PROPAGATION OF SINGULARITIES FOR THE WAVE EQUATION

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Abstract. This expository note gives a digest version of Hörmander's propagation of singularities theorem for the wave equation.

1. Introduction

Let $P = \Box = -\partial_{x_0}^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ be the flat D'Alembertian on $\mathbb{R}^d$, $d = n+1$. We use coordinates $x = (x_0, \ldots, x_n)$. Hörmander's propagation of singularities result for solutions to $\Box u = 0$ says that if u has a given degree of Sobolev regularity at a given position and direction in spacetime $(x, \hat{\xi}) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$, then it must have the same Sobolev regularity at all $(x', \hat{\xi})$ such that x' is on the light ray emanating from x with direction $\hat{\xi}$.

To make precise the notion of a given degree of Sobolev regularity at a given position and direction, we introduce the following definitions. For $m \in \mathbb{R}$, let $S^m = S^m(\mathbb{R}^d)$ be the *Kohn–Nirenberg* class of symbols, defined by the condition that $a \in S^m$ means $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the partial derivatives of a obey the bounds

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}.$$

To each such symbol we associate a *pseudodifferential operator* A given by

$$Au(x) = \frac{1}{(2\pi)^d} \iint e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, dy\, d\xi. \tag{1}$$

The set of A corresponding to some $a \in S^m$ is denoted Ψ^m. If $A \in \Psi^m$, then A is a bounded operator from H^k to H^{k-m} for every k. For a proof of this general statement see Corollary 4.32 of [Hi], but note the following simpler cases: if $a(x,\xi) = a(x)$ is independent of ξ then A is the multiplier $u(x) \mapsto a(x)u(x)$, and if $a(x,\xi) = \langle \xi \rangle^m$ then $\|Au\|_{L^2}$ is the usual norm on the Sobolev space H^m given in terms of the Fourier transform.

To each $a \in S^m$ which is compactly supported in x, we associate an *essential support at fiber infinity*[1] (i.e. a support as the frequency ξ tends to infinity), denoted by ess supp(a), and defined as follows: $(x, \hat{\xi})$ is not in the essential support if and only if there is a neighborhood $U \subset \mathbb{R}^d \times \mathbb{S}^{d-1}$ of $(x, \hat{\xi})$, such that the partial derivatives of a obey the bounds

$$|\partial_x^\alpha \partial_\xi^\beta a(x',\xi')| \le C_{\alpha,\beta,N} \langle \xi' \rangle^{-N}$$

for all N when $(x', \xi'/|\xi'|) \in U$.

We say $a \in S^m$ is *elliptic* at $(x, \hat{\xi})$ if there are positive constants C and ε, and a neighborhood $U \subset \mathbb{R}^d \times \mathbb{S}^{d-1}$ of $(x, \hat{\xi})$, such that

$$|a(x',\xi')| \ge \varepsilon \langle \xi' \rangle^m,$$

[1] One can extend this definition to symbols which need not be compactly supported in x: see Chapter 6 of [Hi] or Section E.2.1 of [DyZw].

when $(x', \xi'/|\xi'|) \in U$ and $|\xi| \geq C$. The set of points in $\mathbb{R}^d \times \mathbb{S}^{d-1}$ at which a is elliptic is denoted $\mathrm{ell}(a)$.

Given symbols b, e, and g, we say that b is *controlled by* e *through* g if for each point $(x, \hat{\xi})$ in $\mathrm{ess\,supp}(b)$, there is a light ray contained in $\mathrm{ell}(g)$ which starts at some point $(x', \hat{\xi}')$ in $\mathrm{ell}(e)$ and ends at $(x, \hat{\xi})$.A *light ray* is an integral curve of the *Hamilton vector field* H_P lying in the *characteristic set* $\Sigma(P) = \{(x, \hat{\xi}) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : \hat{\xi}_0^2 - \sum \hat{\xi}_j^2 = 0\}$. Here the Hamilton vector field is given by

$$H_P = 2\xi_0 \partial_{x_0} - 2\xi_1 \partial_{x_1} - \cdots - 2\xi_n \partial_{x_n}.$$

Note that the Hamilton vector field has no ξ derivatives, so if $(x, \hat{\xi})$ is connected by a light ray to $(x', \hat{\xi}')$, then $\hat{\xi} = \hat{\xi}'$. See Figure 1.

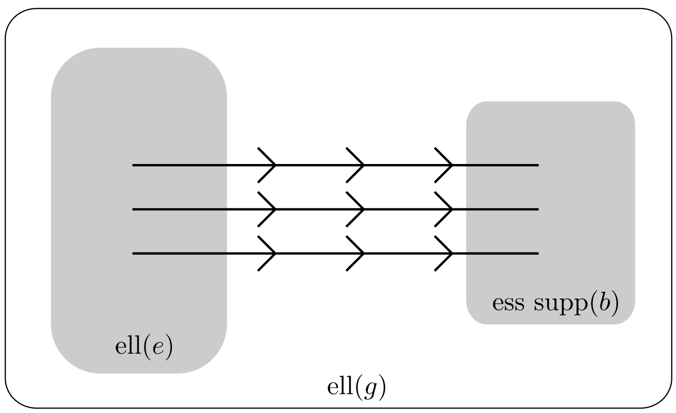

FIGURE 1. The horizontal coordinate s is chosen so that the Hamilton vector field is ∂_s, thus the arrows are integral curves of the Hamilton vector field.

We are now ready to state Hörmander's propagation of singularities theorem.

Theorem. *Let $b \in S^k$, $e \in S^k$, $g \in S^{k-1}$ for some real k be compactly supported in x. Suppose $u \in H^{-N}$, $Eu \in L^2$, and $GPu \in L^2$. If b is controlled by e through g, then $Bu \in L^2$ and there is a constant C (independent of u) such that*

$$\|Bu\|_{L^2} \leq C(\|Eu\|_{L^2} + \|GPu\|_{L^2} + \|u\|_{H^{-N}}). \tag{2}$$

Since B and E both map $H^k \to L^2$, this says in particular that if we have a solution to $\Box u = 0$ which is in some Sobolev space H^{-N}, and if we happen to know that it is in some better Sobolev space H^k in the positions and directions given by the elliptic set of e, then it is also in this better Sobolev space H^k at points and directions emanating from that set. In this sense (2) is a propagation of regularity estimate. Since a singularity is a place where regularity is lacking, we also call (2) a propagation of singularities estimate.

As b is compactly supported in x, without loss of generality we can take e and g also to have compact support in x and then u need only lie in H^{-N} locally. In particular, if $\chi \in C_c^\infty$ satisfies $\chi \equiv 1$ on the support of g, then

$$\|B_0 u\| \leq C\left(\|Eu\| + \|GPu\| + \|\chi u\|_{H^{-N}}\right).$$

For example, let $u = \delta(x_0 - x_1)$. Then $u \in H^s_{\mathrm{loc}}$ if and only if $s < 1/2$. Microlocally, however, one can show that, for almost all $(x, \hat{\xi})$, we have $u \in H^s$ at $(x, \hat{\xi})$ for all s. More precisely, u is in these better Sobolev spaces at $(x, \hat{\xi})$ if and only if $x_0 \neq x_1$ or $|\hat{\xi}_0| \neq 1/\sqrt{2}$ or $\hat{\xi}_1 \neq -\hat{\xi}_0$.

This theorem was first proved by using a Fourier integral operator to microlocally conjugate P to the simpler operator ∂_{x_0}/i, for which the theorem is easy: see [Hö1] and [DuHö], specifically Section 6 of the latter paper. The proof we present here uses a positive commutator argument with a pseudodifferential commutant. It is a special case of Hörmander's proof in Section 3.5 of [Hö2]. Other presentations of this proof in more general settings than the one here can be found in Section E.4 of [DyZw] and Section 8 of [Hi].

The crux of the proof involves bounding the pairing

$$\mathrm{Im}\langle Au, APu\rangle = \frac{1}{2i}\left(\langle PA^*Au, u\rangle - \langle A^*APu, u\rangle\right) = \left\langle \frac{1}{2i}[P, A^*Au]u, u\right\rangle. \tag{3}$$

for a well-chosen $A \in \Psi^{k-\frac{1}{2}}$. We bound (3) from below via a straightforward application of Cauchy–Schwarz, and from above by arranging that the commutator $[P, A^*A]/i$ is almost negative.

As we will see below in Section 2, the principal symbol of $[P, A^*A]/i$ is $H_P(|a|^2)$: this is the fundamental relationship between commutators and Hamilton vector fields. We will see that to arrange that $[P, A^*A]/i$ is almost negative in the right sense we must arrange that $H_P(|a|^2)$ is negative on $\mathrm{ess\,supp}(b) \setminus \mathrm{ell}(e)$, i.e. that $|a|^2$ is decreasing along the Hamilton flow in this region. Such an a is called an *escape function*, a terminology which goes back to [MoRaSt]. Constructing an escape function is straightforward here because of the simple geometry of our spacetime.

Theorem 1 has many generalizations. Most directly, a nearly identical proof strategy applies to operators of real principal type that are not necessarily self-adjoint: i.e. only the principal symbol of P needs to be real, and P can be a variable coefficient differential operator or even a pseudodifferential operator. Constructing an escape function is then done by locally straightening out the Hamilon flow. The additional error term arising in equation (3) from $P - P^*$ is a lower order term. One can also treat cases where the principal part of P is not real, provided the imaginary part has a sign; this sign dictates the direction in which we can propagate regularity.

A common use for such propagation results is to help obtain global estimates for solutions of wave-like equations. As an example, in the non-trapping setting, these estimates let you propagate regularity from where it is a priori known (e.g., in the distant past for the forward propagator). Closing the estimate, however, typically requires additional estimates, which can be more complicated in settings with trapping.

2. Preliminaries from microlocal analysis

The important calculations take place on the level of symbols. To translate the results into statements about operators, we use the following formula for the symbol of the composition of two pseudodifferential operators. If $A_1 \in \Psi^{m_1}$ and $A_2 \in \Psi^{m_2}$ have symbols a_1 and a_2 respectively, then the composition A_1A_2 belongs to $\Psi^{m_1+m_2}$ and has symbol given by $a_3 \in S^{m_1+m_2}$ such that

$$r := a_3 - a_1a_2 - \frac{1}{i}\partial_\xi a_1 \cdot \partial_x a_2 \in S^{m_1+m_2-2}, \tag{4}$$

and $\mathrm{ess\,supp}(r) \subset \mathrm{ess\,supp}(a_1) \cap \mathrm{ess\,supp}(a_2)$.

We define the *principal symbol* $\sigma_m(A)$ of an operator $A \in \Psi^m$ to be the equivalence class of its symbol a in S^m/S^{m-1}. An immediate consequence of the composition formula above is the observation that

$$\sigma_{m_1+m_2}(A_1A_2) = \sigma_{m_1}(A_1)\sigma_{m_2}(A_2).$$

Similarly, for $A_1 \in \Psi^{m_1}$ and $A_2 \in \Psi^{m_2}$, the commutator $[A_1, A_2] = A_1A_2 - A_2A_1$ lies in $\Psi^{m_1+m_2-1}$ and satisfies

$$\sigma_{m_1+m_2-1}([A_1, A_2]) = \frac{1}{i}\left(\frac{\partial\sigma_{m_1}(A_1)}{\partial\xi}\frac{\partial\sigma_{m_2}(A_2)}{\partial x} - \frac{\partial\sigma_{m_2}(A_2)}{\partial\xi}\frac{\partial\sigma_{m_1}(A_1)}{\partial x}\right) = \frac{1}{i}H_{\sigma_{m_1}(A_1)}\left(\sigma_{m_2}(A_2)\right),$$

where the Hamilton vector field $H_{\sigma_{m_1}(A_1)}$ is defined by the equation.

Concretely, the principal symbol of $P = \Box$ is

$$\sigma_2(P) = \xi_0^2 - \xi_1^2 - \cdots - \xi_n^2,$$

and its Hamilton vector field is

$$H_P = 2\xi_0\partial_{x_0} - 2\xi_1\partial_{x_1} - \cdots - 2\xi_n\partial_{x_n}.$$

We also use the following adjoint formula: If $A \in \Psi^m$ has symbol a, then the adjoint operator has symbol $a' \in S^m$ obeying

$$a' - \overline{a} \in S^{m-1}.$$

The composition formula can be found in Theorem 4.16 of [Hi] and the adjoint formula can be found in Corollary 4.13 of [Hi]. They are easy to check for differential operators, i.e. in the case that the symbols are polynomials in ξ. In general they may be checked by writing out the definitions and Taylor expanding.[2]

Our first application of the composition formula is to the following elliptic estimate.

Lemma 1. *Let $a \in S^m$ and $a' \in S^{m'}$ be such that a is compactly supported in x and* ess supp $a \subset$ ell(a'). *For any k and N, if $u \in H^{-N}$ and $A'u \in H^{k+m-m'}$, then $Au \in H^k$ and there is C (independent of u) such that*

$$\|Au\|_{H^k} \le C(\|A'u\|_{H^{k+m-m'}} + \|u\|_{H^{-N}}). \tag{5}$$

Proof. Because a' is elliptic on the essential support of a, there exists $g \in S^{m-m'}$ such that $ga' = a$ when ξ is large.[3] By the composition formula (4), there is $R_1 \in \Psi^{m-1}$ such that

$$GA' = A + R_1.$$

Since $G\colon H^{k+m-m'} \to H^k$ is bounded, it follows that

$$\|Au\|_{H^k} \le C(\|A'u\|_{H^{k+m-m'}} + \|R_1u\|_{H^k}).$$

[2]Observe that if $A_3 = A_1A_2$ then by definition $A_3u(x)$ is given by (1) with $a(x,\xi)$ replaced by

$$a_3(x,\xi) = \frac{1}{(2\pi)^d}\iint e^{-ix\cdot\xi}e^{ix\cdot\eta}e^{-iz\cdot\eta}e^{iz\cdot\xi}a_1(x,\eta)a_2(z,\xi)\,d\eta\,dz = \frac{1}{(2\pi)^d}\iint e^{-iw\cdot\zeta}a_1(x,\xi+\zeta)a_2(x+w,\xi)\,d\zeta\,dw,$$

where we used $e^{-ix\cdot\xi}e^{ix\cdot\eta}e^{-iz\cdot\eta}e^{iz\cdot\xi} = e^{i(x-z)\cdot(\eta-\xi)}$, substituted $w = z - x$ and $\zeta = \eta - \xi$, and ignored issues of convergence. To deduce the expansion (4), Taylor expand

$$a_1(x,\xi+\zeta) = a_1(x,\xi) + \zeta\cdot\partial_\xi a_1(x,\xi) + \cdots, \qquad a_2(x+w,\xi) = a_2(x,\xi) + w\cdot\partial_x a_2(x,\xi) + \cdots,$$

and use the fact that $\frac{1}{(2\pi)^d}\int\int e^{-iw\zeta}w^\alpha\zeta^\beta\,d\zeta\,dw = i^{|\beta|}\int w^\alpha\partial^\beta\delta(w)dw$ which is $(-i)^{|\beta|}$ if $\alpha = \beta$ and 0 otherwise. The convergence issues can be handled by a partition of unity, and the adjoint formula can be proved in the same way: see Chapters 8 and 9 of [Wo] for more.

[3]To construct such a g, for C as in the definition of ellipticity, let $g(x,\xi)a'(x,\xi) = (1-\chi_C(\xi))\psi(x,\hat{\xi})a(x,\xi)$, where $\chi_C \in C_c^\infty(\mathbb{R}^d)$ is identically 1 on the ball of radius C centered at 0, $\psi \in C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$, supp $\psi \subset$ ell(a'), and $\psi = 1$ near ess supp(a).

This gives (5) when $-N = k+m-1$. To get it for larger values of N we apply the same reasoning with A replaced by R_1. That gives $R_2 \in \Psi^{m-2}$ such that

$$\|R_1 u\|_{H^k} \le C(\|A'u\|_{H^{k+m-1-m'}} + \|R_2 u\|_{H^k}),$$

which implies (5) when $-N = k+m-2$. Repeating this argument gives (5) for arbitrary N. $\square$

We further require Gårding's inequality, which states that operators with non-negative principal symbols are non-negative to leading order. In other words, if $A \in \Psi^m$ is compactly supported in x and has $\sigma_m(A) \ge 0$ for $|\xi|$ sufficiently large, then there is some constant C so that

$$\mathrm{Re}\langle Au, u\rangle \ge -C\|u\|^2_{H^{\frac{m-1}{2}}}.$$

The proof of this general form of Gårding's inequality is somewhat involved (see Theorem 18.1.14 of [Hö3]), but is straightforward when the principal symbol of A is a square (or sum of squares). Indeed, if $\sigma_m(A) = b^2$, then by the composition and adjoint formulas, we may write $A = B^*B + R$, where $B \in \Psi^{\frac{m}{2}}$ and $R \in \Psi^{m-1}$. It then follows that

$$\mathrm{Re}\langle Au, u\rangle = \|Bu\|^2 + \langle Ru, u\rangle.$$

The first term on the right side is non-negative. Writing R as the composition of two operators of order $\frac{m-1}{2}$ (such as $R = (I-\Delta)^{\frac{m-1}{4}} \circ (I-\Delta)^{-\frac{m-1}{4}}R$) then shows that this last term is bounded below by $-C\|u\|^2_{H^{\frac{m-1}{2}}}$ for some C. Observe further that combining this proof with the elliptic estimate (5) shows that if $A \in \Psi^m$ is compactly supported in x, has non-negative principal symbol, and $B \in \Psi^0$ has $\mathrm{ess\,supp}\, a \subset \mathrm{ell}(b)$, then for any N there is a constant C with

$$\mathrm{Re}\langle Au, u\rangle \ge -C\|Bu\|^2_{H^{\frac{m-1}{2}}} - C\|u\|^2_{H^{-N}}. \tag{6}$$

3. Proof of theorem

In this section we use the notation (z, ζ) for a variable point of $\mathbb{R}^d \times \mathbb{R}^d$, to avoid a typographical collision with $(x, \hat\xi)$ and $(x', \hat\xi')$ which we use to denote points in the essential support or elliptic set of a particular symbol.

The main lemma in the proof of the theorem is the following similar but weaker statement.

Lemma 2. *Let $b \in S^k$, $e \in S^k$, $g \in S^{k-1}$ for some real k be compactly supported in x. Let $\Lambda_s = (I - \Delta_{\mathbb{R}^d})^{s/2} \in \Psi^s$. If b is controlled by e through g, then there is a constant C such that*

$$\|Bu\|^2 \le C\left(\|\Lambda_{1-k}Gu\|^2_{H^{k-\frac{1}{2}}} + \|Eu\|^2 + \|GPu\|^2 + \|u\|^2_{H^{-N}}\right), \tag{7}$$

for all functions $u \in H^k$ with $Pu \in H^{k-1}$.

Proof. We assume without loss of generality that $\mathrm{ess\,supp}(e) \subset \mathrm{ell}(g)$; this can be arranged to hold without changing the hypotheses by shrinking the essential support of e. An application of the elliptic estimate (5) then yields the result with the original E.

Similarly, by a partition of unity argument, it is enough to prove that, for each point $(x, \hat\xi)$ in $\mathrm{ess\,supp}(b)$, there is $b_1 \in S^k$ which is elliptic at $(x, \hat\xi)$ such that (7) holds with B replaced by B_1.

For $A \in \Psi^{k-\frac{1}{2}}$ to be specified later, consider the pairing

$$\mathrm{Im}\langle Au, APu\rangle = \frac{1}{2i}\left(\langle PA^*Au, u\rangle - \langle A^*APu, u\rangle\right) = \left\langle \frac{1}{2i}[P, A^*Au]u, u\right\rangle.$$

1. We first bound the pairing below. By Cauchy–Schwarz, for any $\epsilon > 0$ we have

$$\mathrm{Im}\langle Au, APu\rangle \geq -\frac{1}{4\epsilon}\|APu\|^2_{H^{-1/2}} - \epsilon\|Au\|^2_{H^{1/2}}.$$

To write this bound in terms of the L^2 pairing we use $\|\Lambda_s u\|_{L^2} = \|u\|_{H^s}$. That gives

$$\left\langle \frac{1}{2i}[P, A^*Au]u, u\right\rangle \geq -\frac{1}{4\epsilon}\|\Lambda_{-1/2}APu\|^2_{L^2} - \epsilon\|\Lambda_{1/2}Au\|^2_{L^2}. \tag{8}$$

2. We next bound the pairing above. Recall that $\frac{1}{2i}[P, A^*A] \in \Psi^{2k}$ and has principal symbol given by $\frac{1}{2}H_P(a^2) = aH_p(a)$. We will construct a in such a way that there exist a compact set $K \subset \mathrm{ell}(e)$ and a $\gamma > 0$ such that

$$-aH_p(a) - \gamma\left(\langle\zeta\rangle^{1/2}a\right)^2 \geq 0, \qquad \text{off of } K. \tag{9}$$

Hence there exists C large enough that[4]

$$Ce^2 - aH_p(a) - \gamma\left(\langle\zeta\rangle^{1/2}a\right)^2 \geq 0.$$

In our construction of a we will also obtain

$$\text{ess supp}(a) \subset \mathrm{ell}(g). \tag{10}$$

Using the version of the Gårding inequality stated in (6)[5] then yields

$$\left\langle \frac{1}{2i}[P, A^*A]u, u\right\rangle \leq C\|Eu\|^2 + C\|\Lambda_{1-k}Gu\|^2_{H^{k-\frac{1}{2}}} + C\|u\|_{H^{-N}} - \gamma\|\Lambda_{1/2}Au\|^2. \tag{11}$$

3. We now combine the two bounds on the pairing. Putting together (8) and (11) and taking $\epsilon = \gamma/2$ yields

$$\|\Lambda_{1/2}Au\|^2 \leq C\left(\|\Lambda_{1-k}Gu\|^2_{H^{k-\frac{1}{2}}} + \|Eu\|^2 + \|\Lambda_{-1/2}APu\|^2 + \|u\|^2_{H^{-N}}\right).$$

To deduce (7) we use the fact that the elliptic estimate (5) and (10) imply $\|\Lambda_{-1/2}APu\|^2 \leq C(\|GPu\|^2 + \|u\|^2_{H^{-N}})$, and moreover we can use $\Lambda_{1/2}A$ as B_1 provided we construct a so that

$$(x, \hat{\xi}) \in \mathrm{ell}(a). \tag{12}$$

Thus is remains to construct $a \in S^{k-\frac{1}{2}}$ such that (9), (10), and (12) all hold.

4. To construct a, use the fact that there is a light ray contained in $\mathrm{ell}(g)$ starting at some $(x', \hat{\xi}) \in \mathrm{ell}(e)$ ending at $(x, \hat{\xi})$, and this light ray can be parametrized by

$$t \mapsto \left(x_0 \pm t\hat{\xi}_0, x_j \mp t\hat{\xi}_j, \hat{\xi}\right),$$

with the choice of sign depending on whether the flow from $(x', \hat{\xi})$ to $(x, \hat{\xi})$ is in the direction the flow of the Hamilton vector field or its opposite. We assume for the argument below that it is with the direction of the Hamilton vector field (and hence the top sign is chosen).

[4] In our setting, $\gamma > 0$ can be arbitrary, but when dealing with error terms γ must be taken sufficiently large.

[5] If we want to avoid invoking the full strength of the sharp Gårding inequality as in Theorem 18.1.14 of [Hö3], we must check that the quantity $Ce^2 - aH_p(a) - \gamma\langle\zeta\rangle a^2$ is a sum of squares. For that, take $\psi \in S^0$ such that $\psi = 0$ near K and $\psi = 1$ near the complement of $\mathrm{ell}(e)$, and use (15) to write $Ce^2 - aH_p(a) - \gamma\langle\zeta\rangle a^2 = a_1^2 + a_2^2$, where

$$a_1 = e\sqrt{C + (1-\psi^2)(-aH_p(a) - \gamma\langle\zeta\rangle a^2)/e^2}, \qquad a_2 = \psi\langle\zeta\rangle^k\chi\sqrt{(-2\varphi' - \gamma\varphi)\varphi}.$$

To see that a_1 is smooth use the fact that the square root of a positive function is smooth. To see that a_2 is smooth, use the explicit formula for φ.

Recall that $(x, \hat{\xi})$ lies on a light ray, and because light rays must lie in the characteristic set of P, $\zeta_0 \neq 0$ on any light ray. Using also the fact that $\mathrm{ell}(g)$ and $\mathrm{ell}(e)$ are open sets, fix $t_0 > 0$, $\delta > 0$, and an open neighborhood $U \subset \mathbb{R}^{d-1} \times \mathbb{S}^{d-1}$ of $(x, \hat{\xi})$ such that $\hat{\zeta}_0$ is non-vanishing on U, such that

$$t \in (-t_0 - \delta, \delta) \text{ and } (y_1, \dots, y_n, \hat{\zeta}) \in U \qquad \Longrightarrow \qquad (x_0 + t\hat{\zeta}_0, y_j - t\hat{\zeta}_j, \hat{\zeta}) \in \mathrm{ell}(g), \tag{13}$$

and such that

$$t \in (-t_0 - \delta, t_0 + \delta) \text{ and } (y_1, \dots, y_n, \hat{\zeta}) \in U \qquad \Longrightarrow \qquad (x_0 + t\hat{\zeta}_0, y_j - t\hat{\zeta}_j, \hat{\zeta}) \in \mathrm{ell}(e). \tag{14}$$

We fix a real function $\chi_1 \in C_c^\infty(U)$ that is identically 1 on a neighborhood of $(x, \hat{\xi})$, and put $\chi = \chi_1^2$. Let

$$\varphi(t) = \exp(-\gamma t + (t - \delta)^{-1} - (t + t_0 + \delta)^{-1}), \qquad \text{when } -t_0 - \delta < t < \delta,$$

and $\varphi(t) = 0$ otherwise.

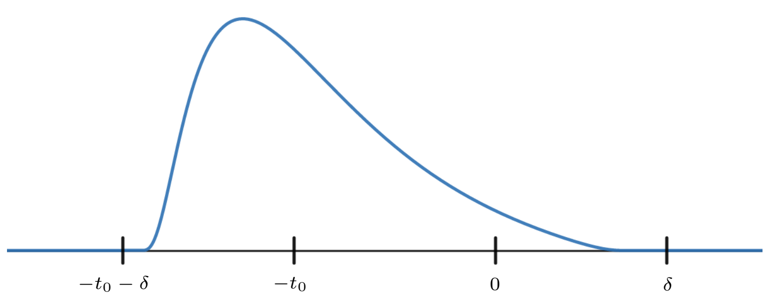

FIGURE 2. A graph of φ, from `https://www.desmos.com/calculator/tqmurytqe9`.

Let

$$a(z, \zeta) = \varphi(t)\chi\left(z_1 + \hat{\zeta}_1 t\,,\, \dots, z_n + \hat{\zeta}_n t\,,\, \hat{\zeta}\right)\langle\zeta\rangle^{k-\frac{1}{2}},$$

where $t = t(z, \zeta) = (z_0 - x_0)/\hat{\zeta}_0$. The expression for a is well-defined as $\hat{\zeta}_0 \neq 0$ on the support of χ and it is straightforward to check that $a \in S^{k-\frac{1}{2}}$. We write more shortly $a = \varphi\chi\langle\zeta\rangle^{k-\frac{1}{2}}$.

Now (10) follows from the property (13) of U, and (12) follows from the fact that $a\langle\zeta\rangle^{-k+\frac{1}{2}} \neq 0$ at (x, ξ). It remains to check (9), and for this we compute $aH_P(a)$.

The computation is made straightforward by the fact that we have defined a to be the product of $\varphi(t)$ and a function which is constant along flows of H_P:

$$aH_P(a) = 2\frac{\zeta_0}{\hat{\zeta}_0}\varphi'\varphi\chi^2\langle\zeta\rangle^{2k-1},$$

which, has principal symbol

$$2\langle\zeta\rangle^{2k}\varphi'\varphi\chi^2,$$

because $\zeta_0/\hat{\zeta}_0 = |\zeta|$. Thus the principal symbol of $-aH_p(a) - \gamma\langle\zeta\rangle a^2$ is

$$-(2\varphi' + \gamma\varphi)\varphi\chi^2\langle\zeta\rangle^{2k}, \tag{15}$$

and the set where this is negative is contained in the set where $-t_0 - \delta \leq t \leq -t_0$ and in the support of χ, i.e. by the property (14) of U above it is contained in a compact subset of $\mathrm{ell}(e)$. $\square$

With Lemma 2 in hand, we now turn our attention to the proof of Theorem 1. We proceed by removing the first term of the right side of the estimate (7) with an inductive argument and then finally relax the regularity hypothesis with a regularization argument.

We first consider the inductive argument. Indeed, we claim that if b is controlled by e through g, then for any m, there is a constant so that

$$\|Bu\|^2 \leq C\left(\|\Lambda_{1-k}Gu\|^2_{H^{k-\frac{m}{2}}} + \|Eu\|^2 + \|GPu\|^2 + \|u\|^2_{H^{-N}}\right). \tag{16}$$

Lemma 2 and the regularization argument above provide the base case $m = 1$.

Suppose now that the inequality (16) holds for m. As $\mathrm{ell}(g)$ and $\mathrm{ell}(e)$ are open and $\mathrm{ess\,supp}(b)$ is closed, we may shrink the support of g to find a new symbol $g_1 \in S^{k-1}$ so that

(1) b is controlled by e through g_1, and
(2) $\langle\zeta\rangle^{1-\frac{m}{2}}g_1$ is controlled by $\langle\zeta\rangle^{-\frac{m}{2}}e$ through $\langle\zeta\rangle^{-\frac{m}{2}}g$.

The inductive hypothesis yields the estimate

$$\|Bu\|^2 \leq C\left(\|\Lambda_{1-k}G_1u\|^2_{H^{k-\frac{m}{2}}} + \|Eu\|^2 + \|G_1Pu\|^2 + \|u\|^2_{H^{-N}}\right).$$

We then apply Lemma 2 to $\langle\zeta\rangle^{1-\frac{m}{2}}g_1 \in S^{k-\frac{m}{2}}$, which is controlled by $\langle\zeta\rangle^{-\frac{m}{2}}e \in S^{k-\frac{m}{2}}$ through $\langle\zeta\rangle^{-\frac{m}{2}}g \in S^{k-1-\frac{m}{2}}$, yielding

$$\|\Lambda_{1-\frac{m}{2}}G_1u\|^2 \leq C\left(\|\Lambda_{1-k}Gu\|^2_{H^{k-\frac{m+1}{2}}} + \|\Lambda_{-\frac{m}{2}}Eu\|^2 + \|\Lambda_{-\frac{m}{2}}GPu\|^2 + \|u\|^2_{H^{-N}}\right).$$

We note first that for any $v \in H^r$, $\|\Lambda_r v\| = \|v\|_{H^r}$. One consequence is the estimate

$$\|\Lambda_{-\frac{1}{2}}v\| = \|v\|_{H^{-\frac{1}{2}}} \leq \|v\|_{L^2}.$$

Similarly,

$$\|\Lambda_{1-k}G_1u\|_{H^{k-\frac{1}{2}}} = \|\Lambda_{\frac{1}{2}}G_1u\|,$$

so that combining the two estimates yields

$$\|Bu\|^2 \leq C\left(\|\Lambda_{1-k}Gu\|^2_{H^{k-\frac{m+1}{2}}} + \|Eu\|^2 + \|GPu\|^2 + \|u\|^2_{H^{-N}}\right),$$

finishing the inductive step.

For $m \geq 2N + 2k$,

$$\|\Lambda_{1-k}Gu\|_{H^{k-\frac{m}{2}}} \leq C\|u\|_{H^{-N}},$$

allowing us to remove this term.

Our regularization argument roughly follows Exercises E.10 and E.31 from [DyZw]. We introduce, for fixed $r > 0$, a family of regularizing operators depending on a parameter $\epsilon \in (0,1)$:

$$\Lambda_{\epsilon,-r} = (1 - \epsilon^2\Delta_{\mathbb{R}^d})^{-r/2}.$$

The inverse of $\Lambda_{\epsilon,-r}$ is given by

$$\Lambda_{\epsilon,r} = (1 - \epsilon^2\Delta)^{r/2}.$$

For each $\epsilon > 0$, $\Lambda_{\epsilon,-r} \in \Psi^{-r}$ with symbol $\langle\epsilon\zeta\rangle^{-r}$; regarded as a subset of S^0, this family of symbols is uniformly bounded in ϵ. In particular, $\Lambda_{\epsilon,-r} \in \Psi^{-r}$ is a uniformly bounded family in Ψ^0 converging to the identity map in Ψ^s for any $s > 0$. Similarly, $\Lambda_{\epsilon,r} \in \Psi^r$ is a uniformly bounded family in Ψ^r.

One useful application of these operators forms the backbone of the regularization argument. The monotone convergence theorem implies that if $u \in H^{s-r}$ and $\|\Lambda_{\epsilon,-r}u\|_{H^s}$ is uniformly bounded in ϵ, then in fact $u \in H^s$.

For each ϵ, the symbol $\langle \epsilon\zeta\rangle^{-r} b$ is controlled by $\langle \epsilon\zeta\rangle^{-r} e$ through $\langle \epsilon\zeta\rangle^{-r} g$. Viewing these symbols as lying in S^k, S^k, and S^{k-1}, respectively, we obtain estimates of the form

$$\|\Lambda_{\epsilon,-r}Bu\|^2 \leq C\left(\|\Lambda_{\epsilon,-r}Eu\|^2 + \|\Lambda_{\epsilon,-r}GPu\|^2 + \|u\|^2_{H^{-N}}\right),$$

provided that $\Lambda_{\epsilon,-r}u \in H^k$ and $\Lambda_{\epsilon,-r}Pu \in H^{k-1}$. Because the symbols above are uniformly bounded in S^k, S^k, and S^{k-1}, an inspection of the proof of Lemma 2 and the inductive argument shows that the constant can be taken to be uniform in ϵ. As u lies in H^{-N} for some N large enough, we may fix r (depending on N) so that $\Lambda_{\epsilon,-r}u \in H^k$ and $\Lambda_{\epsilon,-r}Pu \in H^{k-1}$ and then obtain the bound

$$\|\Lambda_{\epsilon,-r}Bu\|^2 \leq C\left(\|Eu\|^2 + \|GPu\|^2 + \|u\|^2_{H^{-N}}\right).$$

In particular, $\|\Lambda_{\epsilon,-r}Bu\|$ is uniformly bounded in ϵ and so $Bu \in L^2$, finishing the proof.

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