{r^2})\phi, \phi \rangle$ is positive definite for $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. For those values of n and F for which the Hamiltonian fails to be essentially self-adjoint, we use the Friedrichs extension.

In three dimensions, our main result is the following:

Theorem 1.1. *Fix $\chi \in C_c^\infty(\mathbb{R}^3 \setminus 0)$. For ψ the solution of the Dirac–Coulomb system* (1) *and u the solution of the inverse square wave equation* (2) *with $n = 3$, we have, as $t \to \infty$,*

$$\|\chi(\cdot)\psi(t, \cdot)\| \lesssim \langle t \rangle^{-3-\alpha(\mathbf{Z})}, \quad \|\chi(\cdot)u(t, \cdot)\| \lesssim \langle t \rangle^{-2-\beta(\mathsf{F})},$$

where the exponents are given in terms of $\mathbf{Z}$ and F by

$$\alpha(\mathbf{Z}) = 2\sqrt{1 - \mathbf{Z}^2} - 2,$$

and

$$\beta(\mathsf{F}) = \sqrt{1 + 4\mathsf{F}} - 1$$

provided $\sqrt{1 + 4\mathsf{F}}$ is not an odd integer. If $\sqrt{1 + 4\mathsf{F}}$ is an odd integer, then $\beta(\mathsf{F})$ is $\sqrt{9 + 4\mathsf{F}} - 1$.

In contrast, for $0 < \gamma < 1$, the solutions obey the estimates

$$|\psi(t, \gamma t, \theta)| \lesssim t^{-3-\frac{\alpha(\mathbf{Z})}{2}}, \quad |u(t, \gamma t, \theta)| \lesssim t^{-2-\frac{\beta(\mathsf{F})}{2}}.$$

Moreover, all decay rates above are sharp for $\mathbf{Z}, \mathsf{F} \neq 0$ in the permitted range (see Section 3.1).

We emphasize that the main novelty of Theorem 1.1 is the first part; the latter estimates (for $0 < \gamma < 1$) were already known in our prior works [BM22, BBGR21]. Thus we see that *the rates of decay in the two regimes differ*; one holds on bounded spatial sets (here the support of the arbitrary compactly supported function χ) and the other holds on timelike trajectories with nonzero spatial momentum. Note that as the parameters $\mathbf{Z}$ and $\digamma$ vary farther from 0, the contrast between these two behaviors grows. For Dirac–Coulomb, the behavior on compact subsets always decays more slowly than in the $\gamma > 0$ case as $\alpha(\mathbf{Z}) < 0$. In contrast, for the inverse square wave equation, the sign of $\digamma$ determines in which region the solution's decay is faster.

We also note that the results above naturally invite a comparison with t^{-3} (for Dirac–Coulomb) and t^{-2} (for inverse square) rates. These particular rates may seem unnatural, as Price's law would suggest rates of t^{-4} and t^{-3}, respectively, for large classes of smooth perturbations of Minkowski space when $\mathbf{Z} = 0$ or $\digamma = 0$. In the context of these families of equations, however, the rates predicted by Price's law are the outlier and arise from a significant cancellation related to Huygens' principle. A brief discussion of this discrepancy (and related ones) can be found in Section 3 below as well as the first author's prior work [BVW18]. We further refer the reader to the works of Tataru [Tat13] and Morgan [Mor20] for an overview of the t^{-3} decay rate in the asymptotically Minkowski setting.

Remark 1.2. Here we work with the explicit operators of the form $\partial_t^2 - \Delta + \frac{\digamma}{r^2}$, but it is natural to ask about equations of the form $\Box_g + \frac{\digamma}{r^2} + V$ for g an asymptotically flat metric and V a non-singular, lower order perturbation. The effects of these perturbations would most strongly manifest at low frequencies, and hence we conjecture that the decay rates we prove here would still be the expected behavior. Proving this would require adapting the second microlocalization tools of Vasy [Vas21a, Vas21c, Vas21b] as implemented in the wave decay setting in [Hin22] to the conic setting, and is an important topic for future work.

As observed in the previous works of the first author, Vasy, and Wunsch [BVW15, BVW18], the pointwise decay rates of waves are in some contexts governed by the asymptotics of the associated radiation field. For wave equations on both Minkowski and a class of asymptotically Minkowski spaces, the corresponding decay estimate in a neighborhood of null infinity is often what is referred to when one discusses Price's Law. The uniformity of the estimate across timelike infinity there can be interpreted as a consequence of the smoothness of the operator on the radial compactification. In contrast, the singular potentials considered here exhibit a different rate of decay at precisely the region of timelike infinity corresponding to the location of the singularity.

There is a plentitude of recent results on Price's Law in the setting of stationary spacetimes. We refer to Schlag [Sch21] for a more thorough overview than we provide here. Such work goes back at least to the analysis using spherical harmonic decompositions in Donninger–Schlag–Soffer [DSS11] and Schlag–Soffer–Staubach [SSS10a, SSS10b], followed by the work of Tataru [Tat13] that relied upon Fourier localization techniques. The methods of Tataru have now been generalized to a large number of settings [MTT12, MTT17]; in particular, the role of regularity of the metric at infinity was recently studied in the work of Morgan [Mor20] and Morgan–Wunsch [MW21]. The work of Looi [Loo22] considers a closely related problem to those considered by Morgan and Morgan–Wunsch. The results of Hintz [Hin22] also employ microlocal techniques and prove a sharp decay rate for solutions of the wave equation with a smooth decaying potential, albeit in a substantially more geometrically complex setting.

Hintz's work requires that the potential decay at least as fast as r^{-3}, i.e., strictly faster than the decay assumed in our paper. A pointwise decay estimate for the wave equation with slowly decaying potential in one spatial dimension was proved by Donninger–Schlag in [DS10], though their result has regular potentials at $r = 0$ that decay slightly better than $|r|^{-2}$ as $|r| \to \infty$. The result we prove here for inverse square potentials is already in the literature (with substantially different proofs). We refer the reader to the recent work of Gajic [Gaj22], which obtains the same bimodal decay estimate for the inverse square potential as well as its analogues on black hole backgrounds. Gajic's result builds upon the study of late time asymptotics for the wave equation beginning in the works [AAG18a, AAG18b, AAG20]. See also a forthcoming work of Van de Moortel and Gajic cited in [Gaj22] addressing the Minkowski setting. However, the authors are aware of no such results for the massless Dirac–Coulomb equation.

For nonlinear applications, dispersive estimates for wave and Schrödinger equations have been studied quite broadly. For instance, the wave equation with an inverse square potential has been studied in [BPSTZ03, PSTZ03, MZZ13]. The Schrödinger equation with an inverse square potential has been studied in [Duy07, KMV$^+$18, KMV$^+$17] among many others. In particular, the distinction between $\digamma > 0$ and $\digamma < 0$ arises and there is actually a threshold $-1/4 + 1/25 < \digamma^* < 0$ for which local well-posedness theory holds for critical semilinear equations with such potentials. In the long run, we hope that tighter control on pointwise behavior of waves will give insight into further nonlinear applications, see [KMVZ17, AM21] for the most recent literature discussion on this problem to date.

The rest of the paper proceeds as follows. In Section 2, we recall the relevant compactification of our spacetimes and discuss the necessary preliminary analytic results. Section 3 recalls the our main results from prior work and proves the constituent results of the main theorem (including the n-dimensional version for the wave equation with an inverse square potential). Finally, in Appendix A, we provide the resonance calculations for the case of inverse square potentials providing the explicit decay rates in Section 3, the analogous resonances for Dirac-Coulomb already being available from [BBGR21].

2. Compactifications and partial polyhomogeneity

We treat both the Dirac–Coulomb equation and the wave equation with inverse square potential as if they were smooth operators on a manifold with a conic singularity. In particular, as $\{0\} \subset \mathbb{R}^n$ is the singular locus of the potential, we view $\mathbb{R}^n \setminus \{0\}$ as the interior of the infinite Riemannian cone over the sphere $\mathbb{S}^{n-1}$.

We now construct a compactification of the overall spacetime $\mathbb{R}_t \times (\mathbb{R}^n \setminus \{0\}) = \mathbb{R}_t \times (0, \infty)_r \times \mathbb{S}^{n-1}$, treating it exactly as we would the product of a real time axis $\mathbb{R}_t$ with an arbitrary infinite cone $(0, \infty)_r \times Z$. We start by compactifying $\mathbb{R}_t \times (0, \infty)_r$ (and thus $\mathbb{R}_t \times (0, \infty)_r \times Z$) by stereographic projection to a quarter-sphere $\mathbb{S}^2_{++}$ as depicted in Figure 1. In other words, the map $\mathbb{R}_t \times (0, \infty)_r \to \mathbb{S}^2 \subset \mathbb{R}^3$ given by

$$(t, r) \mapsto \frac{(t, r, 1)}{\sqrt{1 + t^2 + r^2}}$$

sends $\mathbb{R}_t \times (0, \infty)_r$ to the interior of the quarter-sphere given by

$$\mathbb{S}^2_{++} = \{(z_1, z_2, z_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 \mid z_2 \geq 0,\ z_3 \geq 0\}.$$

The quarter-sphere $\mathbb{S}^2_{++}$ is a manifold with corners and has two boundary hypersurfaces defined by the boundary defining functions z_2 and z_3. We let cf (or the *conic face*) be the

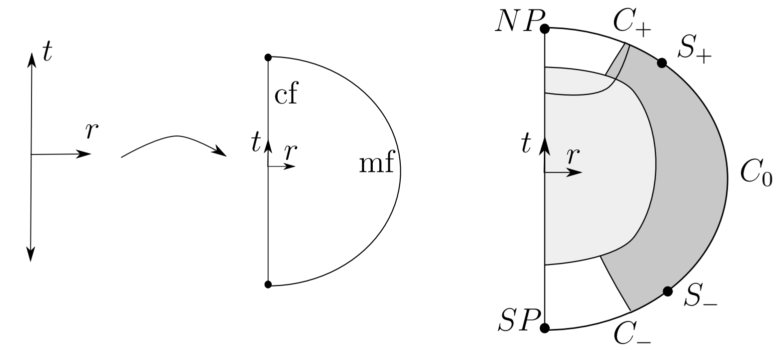

FIGURE 1. The compactification M of $\mathbb{R}_t \times (0,\infty)_r \times Z$ with the Z factor suppressed, mapping on the left with detail on the right. The image of $\mathbb{R}_t \times (0,\infty)_r$ can be thought of either as the right half disk D_+ or the quarter sphere $\mathbb{S}_{++}$. $S_\pm$ are, respectively, collapsed future/past null infinity. $C_\pm$ and C_0 are three subsets of mf corresponding to past and future timelike infinity and spacelike infinity, respectively.

hypersurface connecting NP to SP defined by $\{z_2 = 0\}$, where

$$z_2 = \frac{r}{\sqrt{1+t^2+r^2}},$$

and we let mf (or the *main face*) be the face defined by $\{z_3 = 0\}$, where, over the interior,

$$z_3 = \frac{1}{\sqrt{1+t^2+r^2}},$$

thus $z_3 \to 0$ along rays where either t or r goes to infinity. We now let $M = \mathbb{S}^2_{++} \times \mathbb{S}^{n-1}$ and identify the interior of M with $\mathbb{R}_t \times (\mathbb{R}^n \setminus \{0\})$ by introducing spherical coordinates in the spatial factor and using the above identification to map (t,r) to the interior of $\mathbb{S}^2_{++}$.

Equivalently, we can compactify $\mathbb{R}_t \times (0,\infty)_r$ to a half ball $D_+ = \{w = (w_1, w_2) \in \mathbb{R}^2 : |w| \le 1, w_2 \ge 0\}$ via the mapping

$$(t,r) \mapsto \frac{(t,r)}{1+\sqrt{1+t^2+r^2}}.$$

We leave it to the interested reader to show that these compactifications are equivalent, in the sense that they induce diffeomorphisms of manifolds with corners. In particular, the boundary hypersurface (bhs) $w_2 = 0$ corresponds to $z_2 = 0$ and the bhs $|w| = 1$ corresponds to $z_3 = 0$.

To understand the detailed asymptotic behavior of solutions we perform two blow ups of the compactification M. The first introduces future null infinity, which is the natural domain of definition of the radiation field, while the second blow up introduces the precise region needed to distinguish between the limits of timelike paths with zero and non-zero spatial momentum, thus providing two boundary hypersurfaces which distinguish the two regimes of asymptotic behavior of solutions at timelike infinity.

We recall from previous work [BVW15, BVW18] the first blow up, which induces the construction of the manifold with corners on which the radiation field naturally lives. As in those works, we denote collapsed null infinity by S; this is the intersection of the closure of maximally extended null geodesics with the introduced boundary mf. The submanifold is given by $S = \{v = \rho = 0\}$ where v is a smooth function on M which vanishes simply on

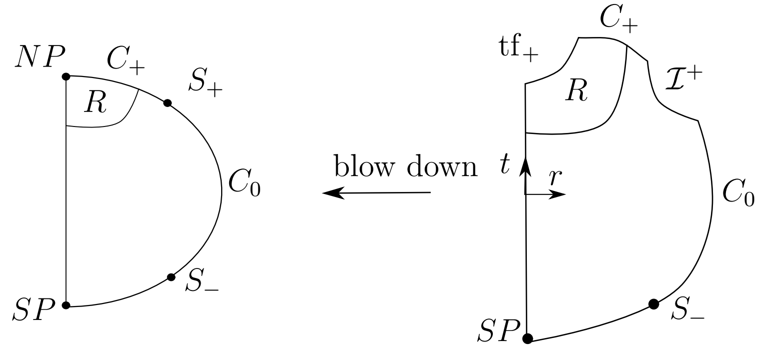

FIGURE 2. A schematic view of the space X (on the right) we use to study Price's Law. The labeled region R is the neighborhood of timelike infinity in which the solutions under consideration exhibit distinct asymptotic behavior at the three boundary hypersurfaces.

the sets $x = \pm 1$. (Such v should be chosen to be independent of $\theta \in \mathbb{S}^2$ so that v, ρ and coordinates on the sphere form local coordinate charts near S.) This submanifold naturally splits into two pieces according to whether $\pm t > 0$ near the component, which we denote by $S_\pm$. The complement of S in mf consists of three regions: the region C_0 being those points in mf where $v < 0$, while the region in mf where $v > 0$ has two components, denoted $C_\pm$ according to whether $\pm t > 0$ nearby.

We now *blow up* S_+ in M by replacing it with its inward pointing spherical normal bundle.[1] In the product cone setting, this is equivalent to blowing up a pair of points in $\mathbb{S}^2_{++}$ and then taking the product with the link Z. This process replaces M with a new manifold $\overline{M} = [M; S_+]$ on which polar coordinates around the submanifold are smooth; the structure of this manifold with corners depends only on the submanifold S_+ (and not on the particular choice of defining functions v and ρ).

The space $\overline{M}$ – which is not yet the final resolution we consider, as we have yet to blow up the north pole (zero momentum future timelike infinity) – is again a manifold with corners and has four boundary hypersurfaces: the lifts of C_+, the union $C_0 \cup C_-$, the lift of cf, again denoted cf, and the new boundary hypersurface consisting of the pre-image of S_+ under the blow-down map. This new boundary hypersurface, which is essentially future null infinity, is denoted $\mathcal{I}^+$. Moreover, $\mathcal{I}^+$ is naturally a fiber bundle over S_+ with fibers diffeomorphic to intervals. Indeed, the interior of the fibers is naturally an affine space (i.e., $\mathbb{R}$ acts by translations, but there is no natural origin). Figure 2 depicts this blow-up construction. Given choices of functions v and ρ, the fibers of the interior of $\mathcal{I}^+$ in $\overline{M}$ can be identified with $\mathbb{R} \times Z$ via the coordinate $s = v/\rho$.

Our focus on the $t > 0$ region of space-time is a matter of convenience only; it is also possible, indeed very useful, to blow up S_- and thereby obtain a boundary hypersurface corresponding to past null infinity. Indeed, the face obtained by the blow up of S_- is the domain of the past radiation field. For Dirac-type equations, one can see distinct behavior (though related by a parity transformation) when comparing $\mathcal{I}^+$ to the corresponding face $\mathcal{I}^-$ over S_-, but these details are irrelevant for our discussion here which treats only the future timelike infinity behavior.

[1]The reader may wish to consult Melrose's book [Mel93] for more details on the blow-up construction.

Finally, we blow up the "north pole" $\mathrm{NP} = \{x = 0, \rho = 0\}$. We thereby obtain a manifold with corners $X = [\overline{M}; \mathrm{NP}]$ with an additional boundary hypersurface tf_+ which separates cf from C_+. Near the intersection of tf_+ with C_+, we can use the boundary defining function $\rho_{\mathrm{tf}_+} = x + \rho$, while a full set of coordinates is given by

$$\rho_{\mathrm{tf}_+} = x + \rho, \quad y = \frac{x - \rho}{x + \rho}, \quad \theta,$$

where θ are local coordinates on $\mathbb{S}^2$.

We now come to the main point of this blow up construction, which can be understood as follows. Still working on $\overline{M}$, the solutions to the differential equations we consider here are tempered distributions which are conormal (on $\overline{M}$) and admit an asymptotic expansion at C_+ with polyhomogeneous coefficients. In this context, conormality[2] is equivalent to the existence of weights $\ell, \ell' \in \mathbb{R}$ such that for all $j, k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{\dim Z}$,

$$(x\partial_x)^j (\rho\partial_\rho)^k \partial_\theta^\alpha (x^{-\ell} \rho^{-\ell'} u) \in L^\infty,$$

while polyhomogeneity at C_+ is the condition that u admits an asymptotic expansion with terms of the form $\rho^s (\log \rho)^k a(x, \theta)$ with $s \in \mathbb{C}$, $k \in \mathbb{N}_0$ and k bounded for $\Re s < L$. The precise nature of the expansion here is the key to our arguments. We describe two notions of polyhomogeneity below. First, we describe the more common notion, here termed (full) polyhomogeneity, which characterizes the joint expansion at $C_+ \cap \mathcal{I}^+$. We then describe the slightly weaker notion governing the expansion at $C_+ \cap \mathrm{cf}$, which we call partial polyhomogeneity. Both notions describe expansions in terms of *index sets*; an index set $\mathcal{E}$ is a discrete subset of $\mathbb{C} \times \mathbb{N}_0$ so that $\{(\sigma, k) \in \mathcal{E} : \operatorname{Im} \sigma > -R\}$ is finite for each R.

Recall that a distribution w on $\overline{M}$ is **(fully) polyhomogeneous** at $C_+ \cap \mathcal{I}^+$ if it admits a joint asymptotic expansion there, meaning that w is conormal and for any L

$$w(v, \zeta, \theta) = \sum_{\substack{(\sigma,k) \in \mathcal{E}, \operatorname{Im} \sigma > -L \\ (\tau,l) \in \mathcal{F}, \operatorname{Im} \tau > -L}} v^{i\sigma} (\log v)^k \zeta^{i\tau} (\log \zeta)^l b_{\sigma k \tau l}(\theta) + (v\zeta)^L w'(v, \zeta, \theta),$$

with w' a bounded conormal distribution, together with separate asymptotics at the two faces. (Here v is a defining function for $\mathcal{I}^+$ while $\zeta = \rho/v = s^{-1}$ is a defining function for C_+.)

The distinction between partially and fully polyhomogeneous distributions lies in the fact that partially polyhomogeneous distributions may not in general have an expansion at cf at all, in particular they may not have a joint expansion. (Incidentally, it is possible to have expansions at the interiors of both faces separately but no joint expansion, a simple example being the function $r = \sqrt{x^2 + y^2}$ on the positive upper half quadrant.)

A fully polyhomogeneous distribution on the whole of $\overline{M}$ admits expansions at all faces and joint expansions at all intersections of faces. In contrast, the partially polyhomogeneous distributions which arise as solutions here admit expansions at all faces except possibly cf. Our analysis therefore focuses on the region near the intersection of cf with C_+. Specifically, given index sets $\mathcal{E}$ and $\mathcal{F}$, we say that a distribution u is **partially polyhomogeneous** at

[2]Here we choose to use L^∞-based conormal spaces as opposed to the H_b^s-based spaces used in [BM22] as the L^∞-based spaces make the proof of Lemma 2.1 slightly clearer. The equivalence of L^∞-based conormality and Sobolev space-based conormality follows from the Sobolev embedding theorem and is discussed in [Mel96] in Chapter 4.

C_+ (near cf) with index set $\mathcal{E}$ at C_+ and $\mathcal{F}$ at cf if, for each L, in sets $\{x < c\}$, i.e. away from $\mathcal{I}_+$,

$$u(x, \rho, \theta) = \sum_{(\sigma,k)\in\mathcal{E}, \operatorname{Im}\sigma > -L} \rho^{i\sigma} (\log \rho)^k a_{\sigma k}(x, \theta) + \rho^L u'(x, \rho, \theta), \tag{3}$$

where the remainder $u'(x, \rho, \theta)$ is a bounded conormal function and the $a_{\sigma k}$ are themselves polyhomogeneous distributions on C^+ with a fixed index set $\mathcal{F}$ at cf. (This means $a_{\sigma k} \sim \sum_{(\tau,l)\in\mathcal{F}} x^{i\tau} (\log x)^p c_{\tau,p}$ as $x \to 0$.) In other words, u enjoys an expansion at C_+ in which each terms is jointly polyhomogeneous at C_+ and cf but the remainder is in principle only conormal at cf. *Our main observation is that such a distribution u pulls back to the blown up space X to be fully polyhomogeneous at C^+ with index set $\mathcal{E}$ and partially polyhomogeneous at* tf_+ *with index set $\mathcal{F} + \mathcal{E}$ at* tf_+ *and index set $\mathcal{F}$ at* cf. This allows us to conclude the distinct asymptotic behavior described above. We formalize this pullback statement in Lemma 2.1.

As polyhomgeneity is essentially a local property near boundary faces of a manifold with corners, we consider now the model of a manifold with codimension two corners. Let

$$\mathbb{R}^2_{++} = \{(z, w) \in \mathbb{R}^2 : z, w \geq 0\}$$

denote the closed upper right quadrant, let $H_1 = \{z = 0\}, H_2 = \{w = 0\}$. Let $v \colon \mathbb{R}^2_{++} \longrightarrow \mathbb{C}$ be a conormal distribution supported near 0, and assume that v is partially polyhomogeneous at H_1 with index sets $\mathcal{E}$ at H_1 and $\mathcal{F}$ at H_2. Let ff denote the new face of the blow up $[\mathbb{R}^2_{++} : (0, 0)]$.

Lemma 2.1. *Let $U \subset \mathbb{R}^N$ be an open set. Let v be a distribution on $\mathbb{R}^2_{++} \times U$ which is partially polyhomogeneous at H_1 with index set $\mathcal{E}$ at H_1 and $\mathcal{F}$ at H_2. Then the pullback of v to the blown-up space $[\mathbb{R}^2_{++}; \{(0, 0)\}] \times U$ is fully polyhomogeneous at H_1 with index set $\mathcal{E}$ and partially polyhomogeneous at* $\mathrm{ff} \times U$ *with index set $\mathcal{E} + \mathcal{F}$ there and index $\mathcal{F}$ at H_2.*

Proof. We suppress the factor U in the domain as the smooth dependence on that parameter is irrelevant in the proof.

By assumption, near $(0, 0)$ we can write, for every $L, M \in \mathbb{R}$,

$$\begin{aligned} v &= \sum_{(\sigma,k)\in\mathcal{E}, \operatorname{Im}\sigma > -L} z^{i\sigma} (\log z)^k a_{\sigma k} + z^L v' \\ &= \sum_{(\sigma,k)\in\mathcal{E}, \operatorname{Im}\sigma > -L} z^{i\sigma} (\log z)^k \Big(\sum_{(\tau,l)\in\mathcal{F}, \operatorname{Im}\tau > -M} w^{i\tau} (\log w)^l c_{\sigma k \tau l} + w^M v'' \Big) + z^L v' \end{aligned} \tag{4}$$

where v', v'' are conormal functions.

Blowing up, the functions $w, s = z/w$ define coordinates near the intersection of ff with H_1. The first line of (4), with $z = sw$, continues to provide a polyhomogeneous expansions with index set $\mathcal{E}$ but now with index set $\mathcal{E} + \mathcal{F}$ at ff since

$$v = \sum_{(\sigma,k)\in\mathcal{E}, \operatorname{Im}\sigma > -L} s^{i\sigma} w^{i\sigma} (\log s + \log w)^k a_{\sigma k}(w) + (sw)^L v'.$$

As pullbacks via blowdown maps preserve conormality [Mel96, Section 4.11], v' is conormal on $[\mathbb{R}^2_{++}; \{(0, 0)\}]$.

Near the intersection of ff with H_2, $z, t = w/z$ define coordinates, with z a defining function of ff and t a defining function of H_2, so using the second line of (4) we get

$$v = \sum_{\substack{(\sigma,k)\in\mathcal{E}, \operatorname{Im}\sigma > -L \\ (\tau,l)\in\mathcal{F}, \operatorname{Im}\tau > -M}} z^{i\sigma+i\tau}(\log z)^k t^{i\tau}(\log z + \log t)^l c_{\sigma k\tau l} + z^R v'''$$

where v''' is a conormal distribution and

$$R < \inf\{L, M - \operatorname{Im}\sigma - \operatorname{Im}\tau : (\sigma, 0) \in \mathcal{E}, (\tau, 0) \in \mathcal{F}\}.$$

This gives the asymptotic expansion with index set $\mathcal{E} + \mathcal{F}$ in z with terms having expansions in index set $\mathcal{F}$ in t, and choosing M, L sufficiently negative gives the result. □

3. Asymptotic expansions and Proof of Theorem 1.1

That solutions of the equations considered here enjoy partial polyhomogeneity near C_+ and cf as well as (full) polyhomogeneity near $C_+ \cap \mathcal{I}^+$ follows from the prior works [BBGR21, BM22]. Those papers describe the asymptotics of the radiation field for this system, i.e., the behavior of solutions near $C_+ \cap \mathcal{I}^+$. In doing so, it characterizes the asymptotic behavior all along C_+ in terms of powers of ρ, x, and hypergeometric functions along C_+.

We first turn our attention to the massless Dirac–Coulomb system [BBGR21]. To that end, we define the relevant index sets[3] for $|\mathbf{Z}| < 1/2, \mathbf{Z} \neq 0$:

$$\mathcal{E}_{\mathrm{DC}} = \left\{ -i\left(2 + \ell + \sqrt{\kappa^2 - \mathbf{Z}^2}\right) : \ell \in \mathbb{N}_0, \kappa \in \mathbb{Z}\setminus\{0\} \right\},$$
$$\mathcal{F}_{\mathrm{DC}} = \left\{ -i\left(-1 + j + \sqrt{\kappa^2 - \mathbf{Z}^2}\right) : j \in \mathbb{N}_0, \kappa \in \mathbb{Z}\setminus\{0\} \right\}.$$

One consequence of that work is the following theorem:

Theorem 3.1. *If ψ is the solution to the massless Dirac–Coulomb system* (1) *with smooth, compactly supported initial data, then, on $\overline{M}$, ψ is (fully) polyhomogeneous at $C_+ \cap \mathcal{I}^+$ with index set $\mathcal{E}_{DC}$ at C_+ and partially polyhomogeneous at C_+ near* cf *with index sets $\mathcal{E}_{DC}$ at C_+ and $\mathcal{F}_{DC}$ at* cf*.*

In particular, from Lemma 2.1, we conclude that, on X, ψ is partially polyhomgeneous at $\mathrm{tf}_+ \cap \mathrm{cf}$ with index sets $\mathcal{E}_{\mathrm{DC}} + \mathcal{F}_{\mathrm{DC}}$ at tf_+ and $\mathcal{F}_{\mathrm{DC}}$ at cf. It is additionally (fully) polyhomogeneous at all other boundary faces; at C_+ the index set is $\mathcal{E}_{\mathrm{DC}}$. The bounds in Theorem 1.1 for solutions of the massless Dirac–Coulomb system then follow by considering the largest term in the expansions at tf_+ and C_+, respectively.

This allows immediately for the proof of Theorem 1.1 for Dirac–Coulomb, in the sense that the theorem statements are merely appropriate interpretations of the polyhomogeneity statements we have derived. Indeed, a distribution ψ which is partially polyhomogeneous at $\mathrm{tf}_+ \cap \mathrm{cf}$ *with index sets $\mathcal{E}_{DC} + \mathcal{F}_{DC}$ necessarily has an asymptotic expansion with leading order $t^{-3-\alpha(\mathbf{Z})}$ at that face; thus, in particular, if $U \Subset \mathrm{tf}_+$ is any open set compactly contained in* tf_+ *then: (1) the flat space coordinate z is a valid coordinate on U and (2) $\psi(z,t)t^{3+\alpha(\mathbf{Z})}$ is a bounded function for $z \in U$. The spatial cutoff χ in the statement of Theorem 1.1 serves exactly to cutoff to such a set U and thus for such χ we have $\chi(z)\psi(z,t)t^{3+\alpha(\mathbf{Z})}$ is bounded. On the other hand, any ray $(t, \gamma t, \theta)$ with $0 < |\theta| < 1$ approaches the interior of the face C^+ as $t \to \infty$, the function $\psi(t, \gamma t, \theta)$ is merely the restriction of ψ to a particular path of*

[3]For the equations considered here, there are no logarithms in the expansions and so we drop the $\mathbb{N}_0$ factor from the index set notation.

approach to this boundary hypersurface; thus the (full) polyhomogeneity statement at that face immediately gives the stated $\sim t^{-3-\alpha(\mathbf{Z})/2}$ *behavior claimed in the theorem.*

For the wave equation (2) on $\mathbb{R} \times \mathbb{R}^n$ with an inverse square potential, we appeal to the results of the earlier paper about conic manifolds [BM22]. That paper describes the asymptotics of solutions of the wave equation on cones, but the techniques apply with no change to the wave equation on $\mathbb{R}^n$ with an inverse square potential. To that end, we define the relevant index sets[4] for the inverse square (IS) problem with $\digamma > -\left(\frac{n-2}{2}\right)^2, \digamma \neq 0$. In the below, we let $\nu_j = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + \digamma}$, where λ_j, $j = 0, 1, \dots$, denote the eigenvalues of the Laplacian, i.e., $\lambda_j = j(j + n - 2)$.

$$\mathcal{E}_{\mathrm{IS}} = \left\{ -i\left(\frac{n}{2} + k + \nu_j\right) : j, k = 0, 1, 2, \dots, \frac{1}{2} + \nu_j \notin \mathbb{Z} \right\},$$

$$\mathcal{F}_{\mathrm{IS}} = \left\{ -i\left(-\frac{n-2}{2} + \ell + \nu_j\right) : j, \ell = 0, 1, 2, \dots, \nu_j \notin \mathbb{Z} \right\}.$$

The methods of the paper [BM22] about product cones, together with the calculation in Appendix A, then establish the following theorem:

Theorem 3.2. *If u is the solution of the wave equation* (2) *with smooth, compactly supported initial data, then, on $\overline{M}$, u is (fully) polyhomogeneous at $C_+ \cap \mathcal{I}^+$ with index set $\mathcal{E}_{IS}$ at C_+ and partially polyhomogeneous at C_+ near* cf *with index sets $\mathcal{E}_{IS}$ at C_+ and $\mathcal{F}_{IS}$ at* cf.

In that paper, the terms in the expansion are characterized by the resonances of the associated Laplacian on a hyperbolic cone. If $\frac{1}{2} + \nu_j$ is an integer, it is a pole of the scattering matrix but not of the resolvent; in terms of the wave equations studied here, this means that the associated term is supported entirely in S_+ and vanishes in C_+ and therefore does not contribute to the expansion at C_+ or tf_+.

Just as in the case of the Dirac–Coulomb problem, Lemma 2.1 allows us to conclude that, after the blowup, u is partially polyhomogeneous at tf_+ near cf with index sets $\mathcal{E}_{\mathrm{IS}} + \mathcal{F}_{\mathrm{IS}}$ at tf_+ and $\mathcal{F}_{\mathrm{IS}}$ at cf. The remaining bounds in Theorem 1.1 again follow by considering the largest terms in the expansions. If $\frac{1}{2} + \nu_0$ is an integer, the leading term is instead

$$\frac{n}{2} + \nu_1 = \frac{n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + n - 1 + \digamma}$$

. As $\digamma \neq 0$, $\frac{1}{2} + \nu_1$ cannot also be an integer.[5] The conclusion of the parts of Theorem 1.1 related to inverse square potentials follow exactly as in the above discussion of Dirac–Coulomb.

3.1. **Sharpness.** We now turn to a discussion of the sharpness of the bounds. If the first term in each expansion is nonzero, the corresponding bound is saturated. In the previous papers [BBGR21, BM22], each term in the expansion arises via the inverse Mellin transform from a finite rank operator applied to the Mellin transform of the inhomogeneous data. (For the initial value problem, one first converts the problem into an inhomogeneous forward

[4]As before, we drop the $\mathbb{N}_0$ factor from the notation.

[5]When $\frac{1}{2} + \nu_0$ is an integer, the radial part (i.e., the projection onto the lowest spherical harmonic) of the solution solves a conjugated odd-dimensional wave equation and therefore has no support at timelike infinity.

problem.) These operators are the residues of the poles of the inverse of a family of operators related to the equation.

To show that the first term is generically non-zero, it therefore suffices to characterize the first operator obtained this way. As described in prior work, the first operator has rank one and is given by $\phi \otimes \psi$, where ϕ is in the kernel of the normal operator and is supported in $\overline{C}_+$, while ψ lies in the kernel of the adjoint of the normal operator and is regular at S_+. Both ϕ and ψ can be described in terms of hypergeometric functions; an energetic reader can follow the arguments of the prior work [BBGR21, Section 7] to verify that ψ is non-trivial in C_+.[6] In particular, for most inhomogeneities (and hence most initial data), the first term in the expansion is a nonzero multiple of ϕ.

These arguments are similar in flavor to those of Hintz [Hin22] demonstrating sharpness. In both cases the leading term arises as the output of a rank one operator applied to the relevant data and thus is generically non-zero.

Appendix A. Resonance calculation

Let $\rho = \frac{1}{t+r}$ and $x = \frac{2r}{t+r}$ so that $r = \frac{x}{2\rho}$. If we set

$$\begin{aligned} L_0 &= \partial_t^2 - \Delta + \frac{\digamma}{r^2} \\ &= \partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} + \frac{\digamma}{r^2}, \end{aligned}$$

then L_0 lifts to

$$\begin{aligned} L_0 = \rho^2(\rho\partial_\rho + x\partial_x)^2 + \rho^2(\rho\partial_\rho + x\partial_x) - \rho^2(2\partial_x - x\partial_x - \rho\partial_\rho)^2 \\ + \rho^2(2\partial_x - x\partial_x - \rho\partial_\rho) - 2\rho^2\frac{n-1}{x}(2\partial_x - x\partial_x - \rho\partial_\rho) - \frac{4\rho^2}{x^2}\Delta_{\mathbb{S}^{n-1}} + \frac{4\digamma\rho^2}{x^2}. \end{aligned}$$

The paper [BM22] and its predecessors consider the reduced normal operator of

$$L = \rho^{-2}\rho^{-\frac{n-1}{2}}L_0\rho^{\frac{n-1}{2}}.$$

As this operator is homogeneous of degree 0 in ρ, the reduced normal operator is given by replacing $\rho\partial_\rho$ with $i\sigma$, yielding

$$\begin{aligned} P_\sigma = -\widehat{N}(L) = 4(1-x)\partial_x^2 + \frac{n-1}{x}4\partial_x - \left(4 + (i\sigma + \frac{n-1}{2}) + 2(n-1)\right)\partial_x \\ + \frac{4}{x^2}\Delta_{\mathbb{S}^{n-1}} - \frac{4\digamma}{x^2} - 2\left(\frac{n-1}{x}\right)(i\sigma + \frac{n-1}{2}). \end{aligned}$$

The poles of the inverse of this operator (on appropriate σ-dependent variable-order Sobolev spaces) yield the exponents seen above. The corresponding resonant states are solutions v of $P_\sigma v = 0$ lying in these same spaces.

For this particular operator, we can see the poles explicitly in terms of the failure of hypergeometric functions to remain linearly independent. Indeed, separating into angular

[6] In the case when $\frac{1}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \digamma} \in \mathbb{Z}$, the hypergeometric function underlying ϕ is in fact a derivative of the delta function and so there is no contribution to the expansion at C_+.

modes ϕ_j with eigenvalues $-\lambda_j$, the radial coefficients v_j of a solution of $P_\sigma v_j = 0$ must satisfy

$$\begin{aligned} 0 &= xP_{\sigma,j}v_j \\ &= 4x(1-x)\partial_x^2 v_j + 4\left(n-1-x(n+i\sigma)\right)\partial_x v_j - 4\left(\frac{\digamma + \lambda_j}{x}\right)v_j - (n-1)(2i\sigma + n - 1)v_j. \end{aligned}$$

Dividing by 4, setting $P = \frac{1}{4}xP_{\sigma,j}$ and letting $w = v_j$, w must satisfy

$$\begin{aligned} Pw &= x(1-x)\partial_x^2 w + (n-1-x(n+i\sigma))\partial_x w - \frac{\digamma + \lambda_j}{x}w - \left(\frac{n-1}{2}\right)(i\sigma + \frac{n-1}{2})w \\ &= 0. \end{aligned}$$

Conjugating by x^α, where

$$\alpha = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \digamma + \lambda_j},$$

yields a hypergeometric equation for $y = x^{-\alpha}w$:

$$x(1-x)\partial_x^2 y + (n-1+2\alpha - x(n+i\sigma+2\alpha))\partial_x y - \left(\alpha + \frac{n-1}{2}\right)\left(\alpha + \frac{n-1}{2} + i\sigma\right)y = 0.$$

This is a hypergeometric differential equation with parameters (see, e.g., [DLMF] for notation)

$$\begin{aligned} a &= \frac{1}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + \digamma}, \\ b &= \frac{1}{2} + i\sigma + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + \digamma}, \\ c &= 1 + 2\sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + \digamma}. \end{aligned}$$

The requirement that the solution v lies locally in H^1 near $x = 0$ implies that y must be a multiple of the hypergeometric function

$$y_1 = F(a, b, c; x) = F\left(\frac{1}{2} + s, \frac{1}{2} + i\sigma + s, 1 + 2s; x\right),$$

where $s = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j + \digamma}$. On the other hand, if a solution is to lie in the appropriate variable order Sobolev space globally, at C_+ it must be a multiple of

$$\begin{aligned} y_4 &= (1-x)^{c-a-b}F(c-a, c-b, c-a-b+1; 1-x) \\ &= (1-x)^{-i\sigma}F\left(\frac{1}{2} + s, \frac{1}{2} - i\sigma + s, 1 - i\sigma; 1-x\right). \end{aligned}$$

The poles under consideration are therefore given by those σ for which y_4 is a multiple of y_1. Kummer's connection formulae [DLMF, 15.10.18] allow us to write y_4 in terms of the basis of solutions y_1 and y_2, where y_2 is given by

$$y_2 = x^{1-c}F(a-c+1, b-c+1, 2-c; x) = x^{-2s}F\left(\frac{1}{2}-s, \frac{1}{2}-s+i\sigma, 1-2s; x\right).$$

In this case, we have

$$y_4 = \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)}y_1 + \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}y_2.$$

In particular, the coefficient of y_2 vanishes precisely when $\frac{1}{2}+s-i\sigma$ is a pole of the gamma function, i.e., when

$$\sigma = \sigma_{j,k} = -i\left(\frac{1}{2}+s+k\right), \quad k = 0, 1, \dots.$$

Moreover, the resonant state v_j associated with $\sigma_{j,k}$ is a multiple of $x^\alpha y_1$ and therefore is polyhomogeneous with leading order behavior

$$x^{-\frac{n-2}{2}+s} \text{ as } x \to 0.$$

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Email address: dbaskin@math.tamu.edu

Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843

Email address: jgell@unimelb.edu.au

School of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia

Email address: marzuola@math.unc.edu

Department of Mathematics, UNC-Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599