ASYMPTOTICS OF RADIATION FIELDS IN ASYMPTOTICALLY MINKOWSKI SPACE

DEAN BASKIN, ANDRÁS VASY, AND JARED WUNSCH

Abstract. We consider a non-trapping n-dimensional Lorentzian manifold endowed with an end structure modeled on the radial compactification of Minkowski space. We find a full asymptotic expansion for tempered forward solutions of the wave equation in all asymptotic regimes. The rates of decay seen in the asymptotic expansion are related to the resonances of a natural asymptotically hyperbolic problem on the "northern cap" of the compactification. For small perturbations of Minkowski space that fit into our framework, our asymptotic expansions yield a rate of decay that improves on the Klainerman–Sobolev estimates.

1. Introduction

In this paper we consider the asymptotics of solutions to the wave equation on a class of spacetimes including asymptotically Minkowski space, as well as more general spacetimes that have compactifications similar to the radial compactification of Minkowski space. Subject to a condition of non-trapping of null geodesics, namely that they tend toward null-infinity, we find a full asymptotic expansion for tempered forward solutions of the wave equation in all asymptotic regimes. Most notably, we find compound asymptotics for the solution near null infinity. The rates of decay seen in the asymptotic expansion are related to the resonances of a natural asymptotically hyperbolic problem on the "northern cap" of the compactification. (This cap corresponds to the interior of the future light cone in Minkowski space.) In the special case of small perturbations of Minkowski space, these expansions imply a rate of decay that improves on the Klainerman–Sobolev estimates.

More specifically, we consider an n-dimensional Lorentzian manifold endowed with the end structure of a "scattering manifold" motivated by the analogous definition for Riemannian manifolds given by Melrose [16]. Our manifolds come equipped with compactifications to smooth manifolds-with-boundary, i.e., we will consider the Lorentzian manifold (M°, g) where M

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is a manifold with boundary denoted $X = \partial M$. The key example is the radial compactification of Minkowski space $\mathbb{R}^{1,n-1}_{(t,x)}$, where X is a "sphere at infinity," with boundary defining function $\rho = (|x|^2 + t^2 + 1)^{-1/2}$. On M the forward and backward light cones emanating from any point $q \in M^\circ$ terminate at ∂M in manifolds $S_\pm$ independent of the choice of q; we call $S_\pm$ the future and past light cones at infinity, and they bound submanifolds (which are open subsets) $C_\pm \subset X$, which we call future (C_+) and past infinity (C_-). In the case of Minkowski space C_+ and C_- are the "north" and "south" polar regions (or caps) on the sphere at infinity (see Figure 1). Further, there is an intermediate region C_0 ("equatorial" on the sphere at infinity in the case of Minkowski space) which has as its two boundary hypersurfaces S_+ and S_-. We assume that the metric g is *non-trapping* in the sense that all maximally extended null-geodesics approach S_- at one end and S_+ at the other. The full set of geometric hypotheses is described in detail in Section 3.2.

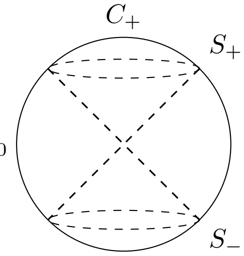

FIGURE 1. The polar and equatorial regions in Minkowski space

We consider solutions w to the wave equation

$$\Box w = f \in \dot{\mathcal{C}}^\infty(M)$$

on such a manifold so that w is tempered and vanishes near the "past infinity" $\overline{C}_-$ (thus f also vanishes near $\overline{C}_-$); here $\dot{\mathcal{C}}^\infty(M)$ denotes $\mathcal{C}^\infty$ functions on M vanishing at ∂M with all derivatives – in the case of Minkowski space this amounts to the set of Schwartz functions. In [22, Section 5] the asymptotic behavior of the solution of the wave equation was analyzed in C_+ on Minkowski space in a manner that extends to our more general setting in a straightforward manner, giving a polyhomogeneous asymptotic expansion in the boundary defining function ρ; the exponents arising in this expansion are related to the *resonances* of the Laplace operator associated to a certain natural *asymptotically hyperbolic* Riemannian metric on C_+.

The main result of this paper is to obtain the *precise asymptotic behavior* of the solution w near the light cone at infinity, $S_+ = \partial C_+$, performing a uniform (indeed, conormal, on an appropriately resolved space) analysis as S_+ is approached in different ways. This amounts to a *blow-up* of S_+

in M. In Minkowski space $(t,x) \in \mathbb{R}^{1+3}$, locally near the interior of this front face (denoted ff), the blow up amounts to introducing new coordinates $\rho = (|x|^2 + t^2 + 1)^{-1/2}$, $s = t - |x|$, $y = x/|x|$, the front face itself being given by $\rho = 0$, so $s = t - |x|$, $y = x/|x|$ are the coordinates on the front face. More generally, if ρ is a defining function for the boundary at infinity of M and v is a defining function for $S_+ \subset X$ with (v,y) a coordinate system on X, we can let $s = v/\rho$ and use s,y as coordinates on the interior of the front face of the blow-up. Thus, s measures the angle of approach to S_+, with $s \to +\infty$ corresponding to approach from C_+, while $s \to -\infty$ corresponds to approach from C_0.

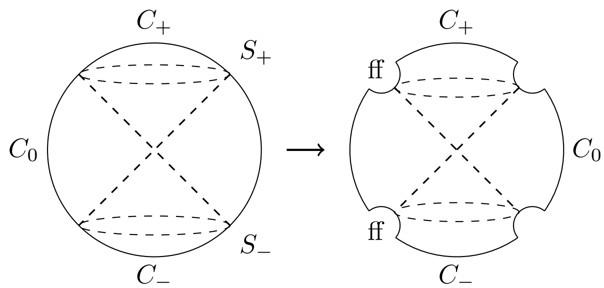

FIGURE 2. The radiation field blow-up of Minkowski space

In order to make a statement without compound asymptotics, we consider the so-called radiation fields. Thus, in this paper we show the existence of the *Friedlander radiation field*, which in the Minkowski setting is given in the coordinates introduced above by

$$\mathcal{R}_+[w](s,y) = \partial_s \rho^{-(n-2)/2} w(\rho, s, y)|_{\rho=0},$$

i.e., by restricting an appropriate rescaling of the derivative of w to the new face obtained by blowing up the future light cone at infinity S_+. (See Section 10 for further discussion.) The function $\mathcal{R}_+$ thus measures the radiation pattern seen by an observer far from an interaction region; in the case of static metrics, it is known to be an explicit realization of the Lax-Phillips translation representation as well as a geometric generalization of the Radon transform [5].

Our main theorem concerns the asymptotics of the radiation field as s, the "time-delay" parameter, tends toward infinity, and more generally the multiple asymptotics of the solution near the forward light cone.

Theorem 1.1. *If (M,g) is a compact manifold with boundary with a non-trapping Lorentzian scattering metric as defined in Section 3.2 and w is a tempered solution of $\Box_g w = f \in \dot{\mathcal{C}}^\infty(M)$ vanishing in a neighborhood of $\overline{C}_-$,* *then the radiation field of w has an asymptotic expansion of the following form as $s \to \infty$:*

$$\mathcal{R}_+[w](s, y) \sim \sum_j \sum_{\kappa \le m_j} a_{j\kappa} s^{-\imath\sigma_j - 1} (\log s)^\kappa$$

Moreover, w has a full asymptotic expansion at all boundary faces with the compound asymptotics given by:

$$w \sim \rho^{\frac{n-2}{2}} \sum_j \sum_{\ell=0}^{N} \sum_{\kappa+\alpha \le \tilde{m}_{j\ell}} \rho^\ell |\log \rho|^\kappa (\log \rho + \log s)^\alpha a_{j\ell\kappa\alpha} s^{-\imath\sigma_j} \tag{1.1}$$

Remark 1.2. Although it may appear in (1.1) that the log terms may obstruct the restriction to $\rho = 0$ and hence the definition of the radiation field, we show in Section 8 that the log terms cancel in the $\ell = 0$ term, enabling this restriction.

We also remark that the power of s in the second formula differs from the previous one by 1 due to a derivative in the definition of the radiation field.

Remark 1.3. In Minkowski space, the requirement that w vanish in a neighborhood of $\overline{C_-}$ implies that w is the *forward* solution of $\Box_g w = f$. One should then think of the vanishing requirement as analogous to taking the forward solution of $\Box_g w = f$.

Remark 1.4. In Minkowski-like settings, namely when the forward and backward problems are well-posed in the sense that given an element f of $\dot{\mathcal{C}}^\infty(M)$ supported away from $\overline{C_+} \cup \overline{C_-}$ there are unique tempered w_+, resp. w_-, with $\Box_g w_\pm = f$ and with w_+, resp. w_- vanishing near $\overline{C_-}$, resp. $\overline{C_+}$, one can turn the solution of the Cauchy problem for appropriate space-like hypersurfaces transversal to C_0 (and intersecting ∂M in C_0 only) with Schwartz initial data into the sum of a "forward solution" to which our theorem applies (supported away from $\overline{C_-}$), and a similar "backward solution" (supported away from $\overline{C_+}$), to which the analogue of our theorem (interchanging C_+ with C_-) applies. Thus, in particular, the asymptotic behavior of solutions of the Cauchy problem with Schwartz initial data is given by our theorem. A statement related to, but slightly weaker than, this forward and backward well-posedness follows from our assumptions (see Remark 3.7), but improving on this, possibly under some additional hypotheses, requires addressing issues beyond the scope of our paper, and thus will be taken up elsewhere.

Remark 1.5. It follows from our arguments that the expansion depends continuously on w and f satisfying a fixed support condition (support in a fixed compact set disjoint from $\overline{C_-}$) in the tempered, respectively $\dot{\mathcal{C}}^\infty(M)$, topologies. In particular, finite expansions follow from imposing finite regularity assumptions, but we do not attempt to make optimal statements in this paper regarding the required regularity assumptions.

A question of considerable interest is, of course, whether the radiation field actually *decays* as $s \to +\infty$ and, more generally, the description of the exponents σ_j. Remarkably, these are the resonance poles of a naturally-defined asymptotically hyperbolic metric defined on C_+. (More precisely, the poles we are interested in are those of the inverses of a family of operators that looks to leading order like an asymptotically hyperbolic Laplacian. It is not in general a *spectral family* of the form $P - \sigma^2$ however: the σ-dependence is more complex.) We denote the family of asymptotically hyperbolic operators by $L_{\sigma,+}$, and record the following corollary:

Corollary 1.6. *If there exists $C > 0$ such that $L_{\sigma,+}^{-1}$ has no poles at σ with $\operatorname{Im}\sigma > -C$ then for all $\epsilon > 0$, the radiation field decays at a rate $O(s^{-C-1+\epsilon})$.*

One class of spacetimes to which our theorem (and corollary) applies is that of *normally short-range* perturbations of Minkowski space, i.e., perturbations of the metric which are, relative to the original metric, $O(\rho^2)$ in the normal-to-the boundary component, $d\rho^2/\rho^4$, $O(1)$ in the tangential-to-the-boundary components, dv^2/ρ^2, dy^2/ρ^2 and $dv\, dy/\rho^2$, and $O(\rho)$ in the mixed components. In particular, note that we are permitted make *large* perturbations of the spherical metric on the cap C_+, hence in these "tangential" metric components our hypotheses allow a much wider range of geometries than even traditional "long-range" perturbations of Minkowski space.

We note here that the long-range structure of the Schwarzschild and Kerr spacetimes near null infinity does not fit into our class of spacetimes. Such spacetimes will be addressed in the follow-up to this paper. Also left to a future paper is the question of how to integrate the decay estimates for geometries with mild (e.g., normally hyperbolic) trapping into our analysis.

In the more restrictive setting of *"normally very short range"* perturbations (defined at the beginning of Section 10.1), we recover the same asymptotically hyperbolic problem at infinity as in the Minkowski case, and therefore exhibit the same order decay as seen on Minkowski space. In particular, in odd spatial dimensions one has rapid decay of solutions of the wave equation away from the light cone. Thus, we obtain the following corollary for "normally very short range" perturbations of Minkowski space:

Corollary 1.7. *If (M, g) is a normally very short range non-trapping perturbation of n-dimensional Minkowski space, w vanishes near $\overline{C}_-$, and $\Box_g w = f \in \dot{\mathcal{C}}^\infty(M)$, then the radiation field of w has an asymptotic expansion of the following form:*

$$\mathcal{R}_+[w](s,\omega) \sim \begin{cases} O(s^{-\infty}) & n \text{ even} \\ \sum_{j=0}^\infty \sum_{\kappa \le j} s^{-\frac{n}{2}-j} (\log s)^\kappa a_{j\kappa} & n \text{ odd} \end{cases}$$

More generally in the case of normally short-range perturbations, given $\epsilon > 0$, if the $O(1)$ metric perturbations of the tangential-to-the-boundary metric components are sufficiently small then the radiation field still decays as $s \to$

$+\infty$:

$$\mathcal{R}_+[w](s,\omega) \lesssim s^{\alpha+\epsilon}, \quad \alpha = -\min(2, n/2).$$

The polynomial decay of solutions of the wave equation may be compared with the Klainerman–Sobolev estimates [13]. (We refer the reader to the book of Alinhac [1, Chapter 7] for a more detailed introduction to such estimates.) In n-dimensional spacetimes where the isometries (and conformal isometries) of Minkowski space (i.e., the translations, rotations, Lorentz boosts, and scaling) are "asymptotic isometries" (or "asymptotic conformal isometries"), then solutions w of the wave equation exhibit decay of the form

$$|\partial w(t,r,\theta)| \lesssim \frac{1}{(t+r)^{(n-2)/2}(t-r)^{1/2}}.$$

In terms of these coordinates, the asymptotic expansion we obtain implies that on our class of Lorentzian manifolds (in particular, on normally short-range perturbations of Minkowski space), there is some α so that solutions w of the wave equation satisfy

$$|\partial w(t,r,\theta)| \lesssim \frac{1}{(t+r)^{(n-2)/2}}(t-r)^{\alpha}.$$

When there are no eigenvalues of the associated asymptotically hyperbolic problem, then $\alpha \leq 0$; in particular, on normally very short range perturbations of Minkowski space (see Section 10.1), $\alpha = -n/2$ if n is odd and $\alpha = -\infty$ if n is even. Further, the resonances of the asymptotically hyperbolic problem depend continuously on perturbations in an appropriate sense. The operator P_σ^{-1} introduced below is stable, but may contain additional poles at certain pure imaginary integers as compared to the asymptotically hyperbolic problem (as is the case in even dimensional Minkowski space). Although such poles do not contribute to the asymptotics of the radiation field, under perturbations they may become poles of $L_{\sigma,+}^{-1}$. Thus, for *small* normally short range perturbations of Minkowski space, α is close to $-\min(2, n/2)$ (rather than $-\infty$). (In higher dimensions, one may improve this statement to obtain α close to $-n/2$ by a careful analysis of resonant states supported exactly at the light cone. As the most interesting case is $n = 4$, when $n/2 = 2$, we do not pursue this improvement here.)

The class of spacetimes we consider is geometrically more general than the class of spacetimes on with the Klainerman–Sobolev estimates hold, but we require a complete asymptotic expansion of the metric (and thus considerably more smoothness at infinity). The methods we employ would, however, allow also for finite expansions when the metric has a finite expansion, using more careful accounting.

Finally, we note that a remarkable extension of the radiation field construction has been obtained by Wang [26] to the nonlinear setting of the Einstein vacuum equations on perturbations of Minkowski space in spacetimes of dimension 5 and higher. This is based in part on very strong estimates of Melrose-Wang [18] in the linear setting for the special case of asymptotically Minkowski metrics.

1.1. **A sketch of the proof of Theorem 1.1.** We start with a tempered solution w of $\Box_g w = f' \in \dot{\mathcal{C}}^\infty(M)$ vanishing identically in a neighborhood of $\overline{C}_-$. We then fix $\chi \in C^\infty(M)$ supported near ∂M so that χ is identically 0 near $\overline{C}_-$, identically 1 near the portion of the boundary where w is non-vanishing. In particular, the support of $w\, d\chi$ is compact in M° and $\chi w = w$ near ∂M. We then consider the function $u = \rho^{-(n-2)/2}\chi w$ and set

$$L = \rho^{-2}\rho^{-(n-2)/2}\Box_g \rho^{(n-2)/2}.$$

The function u then solves $Lu = f$ for some other function $f \in \dot{\mathcal{C}}^\infty(M)$ vanishing near $\overline{C}_-$. A propagation of singularities argument (Section 4) shows that u is conormal to $\{\rho = v = 0\}$.

We now set $P_\sigma = \widehat{N}(L)$, where $\widehat{N}$ is the reduced normal operator, i.e., the family of operators on the boundary at infinity obtained by Mellin transform in the normal variable ρ. If we set $\tilde{u}_\sigma$, $\tilde{f}_\sigma$ to be the Mellin transforms of u, and f, respectively, then $\tilde{u}_\sigma$ solves

$$P_\sigma \tilde{u}_\sigma = \tilde{f}_\sigma + \text{ errors},$$

where the additional correction terms arise because L is not assumed to be dilation-invariant. We show that the operator P_σ fits into the framework of Vasy [22] and modify the argument of that paper to show that P_σ is invertible on certain variable-order Sobolev spaces (Section 5). The argument further shows that P_σ^{-1} is a meromorphic family of Fredholm operators with finitely many poles in each horizontal strip. In fact, the poles of P_σ^{-1} may be identified with resonances for an asymptotically hyperbolic problem (Section 7).

An argument of Haber–Vasy [7] implies that the residues at the poles of P_σ^{-1} are L^2-based conormal distributions. In Sections 6 and 8 we show that they are in fact classical conormal distributions and thus have an expansion in terms of v. We calculate the leading terms of the expansion somewhat explicitly. Inverting the Mellin transform and iteratively shifting the contour of integration in the Mellin inversion (Section 9) realizes these residues as the coefficients of an asymptotic expansion for u in terms of ρ.

A slight complication is that not only do the terms of the expansion become more singular as distributions on ∂M as one obtains more decay (as is indeed necessary for them to contribute to the radiation field in the same way, i.e., letting $\rho \to 0$ with $s = v/\rho$ fixed), but the remainder term also becomes more singular. We use the a priori conormal regularity, as shown in Section 4, to deal with this issue. The philosophy here is that since the algebra of b-pseudodifferential operators, discussed in Section 2 with further references given there, is not commutative to leading order in the sense of decay at ∂M (unlike, say, Melrose's scattering pseudodifferential algebra), one first should obtain regularity in the differential sense, which is the conormal regularity of Section 4, and then proceed to obtain decay estimates.

Finally, rewriting the expansion in terms of the radiation field blow-up $s = v/\rho$ yields an asymptotic expansion at all boundary hypersurfaces. The explicit computation of the leading terms shows that the logarithmic terms match up and thus u may be restricted to the front face of the blow-up, yielding the radiation field (after differentiation), and proving Theorem 1.1 in Section 10.

2. b-geometry and the Mellin transform

2.1. **Introduction to b-geometry.** For a more thorough discussion of b-pseudodifferential operators and b-geometry, we refer the reader to Chapter 4 of Melrose [15].

In this section and the following, we initially take M to be a manifold with boundary with coordinates $(\rho, y) \in [0,1) \times \mathbb{R}^{n-1}$ yielding a product decomposition $M \supset U \sim [0,1) \times \partial M$ of a collar neighborhood of ∂M. In particular, for now we lump the v variable in with the other boundary variables as it will not play a distinguished role.

The space of *b-vector fields*, denoted $\mathcal{V}_b(M)$, is the vector space of vector fields on M tangent to ∂M. In local coordinates (ρ, y) near ∂M, they are spanned over $C^\infty(M)$ by the vector fields $\rho\partial_\rho$ and ∂_y. We note that $\rho\partial_\rho$ is well-defined, independent of choices of coordinates, modulo $\rho\mathcal{V}_b(M)$; one may call this the *b-normal vector field* to the boundary. One easily verifies that $\mathcal{V}_b(M)$ forms a Lie algebra. The set of b-differential operators, $\mathrm{Diff}_{\mathrm{b}}^*(M)$, is the universal enveloping algebra of this Lie algebra: it is the filtered algebra consisting of operators of the form

$$A = \sum_{|\alpha|+j\leq m} a_{j,\alpha}(\rho, y)(\rho D_\rho)^j D_y^\alpha \in \mathrm{Diff}_{\mathrm{b}}^m(M) \tag{2.1}$$

(locally near ∂M) with the coefficients $a_{j,\alpha} \in \mathcal{C}^\infty(M)$. We further define a bi-filtered algebra by setting

$$\mathrm{Diff}_{\mathrm{b}}^{m,l}(M) \equiv \rho^{-l}\,\mathrm{Diff}_{\mathrm{b}}^m(M).$$

The b-pseudodifferential operators $\Psi_{\mathrm{b}}^*(M)$ are the "quantization" of this Lie algebra, formally consisting of operators of the form

$$b(\rho, y, \rho D_\rho, D_y)$$

with $b(\rho, y, \xi, \eta)$ a Kohn-Nirenberg symbol; likewise we let

$$\Psi_{\mathrm{b}}^{m,l}(M) = \rho^{-l}\Psi_{\mathrm{b}}^m(M)$$

and obtain a bi-graded algebra.

Remark 2.1. The convention we use for the sign of the weight exponent l is the opposite of that employed in some other treatments; we have chosen this convention as differential order and the weight order behave similarly: the space increases if either one of these is increased.

The space $\mathcal{V}_b(M)$ is in fact the space of sections of a smooth vector bundle over M, the *b-tangent bundle*, denoted ${}^{\mathrm{b}}TM$. The sections of this bundle are of course locally spanned by the vector fields $\rho\partial_\rho, \partial_y$. The dual bundle to ${}^{\mathrm{b}}TM$ is denoted ${}^bT^*M$ and has sections locally spanned over $\mathcal{C}^\infty(M)$ by the one-forms $d\rho/\rho, dy$. We also employ the *fiber compactification* $\overline{{}^{\mathrm{b}}T^*}M$ of ${}^bT^*M$, in which we radially compactify each fiber. If we let

$$\xi\frac{d\rho}{\rho} + \eta \cdot dy$$

denote the canonical one-form on ${}^bT^*M$ then a defining function for the boundary "at infinity" of the fiber-compactification is

$$\nu = (\xi^2 + |\eta|^2)^{-1/2};$$

a set of local coordinates on each fiber of the compactification near $\{v = \rho = 0\}$ is given by

$$\nu,\ \hat{\xi} = \nu\xi,\ \hat{\eta} = \nu\eta.$$

The symbols of operators in $\Psi_{\mathrm{b}}^*(M)$ are thus Kohn-Nirenberg symbols defined on ${}^bT^*M$. The principal symbol map, denoted σ_{b}, maps (the classical subalgebra of) $\Psi_{\mathrm{b}}^{m,l}(M)$ to ρ^{-l} times homogeneous functions of order m on ${}^bT^*M$. In the particular case of the subalgebra $\mathrm{Diff}_{\mathrm{b}}^{m,l}(M)$, if A is given by (2.1) we have

$$\sigma_{\mathrm{b}}(\rho^{-l}A) = \rho^{-l} \sum_{|\alpha|+j\leq m} a_{j,\alpha}(\rho, y)\xi^j\eta^\alpha$$

where ξ, η are "canonical" fiber coordinates on ${}^bT^*M$ defined by specifying that the canonical one-form be

$$\xi\frac{d\rho}{\rho} + \eta \cdot \frac{dy}{\rho}.$$

In addition to the principal symbol, which specifies high-frequency asymptotics of an operator, we will employ the "normal operator" which measures the boundary asymptotics. For a b-differential operator given by (2.1), this is simply the dilation-invariant operator given by freezing the coefficients of ρD_ρ and D_y at $\rho = 0$, hence

$$N(A) \equiv \sum_{|\alpha|+j\leq m} a_{j,\alpha}(0, y)(\rho D_\rho)^j D_y^\alpha \in \mathrm{Diff}_{\mathrm{b}}^m([0,\infty) \times \partial M).$$

The Mellin conjugate (see Section 2.3 below) of this operator is known as the "reduced normal operator" and is simply the family in σ of operators on ∂M given by

$$\widehat{N}(A) \equiv \sum_{|\alpha|+j\leq m} a_{j,\alpha}(0, y)\sigma^j D_y^\alpha.$$

This construction can be extended to b-pseudodifferential operators, but we will only require it in the differential setting here.

Here and throughout this paper we fix a "b-density," which is to say a density which near the boundary is of the form

$$f(\rho, y)\left|\frac{d\rho}{\rho} \wedge dy_1 \wedge \cdots \wedge dy_{n-1}\right|$$

with $f > 0$ everywhere. Let $L^2_{\mathrm{b}}(M)$ denote the space of square integrable functions with respect to the b-density. We let $H^m_{\mathrm{b}}(M)$ denote the Sobolev space of order m relative to $L^2_{\mathrm{b}}(M)$ corresponding to the algebras $\mathrm{Diff}^m_{\mathrm{b}}(M)$ and $\Psi^m_{\mathrm{b}}(M)$. In other words, for $m \geq 0$, fixing $A \in \Psi^m_{\mathrm{b}}(M)$ elliptic, one has $w \in H^m_{\mathrm{b}}(M)$ if $w \in L^2_{\mathrm{b}}(M)$ and $Aw \in L^2_{\mathrm{b}}(M)$; this is independent of the choice of the elliptic A. For m negative, the space is defined by dualization. (For m a positive integer, one can alternatively give a characterization in terms of $\mathrm{Diff}^m_{\mathrm{b}}(M)$.) Let $H^{m,l}_{\mathrm{b}}(M) = \rho^l H^m_{\mathrm{b}}(M)$ denoted the corresponding weighted spaces.

We recall also that associated to the calculus $\Psi^*_{\mathrm{b}}(M)$ is associated a notion of Sobolev wavefront set: $\mathrm{WF}^{m,l}_{\mathrm{b}}(w) \subset {}^{\mathrm{b}}S^*M$ is defined only for $w \in H^{-\infty,l}_{\mathrm{b}}$ (since $\Psi_{\mathrm{b}}(M)$ is not commutative to leading order in the decay order); the definition is then $\alpha \notin \mathrm{WF}^{m,l}_{\mathrm{b}}(w)$ if there is $Q \in \Psi^{0,0}_{\mathrm{b}}(M)$ elliptic at α such that $Qw \in H^{m,l}_{\mathrm{b}}(M)$, or equivalently if there is $Q' \in \Psi^{m,l}_{\mathrm{b}}(M)$ elliptic at α such that $Q'w \in L^2_{\mathrm{b}}(M)$. We refer to [11, Section 18.3] for a discussion of WF_{b} from a more classical perspective, and [17, Section 3] for a general description of the wave front set in the setting of various pseudodifferential algebras; [23, Sections 2 and 3] provide another discussion, including on the b-wave front set relative to spaces other than $L^2_{\mathrm{b}}(M)$.

2.2. **Scattering geometry.** We now turn to the analogous concepts of "scattering geometry" which will be less used in this paper but which are a useful motivation. For a full discussion of scattering geometry, we refer the reader to Melrose [16].

In analogy to the space of b-vector fields, we define *scattering vector fields* as $\mathcal{V}_{\mathrm{sc}} \equiv \rho\mathcal{V}_b$; that is to say, the vector fields when applied to ρ must return a smooth function divisible by ρ^2. They are locally spanned by $\rho^2\partial_\rho$ and $\rho\partial_y$ over $\mathcal{C}^\infty(M)$. We note that as the b-normal vector field $\rho\partial_\rho$ is well-defined modulo $\rho\mathcal{V}_b$, the *span* of $\rho^2\partial_\rho$ is well-defined modulo $\rho\mathcal{V}_{\mathrm{sc}}$; we call vector fields lying in this span *scattering normal vector fields.* The scattering vector fields form sections of a bundle ${}^{\mathrm{sc}}TM$; the dual bundle, ${}^{\mathrm{sc}}T^*M$ has sections locally spanned by $d\rho/\rho^2,\ dy/\rho$. As motivation for our discussions of the form of the "scattering metrics" below, we remark that if we radially compactify Euclidean space, the *constant* vector fields push forward to be scattering vector fields on the compactification, hence sections of the tensor square of ${}^{\mathrm{sc}}T^*M$ are the natural place for asymptotically Euclidean or Minkowskian metrics to live.

The *scattering differential operators* are those of the form (near ∂M)

$$\sum_{|\alpha|+j\leq m} a_{j,\alpha}(\rho, y)(\rho^2 D_\rho)^j(\rho D_y)^\alpha \in \mathrm{Diff}^m_{\mathrm{sc}}(M).$$

Again, this space of operators can be microlocalized by introducing *scattering pseudodifferential operators* which are formally objects given by

$$b(\rho, y, \rho^2 D_\rho, \rho D_y)$$

with $b(\rho, y, \xi, \eta)$ a Kohn-Nirenberg symbol on the bundle ${}^{\mathrm{sc}}T^*M$. There are of course associated scales of Sobolev spaces, which we will not have occasion to use in this paper, as well as wavefront sets which are described in detail in [16].

2.3. **Mellin transform.** We first recall the definition of the *Mellin transform* on $\mathbb{R}_+$. For a smooth compactly supported function, or indeed a Schwartz function, u on $\mathbb{R}_+$, $\tilde{u}_\sigma = \int_0^\infty \rho^{-\imath\sigma-1}u(\rho)\,d\rho$. (Here Schwartz means that for all $k \in \mathbb{R}$ all derivatives of u are bounded by a multiple of ρ^k both at 0 and at ∞, i.e. they vanish rapidly.) Because u is compactly supported or Schwartz, $\tilde{u}_\sigma$ is an entire function of σ which decays rapidly along each line of constant $\operatorname{Im}\sigma$. We will also use the notation

$$\mathcal{M}u = \tilde{u}$$

for the Mellin transform.

The Mellin transform on $\mathbb{R}_+$ is equivalent to the Fourier transform by the substitution $x = \log\rho$; note that the Schwartz behavior amounts to superexponential decay at $\pm\infty$ in terms of x. In particular, the Plancherel theorem guarantees that it extends to an isomorphism of Hilbert spaces

$$L^2(\mathbb{R}_+; d\rho/\rho) \to L^2(\mathbb{R}),$$

and, more generally, to an isomorphism with a weighted space,

$$\rho^\delta L^2(\mathbb{R}_+; d\rho/\rho) \to L^2(\{\operatorname{Im}\sigma = -\delta\}).$$

Moreover, the Mellin transform intertwines $\rho\partial_\rho$ with multiplication by $\imath\sigma$:

$$\widetilde{(\rho\partial_\rho u)}_\sigma = \imath\sigma\tilde{u}_\sigma$$

The inverse Mellin transform is given by integrating $\tilde{u}_\sigma\rho^{i\sigma}$ along a horizontal line $\{\operatorname{Im}\sigma = C\}$, provided this integral exists.

Near the boundary of M, we use the boundary defining function ρ to obtain a local product decomposition: $M \supset \mathrm{nbhd}(X) = [0, \epsilon)_\rho \times \partial M$. This local product decomposition allows us to define the Mellin transform for functions supported near ∂M via cut-off functions that are identically 1 for $\rho \leq \epsilon/2$. In what follows, this definition suffices, as we may always cut off the functions in which we are interested away from the boundary. Note that this definition of the Mellin transform depends both on the boundary defining function ρ and on the cut-off functions chosen, but this dependence will not make a difference in the sequel.

We additionally recall the space of L^2-based conormal distributions $I^{(s)}$ on the boundary $X = \partial M$. Here we finally split the boundary coordinates locally into $(v, y) \in \mathbb{R} \times \mathbb{R}^{n-2}$ rather than using y to denote all of them. For the hypersurface $Y = \{v = 0\} \subset X$, $u \in I^{(s)}(N^*Y)$ means that $u \in H^s(X)$ and $A_1 \dots A_k u \in H^s$ for all k and for all $A_j \in \Psi^1(X)$ with principal symbol vanishing on N^*Y.

We now record some additional mapping properties of the Mellin transform:

Definition 2.2. Let $\mathbb{C}_\nu$ denote the halfspace $\operatorname{Im}\sigma > -\nu$ and let $\mathcal{H}(\mathbb{C}_\nu)$ denote holomorphic functions on this space. For a Fréchet space $\mathcal{F}$, let

$$\mathcal{H}(\mathbb{C}_\nu) \cap \langle\sigma\rangle^{-k} L^\infty L^2(\mathbb{R}; \mathcal{F})$$

denote the space of g_σ holomorphic in $\sigma \in \mathbb{C}_\nu$ taking values in $\mathcal{F}$ such that each seminorm

$$\int_{-\infty}^{\infty} \left\| g_{\mu+\imath\nu'} \right\|_\bullet^2 \langle\mu\rangle^{2k} \, d\mu$$

is uniformly bounded in $\nu' > -\nu$.

Note the choice of signs: as ν increases, the halfspace gets larger.

We will further allow elements of $\mathcal{H}(\mathbb{C}_\nu)$ to take values in *σ-dependent* Sobolev spaces, or rather Sobolev spaces with σ-dependent norms. In particular, we allow values in the standard semiclassical Sobolev spaces H_h^m on a compact manifold (without boundary), with semiclassical parameter $h = |\sigma|^{-1}$. Recall (see [27, Section 8.3]) that these are the standard Sobolev spaces and up to the equivalence of norms, for h in compact subsets of $(0, \infty)$, the norm is just the standard H^m norm, but the norm is h-dependent: for non-negative integers m, in coordinates y_j, locally the norm $\|g\|_{H_h^m}$ is equivalent to $\sqrt{\sum_{|\alpha|\leq m} \|(hD_{y_j})^\alpha g\|_{L^2}^2}$.

We will require some more detailed information about mapping properties of the Mellin transform acting on b-Sobolev spaces.

Lemma 2.3. *Let $u \in H_{\mathrm{b}}^{m,l}(M)$ be supported in a collar neighborhood $[0, \epsilon)_\rho \times X$ of ∂M Then*

$$\mathcal{M}u \in \mathcal{H}(\mathbb{C}_l) \cap \langle\sigma\rangle^{\max(0,-m)} L^\infty L^2(\mathbb{R}; H^m(X)).$$

If $u \in H_{\mathrm{b}}^{m,l}(M)$ is furthermore conormal to $\rho = v = 0$ then

$$\mathcal{M}u \in \mathcal{H}(\mathbb{C}_l) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(m)}(N^*Y)).$$

The inverse Mellin transform maps

$$\mathcal{H}(\mathbb{C}_\nu) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s)}(N^*Y)))$$

into

$$\rho^{\nu-0} H_b^\infty([0, \infty)_\rho; I^{(s)}(N^*Y))$$

which in turn, for $s < 1/2$, is contained in

$$\rho^{\nu-0} v^{s-1/2-0} L^\infty.$$

Proof. For m a positive integer, the first result follows since lying in $H_{\mathrm{b}}^{m,l}$ implies that

$$\partial_y^\alpha \partial_v^\beta u \in \rho^l L^2_b$$

for all $|\alpha| + |\beta| \leq m$ hence

$$\partial_y^\alpha \partial_v^\beta \mathcal{M} u \in \mathcal{H}(\mathbb{C}_l) \cap L^\infty L^2(\mathbb{R}; L^2(X));$$

the result for general $m \geq 0$ follows by interpolation. For $m < 0$, choose a positive integer $\tilde{m}$ such that $m + \tilde{m} \geq 0$; then u can be written as a finite sum of terms of the product of at most $\tilde{m}$ b-vector fields applied to elements u' of $H_{\mathrm{b}}^{m+\tilde{m},l}(M)$. Now, the Mellin transform of such u' lies in $\mathcal{H}(\mathbb{C}_l) \cap L^\infty L^2(\mathbb{R}; H^{m+m'}(X))$ by the first part; ∂_y and ∂_v act as vector fields on X and thus would lead to the conclusion that u is in $\mathcal{H}(\mathbb{C}_l) \cap L^\infty L^2(\mathbb{R}; H^m(X))$ if only they appeared; however, $\rho\partial_\rho$ Mellin transforms to $\imath\sigma$, and thus we may obtain up to $\tilde{m}$ factors of σ as well, leading to the desired weight when $m < 0$ is an integer; interpolation gives the weight (without a loss) for all $m < 0$.

The proof of the second and third parts is similar; here we use Sobolev embedding, and the fact that regularity under $\rho\partial_\rho$, $v\partial_v$ and ∂_y intertwines under Mellin transform with regularity under σ, $v\partial_v$, and ∂_y. □

We remark further that Mellin transform maps $H_{\mathrm{b}}^{\infty,l}(M)$ into

$$\mathcal{H}(\mathbb{C}_l) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; H^\infty(X)).$$

This map is not onto, as there is no iterated regularity under $\rho\partial_v$ built into the latter space.

3. Geometric set-up

3.1. **Minkowski metric.**

As a preliminary to our discussion of Lorentzian scattering metrics, we record the asymptotic behavior of the Minkowski space on $\mathbb{R}^n$, endowed with the Lorentzian metric with the *mostly minus* sign convention (here we are following the notation of [22]). We take coordinates $t, x_1, \ldots, x_{n-1}$, and set

$$t = \rho^{-1} \cos\theta,$$
$$x_j = \rho^{-1} \omega_j \sin\theta,$$

with $\omega \in \mathbb{S}^{n-2}$. The Minkowski metric is then

$$\begin{aligned} dt^2 - \sum dx^2 &= \left(-\frac{\cos\theta\, d\rho}{\rho^2} - \sin\theta \frac{d\theta}{\rho}\right)^2 - \sum\left(-\omega_j \sin\theta \frac{d\rho}{\rho^2} + \omega_j \cos\theta \frac{d\theta}{\rho} + \sin\theta \frac{d\omega_j}{\rho}\right)^2 \\ &= \cos 2\theta \frac{d\rho^2}{\rho^4} - \cos 2\theta \frac{d\theta^2}{\rho^2} + \sin 2\theta \left(\frac{d\rho}{\rho^2} \otimes \frac{d\theta}{\rho} + \frac{d\theta}{\rho} \otimes \frac{d\rho}{\rho^2}\right) - \sin^2\theta \frac{d\omega^2}{\rho^2}. \end{aligned}$$

Here $d\omega^2$ represents the standard round metric on the sphere.

As the function $\cos 2\theta$ clearly plays an important role here, we set

$$v = \cos 2\theta,$$

replacing the θ coordinate by v, and write

$$g = v\frac{d\rho^2}{\rho^4} - \frac{v}{4(1-v^2)}\frac{dv^2}{\rho^2} - \frac{1}{2}\Big(\frac{d\rho}{\rho^2}\otimes\frac{dv}{\rho} + \frac{dv}{\rho}\otimes\frac{d\rho}{\rho^2}\Big) - \frac{1-v}{2}\frac{d\omega^2}{\rho^2}. \tag{3.1}$$

We remark that this form of the metric in these extremely natural coordinates does not conform to the standard "scattering metric" hypotheses [16] often employed in the Riemannian signature, in which cross terms of the form $(d\rho/\rho^2)\otimes(dy/\rho)$ with y a general smooth function are forbidden.

3.2. **General hypotheses.** Let (M, g) be an n-dimensional manifold with boundary $X = \partial M$ equipped with a Lorentzian metric g over M° such that g extends to be a nondegenerate quadratic form on ${}^{\mathrm{sc}}TM$ of signature $(+, -, \dots, -)$.

Definition 3.1. We say that g is a Lorentzian scattering metric if g is a smooth, Lorentzian signature, symmetric bilinear form on ${}^{\mathrm{sc}}TM$, and there exist a boundary defining function ρ for M, and a function $v \in \mathcal{C}^\infty(M)$ such that

(1) When V is a scattering normal vector field, $g(V, V)$ has the same sign as v at $\rho = 0$,
(2) in a neighborhood of $\{v = 0,\ \rho = 0\}$ we have
$$g = v\frac{d\rho^2}{\rho^4} - \Big(\frac{d\rho}{\rho^2}\otimes\frac{\alpha}{\rho} + \frac{\alpha}{\rho}\otimes\frac{d\rho}{\rho^2}\Big) - \frac{\tilde g}{\rho^2}$$
with α a smooth 1-form on M and $\tilde g$ a smooth symmetric 2-cotensor on M so that
$$\tilde g|_{\mathrm{Ann}(d\rho, dv)} \text{ is positive definite.}$$
We further require that
$$\alpha = \frac{1}{2}dv + O(v) + O(\rho) \text{ near } v = \rho = 0.$$

Remark 3.2. We remark that while it might be tempting to mandate also the vanishing of the dv^2/ρ^2 component at $v = 0$ as we have in the exact Minkowski case, this condition is highly non-invariant, in that it requires a product decomposition of X.

Remark 3.3. The function v must have a non-degenerate 0-level set when restricted to X (and dv, $d\rho$ must be independent at that set), since otherwise our metric would be degenerate at $v = 0$.

We further remark that our hypotheses imply that for *non-trapping* Lorentzian scattering metrics, even if the boundary is disconnected, v vanishes on each component of the boundary, see Remark 3.6. Without the non-trapping assumption this need not be the case: consider $\tilde X \times \overline{\mathbb{R}}$, with $\tilde X$ compact without boundary, $\overline{\mathbb{R}}$ the radial compactification of $\mathbb{R}_r$ (so $\rho = r^{-1}$ works for $r \gg 1$). Then $dr^2 - (1+r^2)h$, h a metric on $\tilde X$, is a Lorentzian scattering metric if one chooses $v \equiv 1$; X is then the disjoint union of two copies of $\tilde X$.

Remark 3.4. Note that near $v = 0$, $V = \rho^2\partial_\rho$ gives $g(V, V) = v$, which has the same sign as v, so the first and second parts of the definition are consistent, with the second part refining the first near $v = 0$.

Before proceeding, note that the rescaled, or scattering, Hamilton vector field of the metric function on ${}^{\mathrm{sc}}T^*M \setminus o$ is a $\mathcal{C}^\infty$ vector field, tangent to the boundary. The integral curves of this Hamilton vector field within the zero set of the metric function (i.e., the null bicharacteristics) over the interior of M project to reparameterized null-geodesics; indeed, they are exactly the appropriately reparameterized null-geodesics lifted to T^*M°. We show later in Section 3.6 that over $S = \{v = 0, \rho = 0\}$ the Hamilton flow has sources and sinks; there we shift to the b-framework, and these sources and sinks are located at the "b-conormal bundle" of S, denoted by $\mathcal{R}$.

With this in mind, we make two additional global assumptions on the structure of our spacetime:

Definition 3.5. A Lorentzian scattering metric is *non-trapping* if

(1) The set $S = \{v = 0, \rho = 0\} \subset X$ splits into S_+ and S_-, each a disjoint union of connected components; we further assume that $\{v > 0\}$ splits into components $C_\pm$ with $S_\pm = \partial C_\pm$. We denote by C_0 the subset of X where $v < 0$.
(2) The projections of all null bicharacteristics on ${}^{\mathrm{sc}}T^*M \setminus o$ tend to $S_\pm$ as their parameter tends to $\pm\infty$ (or vice versa). (A discussion of the flow near $S_\pm$ is contained in Sections 3.4-3.6.)

In particular, this implies the time-orientability of (M, g) by specifying the future light cone as the one from which the forward (in the sense of the Hamilton flow) bicharacteristics tend to S_+.

Remark 3.6. For non-trapping Lorentzian scattering metrics, v must necessarily vanish on each component of X. To see this, note that on ${}^{\mathrm{sc}}T^*M$ a Lorentzian metric has non-trivial characteristic set over each point, in particular over each point in X. Since the scattering Hamilton vector field is a smooth vector field on ${}^{\mathrm{sc}}T^*X$ tangent to the boundary, the bicharacteristics through a point in a connected component of X stay in that component. Thus, for non-trapping metrics the zero set of v within each connected component must be non-trivial.

Remark 3.7. C stands for "cap" as in the Minkowski case C_+ is simply the spherical cap $|\theta| < \pi/4$. The assumption the S_+ bounds a cap is in fact not necessary for us to prove any of the Fredholm properties in Section 5; however it is of course necessary to recognize the poles of the resulting operator as resonance poles on a cap, and hence in order to know that there are finitely many resonances in any horizontal strip in $\mathbb{C}$, which is crucial to the development of our asymptotic expansion.

Given Schwartz f supported away from $\overline{C_-}$, it is natural to consider "forward tempered" solutions of $\Box_g u = f$ with $u = 0$ near $\overline{C_-}$. It is unclear whether our hypotheses guarantee that these always exist, though it is not hard to show that there are only finite dimensional obstructions to solvability and uniqueness in fixed weighted spaces.

Near $v = 0$, which is away from the critical points of $v|_X$, we may choose $y_1, \dots, y_{n-2} \in \mathcal{C}^\infty(M)$ so that (v, y) constitute a coordinate system on $X = \partial M$ and (ρ, v, y) thus give coordinates on M in a neighborhood of X. Moreover, (ρ, v, y) also provide a product decomposition of that neighborhood into $[0, \epsilon)_\rho \times X$. In the frame

$$\rho^2\partial_\rho,\ \rho\partial_v,\ \rho\partial_y,$$

associated to these coordinates, the metric (when restricted to the boundary $\{\rho = 0\}$) thus has the block form

$$G_0 = \begin{pmatrix} v & -\frac{1}{2} + a_0 v & a_1 v & \dots & a_{n-2} v \\ -\frac{1}{2} + a_0 v & b & c_1 & \dots & c_{n-2} \\ a_1 v & c_1 & -h_{1,1} & \dots & -h_{n-2,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2} v & c_{n-2} & -h_{1,n-2} & \dots & -h_{n-2,n-2} \end{pmatrix}, \tag{3.2}$$

with the lower $(n-1)\times(n-1)$ block negative definite, hence h_{ij} is positive definite.

Blockwise inversion shows that in the frame

$$\frac{d\rho}{\rho^2},\ \frac{dv}{\rho},\ \frac{dy}{\rho},$$

the inverse metric when restricted to the boundary has the block form (the α here is a function and should not be confused with the 1-form α in the definition of the metric)

$$G_0^{-1} = \begin{pmatrix} -q & -2 + \alpha v & -\frac{1}{2}\Upsilon^{\mathrm{T}} + O(v) \\ -2 + \alpha v & -4v + \beta v^2 & -v\Upsilon^{\mathrm{T}} + O(v^2) \\ -\frac{1}{2}\Upsilon + O(v) & -v\Upsilon + O(v^2) & -h^{-1} + O(v) \end{pmatrix}.$$

In the above, $h^{-1} = h^{ij}$ is the inverse matrix of h_{ij}, q, α, β, and Υ_j are smooth near $v = \rho = 0$, and A^{T} denotes the transpose of the matrix A.

In a neighborhood of the boundary, i.e., at $\rho \neq 0$, there are further correction terms in the inverse metric as the actual metric is given by

$$G = G_0 + H,$$
$$H = \begin{pmatrix} O(\rho^2) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \end{pmatrix}.$$

Thus in the inverse frame above,

$$G^{-1} = G_0^{-1} + \begin{pmatrix} O(\rho) & O(\rho) & O(\rho) \\ O(\rho) & O(\rho^2) + O(\rho v) & O(\rho) \\ O(\rho) & O(\rho) & O(\rho) \end{pmatrix}. \tag{3.3}$$

Thus in the coordinate frame ∂_ρ, ∂_v, ∂_y, the dual metric becomes

$$\begin{pmatrix} g^{\rho\rho}\rho^4 + O(\rho^5) & g^{\rho v}\rho^3 + O(\rho^4) & g^{\rho y}\rho^3 + O(\rho^4) \\ g^{\rho v}\rho^3 + O(\rho^4) & g^{vv}\rho^2 + O(\rho^4) + O(\rho^3 v) & g^{vy}\rho^2 + O(\rho^3) \\ g^{\rho y}\rho^3 + O(\rho^4) & g^{vy}\rho^2 + O(\rho^3) & g^{yy}\rho^2 + O(\rho^3) \end{pmatrix}, \tag{3.4}$$

where $g^{\bullet\bullet}$ are given by:

$$\begin{aligned} g^{\rho\rho} &= -q & g^{\rho v} &= -2 + \alpha v & g^{\rho y} &= -\frac{1}{2}\Upsilon + O(v) \\ g^{vv} &= -4v + \beta v^2 & g^{vy} &= -v\Upsilon + O(v^2) & g^{yy} &= -h^{-1} - O(v) \end{aligned} \tag{3.5}$$

Again all terms are smooth.

Cofactor expansion of equation (3.2) scaled to the frame ∂_ρ, ∂_v, ∂_y shows that the determinant of the metric is

$$|g| = \rho^{-2(n+1)}|G| = \rho^{-2(n+1)}\left((f^2 - qv)|h| + O(\rho)\right)$$

In particular,

$$\begin{aligned} \frac{1}{2}\partial_\rho \log|g| &= -(n+1)\rho^{-1} + O(1) \\ \frac{1}{2}\partial_v \log|g| &= O(1) \\ \frac{1}{2}\partial_y \log|g| &= O(1). \end{aligned}$$

3.2.1. *Induced metrics.* In this section we describe induced metrics on the "caps" $C_\pm$ (the components of $\{v > 0\}$ bounded by $S_\pm$) and on the "side" C_0 ($\{v < 0\}$).

We define the metric K on T^*X via the inclusion $r^* : T^*X \hookrightarrow {}^{\mathrm{b}}T^*_X M$ (which is dual to the restriction map $r : {}^{\mathrm{b}}T_X M \to TX$). As $\rho^2 g$ is a b-metric, we define for $\omega, \eta \in T^*(X)$ the dual metric K^{-1} by

$$K^{-1}(\omega, \eta) = -(\rho^2 g)^{-1}(r^*\omega, r^*\eta)|_{\rho=0}.$$

Observe that K^{-1} is the restriction of $-(\rho^2 g)^{-1}$ to the annihilator of $\rho\partial_\rho$ (the "b-normal" vector field) at $\rho = 0$.

The components of the dual metric K^{-1} are given in the frame ∂_v, ∂_y by

$$\begin{pmatrix} K^{vv} & K^{vy} \\ K^{vy} & K^{yy} \end{pmatrix} = \begin{pmatrix} -g^{vv} & -g^{vy} \\ -g^{vy} & -g^{yy} \end{pmatrix},$$

where $g^{\bullet\bullet}$ are the components of the dual metric of g in the frame $\rho^2\partial_\rho$, $\rho\partial_v$, and $\rho\partial_y$.

Because $\rho^2\partial_\rho$ is time-like near $C_\pm$ and K^{-1} is the restriction of $-(\rho^2 g)^{-1}$ to the annihilator of $\rho\partial_\rho$, K^{-1} is nondegenerate, and, in fact, Riemannian in $C_\pm$. In coordinates (v, y), the metric K on TX is given by

$$K = \frac{1}{4v}\left(1 + O(v)\right) dv^2 + \sum_{j=1}^{n-2} O(1)\left(dv \otimes dy_j + dy_j \otimes dv\right) + \sum_{i,j=1}^{n-1} K_{ij}\, dy_i \otimes dy_j.$$

It is easy to see from the above expression that the metric

$$k_\pm = \frac{1}{v} K|_{C_\pm}$$

is an asymptotically hyperbolic metric (in the sense of Vasy [22]) on $C_\pm$. Setting $v = x^2$ in this region ensures that k is an asymptotically hyperbolic metric (in the sense of Mazzeo–Melrose [14]) which is *even* in its boundary defining function (cf. the work of Guillarmou [6]).

Similarly, because $\rho^2\partial_\rho$ is space-like near C_0, $K^{-1}|_{C_0}$ is Lorentzian (with the "mostly-plus" convention), and

$$k_0 = \frac{1}{v} K|_{C_0}$$

is an even *asymptotically de Sitter* metric (with the "mostly-minus" convention, as $v < 0$ here) on C_0. Indeed, if $v = -x^2$, then the metric has the form used by Vasy [24]. The non-trapping assumption (2) above implies that the metric satisfies the conditions in Vasy's definition of an asymptotically de Sitter metric.

The ρ components of the dual metric of g are also related to the components of the dual metric of K. In the $\rho^2\partial_\rho, \rho\partial_v, \rho\partial_y$ frame for g and the ∂_v, ∂_y frame for K, we have

$$g^{\rho\rho} = -\frac{1}{v}\left(4qK^{vv} + O(v^2)\right), \quad g^{\rho v} = -\frac{1}{2v}\left(K^{vv} + O(v^2)\right), \quad g^{\rho y} = -\frac{1}{2v}\left(K^{vy} + O(v^2)\right).$$

As K^{-1} is the lower-right block of $-g^{-1}$ and $g_{\rho\rho} = v$, the volume forms of g and K (and hence the asymptotically hyperbolic and asymptotically de Sitter metrics $k_\pm, k_0$) are also related:

$$\sqrt{g} = \rho^{-(n+1)}\left(v^{1/2}\sqrt{|K|} + O(\rho)\right) = \rho^{-(n+1)}\left(v^{n/2}\sqrt{|k_{\pm,0}|} + O(\rho)\right).$$

3.3. **The form of the d'Alembertian.** In this section we compute the form of the operator $\Box_g$ and its normal operator $\widehat{N}(\rho^{-2}\Box_g)$.

Putting the calculations of the metric components and the volume form in Section 3.2 together, we compute the form of $\Box_g$ near $\rho = 0$ (here we use $\sqrt{G} = \rho^{n+1}\sqrt{g}$ and recall that $g^{\bullet\bullet}$ are given by (3.5)):

$$\begin{aligned}
-\Box_g = \rho^2 \Big[&\left(g^{\rho\rho} + O(\rho)\right)(\rho\partial_\rho)^2 + \left(g^{\rho v} + O(\rho)\right)(\rho\partial_\rho)\partial_v + \left(g^{\rho y} + O(\rho)\right)(\rho\partial_\rho)\partial_y \\
&+ (2-n)\left(\left(g^{\rho\rho} + O(\rho)\right)\rho\partial_\rho + \left(g^{\rho v} + O(\rho)\right)\partial_v + \left(g^{vy} + O(\rho)\right)\partial_y\right) \\
&+ \frac{1}{\sqrt{G}}\partial_v\left(\left(g^{\rho v} + O(\rho)\right)\sqrt{G}\rho\partial_\rho + \left(g^{vv} + O(\rho)\right)\sqrt{G}\partial_v + \left(g^{vy} + O(\rho)\right)\sqrt{G}\partial_y\right) \\
&+ \frac{1}{\sqrt{G}}\partial_y\left(\left(g^{\rho y} + O(\rho)\right)\sqrt{G}\rho\partial_\rho + \left(g^{vy} + O(\rho)\right)\sqrt{G}\partial_v + \left(g^{yy} + O(\rho)\right)\sqrt{G}\partial_y\right)\Big]
\end{aligned}$$

Adopting now the notation of Vasy [22],

$$\begin{aligned}-\widetilde{P}_\sigma &= -\widehat{N}(\rho^{-2}\Box_g)\\ &= \frac{1}{\sqrt{G}}\left[\partial_v\left(g^{vv}\sqrt{G}\partial_v + g^{vy}\sqrt{G}\partial_y\right) + \partial_y\left(g^{vy}\sqrt{G}\partial_v + g^{yy}\sqrt{G}\partial_y\right)\right]\\ &\quad + g^{\rho v}\left(2\imath\sigma + 2 - n\right)\partial_v + g^{\rho y}\left(2\imath\sigma + 2 - n\right)\partial_y\\ &\quad + \imath\sigma\left[\frac{1}{\sqrt{G}}\partial_v\left(g^{\rho v}\sqrt{G}\right) + \frac{1}{\sqrt{G}}\partial_y\left(g^{\rho y}\sqrt{G}\right) + g^{\rho\rho}\imath\sigma\right]\end{aligned}$$

In particular, near $v = 0$,

$$\begin{aligned}-\widetilde{P}_\sigma = \left(-4v + O(v^2)\right)\partial_v^2 + O(v)\partial_v\partial_y - \left(h^{ij} + O(v)\right)\partial_{y_i}\partial_{y_j} + O(1)\partial_y\\ + 2\left(n - 4 - 2\imath\sigma + O(v)\right)\partial_v + q(\sigma),\end{aligned}$$

with q a smooth function in v and y with values in quadratic polynomials in σ.

In our asymptotic expansions (and in the analysis of the radiation field), it is more convenient to deal with

$$P_\sigma \equiv \widehat{N}\left(\rho^{-(n-2)/2}\rho^{-2}\Box_g\rho^{(n-2)/2}\right) \tag{3.6}$$

than with $\widetilde{P}_\sigma$, in part to more directly correspond to the setting of [22]. To this end we note simply that

$$P_\sigma = \widetilde{P}_{\sigma - \imath(n-2)/2} \tag{3.7}$$

hence since the ∂_v^2 and ∂_v terms of $\widetilde{P}_\sigma$ may be written

$$-4\left((v + O(v^2))D_v^2 + \left(\frac{\imath}{2}(n - 4 - 2\imath\sigma) + O(v)\right)D_v\right), \tag{3.8}$$

we have

$$P_\sigma = -4\left(\left(v + O(v^2)\right)D_v^2 + \left((\sigma - \imath) + O(v)\right)D_v\right) + O(1)\partial_y^2 + O(1)\partial_y + O(v)\partial_v\partial_y + O(\sigma^2). \tag{3.9}$$

3.3.1. *Relationship with the induced metrics.* In the regions $C_\pm$ and C_0 of the boundary, P_σ may be written in terms of the metrics $k_\pm$ and k_0.

We first work near $C_\pm$. By an explicit computation, there is a (σ-dependent) vector field $\mathcal{X}(\sigma)$ tangent to $v = 0$ and a (σ-dependent) smooth potential $V(\sigma) \in C^\infty(X)$ so that:

$$v^{\frac{1}{2}}v^{\frac{n}{4}+\frac{\imath\sigma}{2}}P_\sigma v^{-\frac{n}{4}-\frac{\imath\sigma}{2}}v^{\frac{1}{2}} = -\Delta_{k_\pm} + \left(\sigma^2 + \frac{(n-2)^2}{4}\right) + v\mathcal{X}(\sigma) + vV(\sigma). \tag{3.10}$$

(In terms of the variable x given by $v = x^2$, the vector field $\mathcal{X}$ is in fact a 0-vector field in the sense of Mazzeo–Melrose [14].) Moreover, if all a_j and q vanish identically on X (as is the case in Minkowski space and the normally very short range perturbations of Minkowski space defined in Section 10.1) then $\mathcal{X} = 0$ and $V = 0$.

Remark 3.8. The result of this computation should not be too surprising, as the entries of the inverse metric of k agree up to a factor of v with a block of the inverse metric of g, accounting for the second-order terms. Moreover, it is easy to check that the operator on the left side is a b-differential operator on X. The remainder of the computation requires checking only that the b-normal operators of the two sides agree. A similar computation is carried out in [22, Section 5].

We now consider C_0. The same calculation as above implies that

$$|v|^{\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma}{2}} P_\sigma |v|^{\frac{1}{2}-\frac{n}{4}-\frac{\imath\sigma}{2}} = \Box_{k_0} - \left(\sigma^2 + \frac{(n-2)^2}{4}\right) + v\mathcal{X}(\sigma) + vV(\sigma), \tag{3.11}$$

where $\mathcal{X}$ and V are as above. *In particular, note that P_σ is a hyperbolic operator on C_0 and an elliptic operator on $C_\pm$.*

3.4. **Location of radial points.** We now study the flow associated to the Hamilton vector field of P_σ. In particular, we are interested in the radial points of the vector field, i.e., those points in the characteristic set where it is proportional to the fiber-radial vector field. As P_σ is hyperbolic for $v < 0$ and elliptic for $v > 0$, the only possible radial points must occur when $v = 0$.

As 0 is not a critical point of v, we may take

$$\gamma dv + \eta \cdot dy$$

to be the canonical one-form on T^*X. The principal symbol of P_σ is (employing summation convention) given by

$$\sigma(P_\sigma) = -(4v - \beta v^2)\gamma^2 - (2v\Upsilon + O(v^2))\gamma\eta - (h^{ij} + O(v))\eta_i\eta_j.$$

Letting H denote the resulting Hamilton vector field on T^*X, we have

$$\frac{1}{2}\mathsf{H} = (-4v\gamma + \beta v^2\gamma + v\eta\cdot\Upsilon)\partial_v + (v\gamma\Upsilon_j + g^{y_iy_j}\eta_i)\partial_{y_j} + \bullet\partial_\gamma + \bullet\partial_\eta, \tag{3.12}$$

with the $\bullet$ terms homogeneous of degree 2 in the fiber variables. We now analyze the radial points of the vector field. The components in the base variables are given by

$$(-4v\gamma + \beta v^2\gamma + v\eta\cdot\Upsilon)\partial_v + (v\gamma\Upsilon_j + g^{y_iy_j})\partial_{y_j}.$$

These coefficients must vanish at the radial set, which we have already observed to lie over $v = 0$. In particular, we must have

$$g^{y_iy_j}\eta_i = 0$$

for all j. As $g^{y_iy_j}$ is nondegenerate at $v = 0$, we must have $\eta = 0$ on the radial set.

We now easily verify that indeed the vector field at points

$$v = 0,\ \ \eta = 0$$

is radial; hence these are in fact precisely the radial points.

3.5. **Structure near radial points.** We now verify several of the hypotheses of [22] near the radial points. We have established that the radial points occur at

$$\Lambda^{\epsilon_1}_{\epsilon_2} \equiv \{v = 0,\ \eta = 0, \epsilon_2\gamma > 0\} \cap \pi^{-1}(S_{\epsilon_1}) \subset T^*X,\ \epsilon_i = \pm; \tag{3.13}$$

thus the $\pm$ in the superscript distinguishes "past" from "future" null infinity, while that in the subscript separates the intersections with the two components of the characteristic set. We will write

$$\Lambda^{\pm} = \Lambda^{\pm}_{+} \cup \Lambda^{\pm}_{-}, \quad \Lambda_{\pm} = \Lambda^{+}_{\pm} \cup \Lambda^{-}_{\pm}.$$

We must now verify the following:

(1) For a degree -1 defining function ρ_∞ of S^*X inside the fiber-radial-compactification of T^*X, we have

$$\rho_\infty \mathsf{H} \rho_\infty\big|_{\Lambda_\pm} = \mp\beta_0,\ \beta_0 \in \mathcal{C}^\infty(\Lambda_\pm),\ \beta_0 > 0$$

(equation (2.3) of [22]).

(2) There exists a non-negative homogeneous degree 0 function ρ_0 vanishing quadratically and non-degenerately exactly at $\Lambda_\pm$ and a $\beta_1 > 0$ such that

$$\mp\rho_\infty \mathsf{H}\rho_0 - \beta_1\rho_0 \geq 0 \text{ modulo cubic terms vanishing at } \Lambda_\pm$$

(equation (2.4) of [22]).

To deal with the first property, we remark that from (3.12), we have

$$\frac{1}{2}\mathsf{H} = (2\gamma^2 + O(v) + O(\eta))\partial_\gamma + (O(\eta^2) + O(v\eta) + O(v^2))\partial_\eta + (-4v\gamma + \beta v^2\gamma + v\eta\cdot\Upsilon)\partial_v + \bullet\partial_y$$

where the big-Oh terms all have the homogeneities in γ, η required to make the overall vector field homogeneous of degree 1. Near $\eta = 0$ we may employ the homogeneous coordinates

$$\rho_\infty = \frac{1}{|\gamma|}, N = \frac{\eta}{|\gamma|}$$

on the radial compactification of T^*X, hence we compute that near $\Lambda_\pm$

$$\begin{aligned} \frac{1}{2}\mathsf{H} = \rho_\infty^{-1}\big(&(\mp 2 + O(v) + O(N))\rho_\infty\partial_{\rho_\infty} \\ &+ (\mp 2N + O(v^2) + O(vN) + O(N^2))\partial_N \\ &+ (\mp 4 \pm \beta v + N\cdot\Upsilon)v\partial_v + \bullet\partial_y\big), \end{aligned} \tag{3.14}$$

hence

$$\mp\rho_\infty \mathsf{H}\rho_\infty\big|_{\Lambda_\pm} = 4,$$

i.e., the first property holds with

$$\beta_0 = 4.$$

To verify the second property, we take, in our compactified coordinates,

$$\rho_0 = v^2 + N^2.$$

Applying (3.14) yields

$$\rho_\infty \mathsf{H}(\rho_0) = \mp(16v^2 + 8N^2) + \text{cubic terms in } (v, N),$$

i.e.,

$$\mp\rho_\infty \mathsf{H}(\rho_0) - 8\rho_0 = \text{cubic terms in } (v, N),$$

hence the second property is satisfied with $\beta_1 = 8$.

We may thus compute the subprincipal symbol of $\widetilde{P}_\sigma$ (and hence of P_σ) in terms of β_0. Indeed, we compute

$$\begin{aligned} &-(2\imath)^{-1}(\widetilde{P}_\sigma - \widetilde{P}_\sigma^*) \\ &\quad = (2\imath)^{-1}\big((8 + 4(n-4+2\operatorname{Im}\sigma) + O(v))\partial_v + O(1)\partial_y\big) + O(1) \\ &\quad = -2\imath\big((n-2+2\operatorname{Im}\sigma) + O(v)\big)\partial_v + O(1)\partial_y + O(1) \end{aligned} \tag{3.15}$$

and consequently

$$\begin{aligned} \sigma\big((2\imath)^{-1}(\widetilde{P}_\sigma - \widetilde{P}_\sigma^*)\big)\big|_{v=0,\,\eta=0} &= \pm 4\left(-\frac{n-2}{2} - \operatorname{Im}\sigma\right)|\gamma| \\ &= \pm\beta_0\left(-\frac{n-2}{2} - \operatorname{Im}\sigma\right)|\gamma|. \end{aligned} \tag{3.16}$$

Note that, even apart from the shift by $(n-2)/2$, the sign of $(2\imath)^{-1}(\widetilde{P}_\sigma - \widetilde{P}_\sigma^*)$ is switched as compared to [22] (where $\pm\beta_0 \operatorname{Im}\sigma|\gamma|$ was used with the present notation). Switching the roles of $\widetilde{P}_\sigma$ and $\widetilde{P}_\sigma^*$ reverses this sign, and thus what we do here corresponds to what was discussed in [22] for the adjoint operator in the context of the general theory, though this reversal was pointed out there already in the Minkowski context in Section 5 of [22].

Returning to the operator P_σ itself, we compute for later reference that by (3.16) and (3.7),

$$\begin{aligned} \hat\beta^\pm(\sigma) &\equiv \pm\frac{\rho_\infty}{2\imath\beta_0}\sigma_1(P_\sigma - P_\sigma^*)|_{\Lambda^\pm} \\ &= \Big(-\frac{(n-2)}{2} - \operatorname{Im}(\sigma - \imath(n-2)/2)\Big) = -\operatorname{Im}\sigma. \end{aligned} \tag{3.17}$$

3.6. **b-radial points.** It is also useful to compute the full b-structure of the radial set of

$$L = \rho^{-(n-2)/2}\rho^{-2}\Box_g\rho^{(n-2)/2} \in \operatorname{Diff}^2_{\mathrm{b}}(M).$$

in ${}^{\mathrm{b}}T^*M$. Note that the powers are chosen here so that L is formally self-adjoint with respect to the b-density

$$\rho^n\,|dg|.$$

The b-principal symbol of L is the "same" as the sc-principal symbol of $\Box_g$ under the identification of ${}^{\mathrm{b}}T^*M$ and ${}^{\mathrm{sc}}T^*M$, namely

$$\begin{aligned} \lambda = \sigma_b(L) = g^{\rho\rho}\xi^2 &- (4v - \beta v^2 + O(\rho v) + O(\rho^2))\gamma^2 - 2(2 - \alpha v + O(\rho))\xi\gamma \\ &+ 2g^{\rho y}\cdot\eta\xi + \big(2v\Upsilon + O(\rho)\big)\cdot\eta\gamma + g^{y_iy_j}\eta_i\eta_j, \end{aligned} \tag{3.18}$$

where we write b-covectors as

$$\xi\frac{d\rho}{\rho}+\gamma\,dv+\eta\,dy. \tag{3.19}$$

The b-Hamilton vector field of a symbol λ is

$$(\partial_\xi\lambda)(\rho\partial_\rho)+(\partial_\gamma\lambda)\partial_v+(\partial_\eta\lambda)\partial_y-(\rho\partial_\rho\lambda)\partial_\xi-(\partial_v\lambda)\partial_\gamma-(\partial_y\lambda)\partial_\eta, \tag{3.20}$$

so in our case we obtain

$$\begin{aligned}\mathsf{H}_\lambda =&\big(2g^{\rho\rho}\xi+2g^{\rho y}\eta-2\gamma(2-\alpha v+O(\rho))\big)(\rho\partial_\rho)\\ &-2\big((4v-\beta v^2+O(\rho v)+O(\rho^2))\gamma\\ &\qquad\qquad+(2-\alpha v+O(\rho))\xi+(v\Upsilon+O(\rho))\eta\big)\partial_v\\ &+2\big(g^{\rho y}\xi+(v\Upsilon+O(\rho))\gamma+g^{y_iy_j}\eta_j\big)\partial_y\\ &-(\rho\partial_\rho\lambda)\partial_\xi-(\partial_v\lambda)\partial_\gamma-(\partial_y\lambda)\partial_\eta.\end{aligned} \tag{3.21}$$

We now investigate when this vector field has radial points when restricted to $\rho=0$. The symbol λ is a nondegenerate (Lorentzian) metric on the fibers of ${}^bT^*M$, hence the projection to the base of H_λ must be a nonvanishing b-vector field; for $\pi(\mathsf{H}_\lambda)$ to vanish over the boundary, then, it must lie in the span of $\rho\partial_\rho$. On the other hand, letting g_b denote our induced b-metric given by (the dual of) λ and letting $p\in{}^bT^*M$ we have $g_b(\pi\mathsf{H}_\lambda|_p,\pi\mathsf{H}_\lambda|_p)=\lambda(p)=0$ assuming p lies in the characteristic set. Thus, at a radial point over $\rho=0$, $\rho\partial_\rho$ must be a null vector field, hence we must have $v=0$.

We further see by examining the coefficients of ∂_v, ∂_y that the radial set $\mathcal{R}$ within $\rho=0$ is exactly $v=0$, $\eta=0$, $\xi=0$. Further, there are no radial points in $\rho>0$, since the metric is a standard Lorentzian metric there (and there is no distinction between b-metrics and standard metrics in the interior). Now, on the fiber compactification of ${}^{\mathrm{b}}T^*M$ near $\mathcal{R}$ we can use local coordinates,

$$\nu=\frac{1}{\gamma},\ \hat\xi=\frac{\xi}{\gamma},\ \hat\eta=\frac{\eta}{\gamma}, \tag{3.22}$$

to obtain the linearization of H_λ at $\mathcal{R}$. That is, $\nu\mathsf{H}_\lambda\in\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{b}}T^*}M)$, i.e. is tangent to both $\rho=0$, defining ∂M, and $\nu=0$, defining fiber infinity, vanishes at $\partial\mathcal{R}$ (fiber infinity of the radial set), thus maps the ideal $\mathcal{I}$ of $\mathcal{C}^\infty$ functions vanishing at a point $q\in\partial\mathcal{R}$ to themselves, and thus $\mathcal{I}^2$ to $\mathcal{I}^2$, so it acts on $\mathcal{I}/\mathcal{I}^2\cong T^*_q\overline{{}^{\mathrm{b}}T^*}M$. In computing this, terms of $\nu\mathsf{H}_\lambda$ which vanish quadratically at $\partial\mathcal{R}$ can be neglected; modulo these we have

$$\begin{aligned}\nu\mathsf{H}_\lambda=-4\rho\partial_\rho+(-8v-4\hat\xi)\partial_v+2(g^{\rho y_i}\hat\xi+v\Upsilon+\rho c+g^{y_iy_j}\hat\eta_j)\partial_{y_i}&\\ -4(\nu\partial_\nu+\hat\xi\partial_{\hat\xi}+\hat\eta\partial_{\hat\eta})+\mathcal{I}^2\mathcal{V}(\overline{{}^{\mathrm{b}}T^*}M),&\end{aligned}$$

with c smooth. Correspondingly, the eigenvectors and eigenvalues of $\nu\mathsf{H}_\lambda$ are

$$\begin{aligned}
&dv + d\hat{\xi}, \text{ with eigenvalue } -8,\\
&d\rho, d\nu, d\hat{\xi}, d\hat{\eta}, \text{ with eigenvalue } -4,\\
&2\,dy + g^{\rho y}\,d\hat{\xi} + \Upsilon\,dv - c\,d\rho + g^{y_i y_j}\,d\hat{\eta}_j, \text{ with eigenvalue } 0.
\end{aligned} \tag{3.23}$$

3.7. **Semiclassical symbol and flow.** In this section we record the relationship of the computations performed above in in the b-cotangent bundle to the semiclassical results on P_σ that we will require below. Fortunately, these computations are nearly identical: as P_σ is obtained from L by Mellin transform, if we let $\sigma_\hbar$ denote the semiclassical principal symbol of an operator with parameter σ, where $\pm\operatorname{Re}\sigma$ is the semiclassical parameter, then we have simply

$$\sigma_\hbar(|\sigma|^{-2}P_\sigma) \cong \sigma_{\mathrm{b}}(L)|_{\xi=\pm 1},\ \operatorname{Re}\sigma \to \pm\infty.$$

The computation of the semiclassical Hamilton flow is similarly simple: the vector field in (3.21) is tangent to ${}^{\mathrm{b}}T^*_{\partial M}M$, with a vanishing ∂_ξ component at $\rho = 0$; thus the semiclassical flow associated to P_σ is given precisely by (3.21), restricted to $\rho = 0$ and with $\xi = \pm 1$.

We let $\Sigma_\hbar$ denote the semi-classical characteristic set, where $\sigma_\hbar(|\sigma|^{-2}P_\sigma) = 0$, and let $\Sigma_{\hbar,\pm}$ denote its two components; by the above discussion, these are simply the same as the characteristic sets of the rescaling of $\Box$ viewed as a b-operator.

For later use, we also recall the *semi-classical Sobolev* spaces appropriate to our problem. These spaces are denoted $H^s_{|\sigma|^{-1}}$ and when s is a positive integer they are given in local coordinates by

$$u \in H^s_{|\sigma|^{-1}} \Longleftrightarrow |\sigma|^{-|\alpha|}D^\alpha u \in L^2, \quad \text{for all } |\alpha| \le s.$$

The definition can be extended to non-integer s via interpolation and duality, or else by using the semiclassical pseudodifferential calculus with parameter $h = |\sigma|^{-1}$. We refer the reader to §2.8 of [22] for details.

3.8. **The radiation field blow-up.** Having described our geometric set-up, we now digress slightly to revisit the "usual" construction of the radiation field in the context at hand. This section is not necessary in the logical development of the paper but rather serves to situation our results in the context of prior theorems.

In particular, although the existence of the radiation field for tempered solutions of $\Box_g w = f \in \dot{\mathcal{C}}^\infty(M)$ with appropriate support properties is a consequence of our main theorem, in this section we recall the definition of the radiation field for metrics of the form in Section 3.2. In order to do this we will also need to assume an additional support condition on the solution w analogous to the one satisfied by the forward fundamental solution in more familiar contexts. In particular, then, assume that g is a non-trapping Lorentzian scattering metric as described above, and further assume that

> The function w solves $\Box_g w = f \in C_c^\infty(M^\circ)$ and there is an s_0 so that near S_+, w vanishes identically for $v/\rho \geq s_0$.

We now *blow up* $S = \{v = \rho = 0\}$ by replacing it with its inward pointing spherical normal bundle. (The reader may wish to consult [15] for more details on the blow-up construction than we give here.) This process replaces M with a new manifold $\overline{M} = [M; S]$ on which polar coordinates around the submanifold are smooth, and depends only on S (not the actual functions v and ρ). The blow-up comes equipped with a natural blow-down map $\overline{M} \to M$ which is a diffeomorphism on the interior. $\overline{M}$ is a manifold with corners with two boundary hypersurfaces: bf, the closure of the lift of $X \setminus S$ to $\overline{M}$; and ff, the lift of S to $\overline{M}$. Further, the fibers of ff over the base, S, are diffeomorphic to intervals, and indeed, the interior of the fibers is naturally an affine space (i.e. these interiors have $\mathbb{R}$ acting by translations, but there is no natural origin). Figure 2 depicts this blow-up construction.

Given v and ρ, the fibers of the interior of ff in $[M; S]$ can be identified with $\mathbb{R}$, via the coordinate $s = v/\rho$. In particular, ∂_s is a well-defined vector field on the fibers.

We define "polar coordinates"

$$R = \left(v^2 + \rho^2\right)^{1/2} \in [0, \infty), \quad \Theta = \frac{(\rho, v)}{r} \in \mathbb{S}^1_+,$$

which are smooth on $\overline{M}$. Near the interior of ff, we use the projective coordinates $\rho, s = v/\rho$ as well as local coordinates y on S. In these coordinates, a simple computation shows that the unbounded terms of $\rho^2 g$ cancel near $\rho = 0$ and hence $\rho^2 g$ is a smooth Lorentzian metric in a neighborhood of the interior of ff (i.e., down to $\rho = 0$).

Given a solution $w(\rho, v, y)$ of $\Box_g w = f$ with f smooth and compactly supported, we define the function

$$u(\rho, s, y) = \rho^{-\frac{n-2}{2}} w(\rho, \rho s, y).$$

The wave operators for the metrics g and $\rho^2 g$ are related by the somewhat remarkable identity

$$\begin{aligned}\rho^{\frac{2-n}{2}} \Box_g w = \rho^{\frac{2-n}{2}} \Box_g \left(\rho^{\frac{n-2}{2}} u\right) &= \rho^2 \Box_{\rho^2 g} u - \left(\rho^{\frac{n+2}{2}} \Box_{\rho^2 g} \rho^{\frac{2-n}{2}}\right) u \\ &= \rho^2 \Box_{\rho^2 g} u - \rho^2 \gamma u;\end{aligned}$$

we refer the reader to [5] for the details of this computation. Note that γ is smooth on M because ρ is. Moreover, $\rho^2 g$ is a nondegenerate metric near the interior of ff, and so $\Box_{\rho^2 g} - \gamma$ is a nondegenerate hyperbolic operator near ff. This calculation thus shows that if w is a solution of $\Box_g w = f$ with smooth compactly supported f, vanishing identically for $s \leq s_0$, then the argument of Friedlander [5, Section 1] shows that w may be smoothly extended across ff. In particular, w and its derivatives may be restricted to ff. Note that the condition on the support of w is analogous to the support condition satisfied by forward solutions of the inhomogeneous equation. (The argument applies equally well to solutions of the homogeneous initial value problem with the same support property on globally hyperbolic spacetimes of this form.)

Thus, if w is a solution of $\Box_g w = f$ satisfying the above support property, with f smooth and compactly supported, we may define the (forward) *radiation field* of w by

$$\mathcal{R}_+[w](s, y) = \partial_s u(0, s, y).$$

This agrees with Friedlander's original construction. We will later show that we may make this definition even without the hypothesis on the support of w in the s variable.

Remark 3.9. Note that the smooth expansion of w across ff implies that it does not have singularities at $s = 0$.

4. Propagation of b-regularity

In this section we prove an initial conormal estimate for tempered solutions w of $\Box_g w = f \in \dot{\mathcal{C}}^\infty(M)$ vanishing near $\overline{C}_-$. This estimate is used to begin the iterative scheme in Section 9. Our goal is conormal regularity at Λ^+.

The basic background in this section is the propagation of b-regularity away from radial points (see, e.g., [23]), which we briefly recall here. Let $L \in \Psi_{\mathrm{b}}^{s,r}(M)$, and let $\Sigma \subset {}^{\mathrm{b}}S^*M$ denote the characteristic set of L, λ denote the principal symbol of L in $\Psi_{\mathrm{b}}^{s,r}(M)$.

Proposition 4.1. *Suppose $w \in H_{\mathrm{b}}^{-\infty,l}(M)$. Then*

(1) *Elliptic regularity holds away from Σ, i.e.,*

$$\mathrm{WF}_{\mathrm{b}}^{m,l}(w) \subset \mathrm{WF}_{\mathrm{b}}^{m-s,l-r}(Lw) \cup \Sigma,$$

(2) *In Σ, $\mathrm{WF}_{\mathrm{b}}^{m,l}(w) \setminus \mathrm{WF}_{\mathrm{b}}^{m-s+1,r-l}(Lw)$ is a union of maximally extended bicharacteristics, i.e., integral curves of H_λ.*

Note that the order in $\mathrm{WF}_{\mathrm{b}}^{m-s+1,r-l}(Lw)$ is shifted by 1 relative to the elliptic estimates, corresponding to the usual hyperbolic loss. This arises naturally in the positive commutator estimates used to prove such hyperbolic estimates: commutators in $\Psi_{\mathrm{b}}(M)$ are one order lower than products in the differentiability sense (the first index), but not in the decay order (the second index); hence the change in the first order relative to elliptic estimates but not in the second.

We now turn to the radial set $\mathcal{R}$, where Proposition 4.1 does not yield any interesting statements, and more refined arguments are needed.

Definition 4.2. Let $\mathcal{M} \subset \Psi_{\mathrm{b}}^1(M)$ denote the $\Psi_{\mathrm{b}}^0(M)$-module of pseudodifferential operators with principal symbol vanishing on the radial set $\mathcal{R} = \{\rho = 0,\ v = 0,\ \xi = 0,\ \eta = 0\}$.

Note that a set of generators for $\mathcal{M}$ over $\Psi_{\mathrm{b}}^0(M)$ is $\rho\partial_\rho, \rho\partial_v, v\partial_v, \partial_y$ (with symbols $\xi, \rho\gamma, v\gamma, \eta$; γ enters to convert ρ and v to first order operators) and I. (Recall that these fiber variables are defined by the canonical one-form (3.19).)

Lemma 4.3. *The module $\mathcal{M}$ is closed under commutators.*

Proof. While this can be checked directly from (3.20), a more conceptual proof is as follows.

We observe from the formula (3.20) that whenever $f \in \mathcal{C}^\infty({}^bT^*M)$ (or just defined on an open subset, such as ${}^bT^*M \setminus o$), the b-Hamilton vector field H_f is tangent to the submanifold $\{\rho = \xi = 0\} \subset {}^bT^*M$. This submanifold is the image of $T^*_X M$ in ${}^bT^*M$ via the canonical map dual to the inclusion of $\mathcal{V}_{\mathrm{b}}(M)$ in $\mathcal{V}(M)$, and is thus canonically identified with T^*X; we tacitly use this identification from now on. Correspondingly, it is a symplectic manifold with symplectic form $d\gamma \wedge dv + d\eta \wedge dy$, and further, the restriction of the b-Hamilton vector field H_f of $f \in \mathcal{C}^\infty({}^bT^*M)$ as above is equal to the standard Hamilton vector field of $f|_{T^*X} \in \mathcal{C}^\infty(T^*X)$. Since $\mathcal{R}$ is a *Lagrangian* submanifold of T^*X, if $f|_{T^*X}$ vanishes on $\mathcal{R}$, the Hamilton vector field of f is tangent to $\mathcal{R}$. Correspondingly the set of $\mathcal{C}^\infty$ functions vanishing on $\mathcal{R}$ is closed under Poisson brackets; taking into account that the Poisson bracket of homogeneous degree one functions on ${}^bT^*M \setminus o$ is also such and that the principal symbol of the commutator of two b-pseudodifferential operators is given by Poisson brackets, we immediately conclude that $\mathcal{M}$ is closed under commutators. $\square$

Proposition 4.4. *Let $L = \rho^{-(n-2)/2}\rho^{-2}\Box_g\rho^{(n-2)/2} \in \mathrm{Diff}^2_{\mathrm{b}}(M)$. If $w \in H_{\mathrm{b}}^{-\infty,l}(M)$ for some l, $Lw \in H_{\mathrm{b}}^{m-1,l}$ and $w \in H_{\mathrm{b}}^{m,l}$ on a punctured neighborhood $U \setminus \partial\mathcal{R}$ of $\partial\mathcal{R}$ in ${}^{\mathrm{b}}S^*M$ (i.e. $\mathrm{WF}_{\mathrm{b}}^{m,l}(w) \cap (U \setminus \partial\mathcal{R}) = \emptyset$) then for $m' \leq m$ with $m' + l < 1/2$, $w \in H_{\mathrm{b}}^{m',l}(M)$ at $\partial\mathcal{R}$ (i.e. $\mathrm{WF}_{\mathrm{b}}^{m',l}(w) \cap \partial\mathcal{R} = \emptyset$), and for $N \in \mathbb{N}$ with $m' + N \leq m$ and for $A \in \mathcal{M}^N$, Aw is in $H_{\mathrm{b}}^{m',l}(M)$ at $\partial\mathcal{R}$ (i.e. $\mathrm{WF}_{\mathrm{b}}^{m',l}(Aw) \cap \partial\mathcal{R} = \emptyset$).*

Remark 4.5. In the situation that we care about, $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ splits into two components ("future" and "past") and we note that the proof in fact shows that the result holds at each component separately.

This result is analogous to [7], except $\rho = 0$ produces an extra boundary (so we are in codimension 2), and $\mathcal{R}$ is not Lagrangian (${}^{\mathrm{b}}T^*M$ is not symplectic at the boundary). The relevant input of the Lagrangian nature in [7] is the eigenvectors and eigenvalues of the linearization, hence much the same proof goes through. This is also analogous to the "easy" part, Section 11, of [17], describing the propagation of edge singularities, except here we have a source/sink rather than a saddle point, and thus the treatment is simpler.

Proof. First we ignore the module. We will inductively show that $\mathrm{WF}_{\mathrm{b}}^{\tilde{m},l}(w) \cap \partial\mathcal{R} = \emptyset$ assuming that we already has shown $\mathrm{WF}_{\mathrm{b}}^{m'',l}(w) \cap \partial\mathcal{R} = \emptyset$ with

$m'' = \tilde{m} - 1/2$. We may start with $\tilde{m} = \min(m_0 + 1/2, m')$, and increasing $\tilde{m}$ by $\leq 1/2$, we reach $\tilde{m} = m'$ in finitely many steps.

Thus, consider $A \in \Psi_{\mathrm{b}}^{s,r}(M) = \rho^{-r}\Psi_{\mathrm{b}}^{s}(M)$. Then

$$\imath[L, A] \in \Psi_{\mathrm{b}}^{s+1,r}(M),\ \sigma_{\mathrm{b}}(\imath[L, A]) = \mathsf{H}_\lambda a,\ a = \sigma_{\mathrm{b},r,s}(A).$$

We choose

$$a = \rho^{-r}\nu^{-s}\phi^2,$$

where $\phi \geq 0$, $\phi \equiv 1$ near $\mathcal{R}$, supported in U ($\operatorname{supp}\phi$ will be further constrained below). By (3.23), $\nu\mathsf{H}_\lambda a = (4(r+s)+c)a + e$, where c vanishes at $v = 0$, and e is supported in $\operatorname{supp} d\phi$. We take $r + s < 0$, and we choose the support of ϕ so that $|c| < |r + s|$ on the support of ϕ. Note that $r + s < 0$ means $\nu\mathsf{H}_\lambda a$ necessarily has negative sign at least in some place on $\operatorname{supp} d\phi$, since ϕ has to increase along the flow as it approaches $\mathcal{R}$. Then we have $\nu\mathsf{H}_\lambda a = -b^2 + e$, with b elliptic near $\mathcal{R}$. Then with $B \in \Psi_{\mathrm{b}}^{(s+1)/2,r/2}(M)$ with principal symbol b and $\mathrm{WF}'_{\mathrm{b}}(B) \subset \operatorname{supp} b \cap {}^{\mathrm{b}}S^*M$ (so for instance B can be a quantization of b), and similarly with $E \in \Psi_{\mathrm{b}}^{s+1,r}(M)$,

$$\imath[L, A] = -B^*B + E + F,\ F \in \Psi_{\mathrm{b}}^{s,r}(M).$$

This gives an estimate

$$\|Bw\|^2 \leq |\langle Ew, w\rangle| + |\langle Fw, w\rangle| + 2|\langle Lw, Aw\rangle| \tag{4.1}$$

when w is a priori sufficiently regular. Given $\tilde{m}, l$, we now take $s = 2\tilde{m} - 1$, $r = 2l$, so $s + r < 0$ indeed. Note that F has order $\leq 2m''$, so the inductive assumption gives a bound for $|\langle Fw, w\rangle|$. A standard regularization argument can be used to complete the proof by allowing us to apply (4.1) to any w for which the right-hand-side is a priori finite: for instance one can use a regularizer $\psi_\epsilon(\nu) = (1 + \epsilon\nu^{-1})^{-1} = \frac{\nu}{\nu+\epsilon}$, $\epsilon > 0$, which is in S^{-1} for $\epsilon > 0$ and is uniformly bounded in S^0 for $\epsilon \in (0, 1]$. Thus, one lets

$$a_\epsilon = a\psi_\epsilon(\nu)^2;$$

then $\nu\mathsf{H}_\lambda\psi_\epsilon = \epsilon\nu^{-2}\psi_\epsilon^2(\nu\mathsf{H}_\lambda\nu)$ shows that the contribution of the regularizer to the principal symbol of the commutator is the negative of a square, provided again that ϕ has sufficiently small support, i.e. adds another "good term" beside $-b^2$. One can drop the corresponding term in the inequality given by quantized version,

$$\|B_\epsilon w\|^2 \leq |\langle E_\epsilon w, w\rangle| + |\langle F_\epsilon w, w\rangle| + 2|\langle Lw, A_\epsilon w\rangle|,$$

where the calculation (involving the pairing) now makes sense for $\epsilon > 0$. Now letting $\epsilon \to 0$ the right hand side remains bounded, while $B_\epsilon \to B$ strongly in $\mathcal{L}(L^2_{\mathrm{b}}(M))$, so one concludes $Bw \in L^2_{\mathrm{b}}(M)$ and obtains the desired inequality. This completes the proof in the case when $N = 0$, i.e., when we have included no factors from the module $\mathcal{M}$ in the test operator.

In the general case $N \geq 0$ one employs the methods developed by Hassell, Melrose and Vasy [8, 9], adapted to a similar, but different (edge), setting by Melrose, Vasy and Wunsch in the appendix of [17]. For this purpose one uses generators of the module, denoted by $G_0 = \mathrm{I}, G_1, \dots, G_n, G_{n+1} = \Lambda L$, where $\Lambda \in \Psi_{\mathrm{b}}^{-1}$ is elliptic near $\mathcal{R}$ and $G_1, \dots, G_n \in \Psi_{\mathrm{b}}^1(M)$. A sufficient condition for these methods is that for $i = 1, \dots, n$,

$$\imath\Lambda[G_i, L] = \sum_j C_{ij} G_j, \tag{4.2}$$

where

$$\sigma_{\mathrm{b},0,0}(C_{ij})|_{\mathcal{R}} = 0. \tag{4.3}$$

In our case this sufficient condition is satisfied by choosing dg_i to be an eigenvector of $\nu\mathsf{H}_\lambda$ at $\mathcal{R}$, with eigenvalue -4 (cf. (3.23)), where G_i has principal symbol $\nu^{-1}g_i$. For instance, we may take the g_i to be $\rho, \hat{\xi}, \hat{\eta}$ since these, together with $\lambda\nu^2$ cut out $\mathcal{R}$ in the cosphere bundle. Since $d\nu$ and dg_i have equal eigenvalues then, the conclusion for C_{ij} follows. (We note that strictly speaking, because $\hat{\eta}$ is not globally defined, we must include additional generators to account for different coordinate charts in the tangential variables. Including additional generators does not cause any problem, as we do not require the generating set to be independent.)

We thus prove iterative regularity under $\mathcal{M}$ inductively in the power of the module as follows: we repeat the previous commutator argument, but with the commutant A replaced by

$$\mathrm{Op}(\sqrt{a})^*(G^\alpha)^*(G^\alpha)\,\mathrm{Op}(\sqrt{a})$$

where $G^\alpha = G_1^{\alpha_1} \dots G_{n+1}^{\alpha_{n+1}}$ denotes a product of powers of the generators of $\mathcal{M}$, hence $G^\alpha \in \mathcal{M}^{|\alpha|}$. Considering all of these commutators at once, as G^α runs over a basis of $\mathcal{M}^N/\mathcal{M}^{N-1}$, we then follow the same argument as used when $N = 0$ but now with systems of operators, taking values in $\mathbb{C}^d$ with $d = \dim \mathcal{M}^N/\mathcal{M}^{N-1}$. The main term in the commutator, arising from the commutators $[L, \mathrm{Op}(\sqrt{a})]$, is diagonal and positive, just as before (again, because the factor $4(r+s)+c$ is negative). Moreover the condition (4.3) permits us to absorb into this positive term those new terms that arise from commutators of L with G^α and that have the maximum number of module factors. Thus we are in the end able to estimate the terms $\|BG^\alpha w\|^2$ (with B as before) where $|\alpha| = N$ by terms microsupported away from $\mathcal{R}$ and by terms involving $G^\beta w$ with $|\beta| \leq N-1$, thus proving the result inductively. $\square$

Putting Propositions 4.4 and 4.1 together then yields the following:

Corollary 4.6. *Let $L = \rho^{-(n-2)/2}\rho^{-2}\Box_g\rho^{(n-2)/2} \in \mathrm{Diff}_{\mathrm{b}}^2(M)$, and let $\pi : {}^{\mathrm{b}}T^*M \to M$ be the projection. Suppose $w \in H_{\mathrm{b}}^{-\infty,l}(M)$ for some l, $Lw \in H_{\mathrm{b}}^{m-1,l}(M)$. Suppose $\mathcal{U}$ is a neighborhood of $\pi(\partial\mathcal{R})$ and that all bicharacteristics (in Σ) of L that enter $\mathcal{U}$, other than those in $\mathcal{R}$, possess a point disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(w)$. Then for $m' \leq m$ with $m' + l < 1/2$, w is in $H_{\mathrm{b}}^{m',l}(M)$ on $\mathcal{U}$ and for $N \in \mathbb{N}$ with $m' + N \leq m$ and for $A \in \mathcal{M}^N$, Aw is in $H_{\mathrm{b}}^{m',l}(M)$ on $\mathcal{U}$.*

Note that the hypotheses of the corollary at the future radial set hold automatically if L is non-trapping, i.e. all bicharacteristics tend to the future and past radial sets in the two directions of flow, and if w vanishes near S_-.

Remark 4.7. Corollary 4.6 implies that when Lw is in $H_{\mathrm{b}}^{\infty,l}(M)$ (so in particular if $Lw \in \dot{\mathcal{C}}^\infty(M)$) and w vanishes near $\overline{C_-}$ then w is in fact *conormal* to the front face of the blow-up defined in Section 3.8 since we obtain $H_{\mathrm{b}}^{\infty,l}$ regularity sufficiently far along all bicharacteristics (indeed, the solution vanishes in a neighborhood of the boundary there by hypothesis). In particular, this implies that the coefficients in the asymptotic expansion of Theorem 1.1 may be taken to be smooth.

5. The mapping properties of P_σ

Having verified that the operator P_σ satisfies many of the hypotheses of the theorem of Vasy [22], we now show that P_σ is Fredholm on appropriate function spaces. In this section we modify the argument of [22] to our current setting.

Recall that under our global assumptions, the characteristic set of P_σ in S^*X has two parts $\Sigma_\pm$ (each of which is a union of connected components) such that the integral curves of the Hamilton flow in $\Sigma_\pm$ tend to $S_\pm$ as the parameter tends to $+\infty$. Writing the radial sets at future, resp. past, infinity as Λ^+, resp. Λ^- (and the components of each as $\Lambda^\pm_\pm = \Lambda^\pm \cap \Sigma_\pm$), one is interested in the following two kinds of Fredholm problems, in which one requires a relatively high degree of regularity at Λ^+, resp. Λ^-, but allows very low regularity at the other radial set, Λ^-, resp. Λ^+.

To make this into a Fredholm problem it is convenient to introduce variable order Sobolev spaces and variable order pseudodifferential operators. This was originally done by Visik, Eskin [25], Unterberger [19] and Duistermaat [2], and we recall this theory in Appendix A. More recently, Faure–Sjöstrand [4] used variable-order Sobolev spaces in a manner similar to ours in their work on Ruelle resonances for Anosov flows. The main result that we use is Proposition A.1, which shows that standard propagation of singularities arguments along forward null-bicharacteristics hold with respect to the spaces H^s with $s \in \mathcal{C}^\infty(S^*X)$ defining the variable order, provided s is non-decreasing along the Hamilton flow.

Remark 5.1. We recall that in [22] such issues were avoided by using complex absorption arranged so that the resulting operator is elliptic at one of the radial sets, say Λ^-, but is unchanged near Λ^+. Thus, each bicharacteristic enters the complex absorption region in either the forward or backward direction; in this region the operator becomes elliptic due to the imaginary part of its principal symbol, hence only Λ^+ acts as a radial set for the operator with complex absorption added, and one could use standard Sobolev spaces as one did not have to deal with different regularity thresholds at Λ^+ and Λ^-.

Now we recall, as computed in (3.17), that the quantity

$$\hat\beta^{\pm}(\sigma) = \pm\frac{\rho_\infty}{2\imath\beta_0}\sigma_1(P_\sigma - P_\sigma^*)|_{\Lambda^\pm}$$

is given the "constant" value $-\operatorname{Im}\sigma$: it is independent of the point in $\Lambda^\pm$. *Here the $\pm$ at the front of the right hand side corresponds to $\Sigma_\pm$, i.e. the subscript of $\Lambda^\pm_\pm$.* Let

$$\bar s^{\pm}(\sigma) = \frac{1}{2} - \hat\beta^{\pm}(\sigma) = \frac{1}{2} + \operatorname{Im}\sigma$$

denote the threshold Sobolev exponents at $\Lambda^\pm$, i.e. at the future and past radial sets. Thus,

$$\bar s^{+}(\sigma) = \bar s^{-}(\sigma),$$

but this is actually not important below. Let s_{ftr} be a function on S^*X, such that

(1) s_{ftr} is constant near $\Lambda^\pm$,
(2) s_{ftr} is decreasing along the H_p-flow on Σ_+, increasing on Σ_-,
(3) s_{ftr} is less than the threshold exponents at Λ^+, towards which we propagate our estimates, i.e. $s_{\mathrm{ftr}}|_{\Lambda^+} < \bar s^{+}(\sigma)$,
(4) s_{ftr} is greater than the threshold value at Λ^-, away from which we propagate our estimates, i.e. $s_{\mathrm{ftr}}|_{\Lambda^-} > \bar s^{-}(\sigma)$.

Since $U \in H^{s_{\mathrm{ftr}}}$ near Λ^- a priori (indeed it is residual there), one can propagate regularity and estimates from Λ^- to Λ^+ as in [22, Section 2.4], and for all N (in practice taken very large) obtain estimates for such U

$$\|U\|_{H^{s_{\mathrm{ftr}}}} \leq C(\|P_\sigma U\|_{H^{s_{\mathrm{ftr}}-1}} + \|U\|_{H^{-N}}). \tag{5.1}$$

(More generally, the Sobolev exponent on the first term on right hand side would be $s_{\mathrm{ftr}} - m + 1$ where m is the order of P_σ; here $m = 2$.)

On the other hand, if s_{past} is a function on S^*X, such that

(1) s_{past} is constant near $\Lambda^\pm$,
(2) s_{past} is increasing along the H_p-flow on Σ_+, decreasing on Σ_-,
(3) s_{past} is less than the threshold exponents at Λ^-, towards which we propagate our estimates, i.e. $s_{\mathrm{past}}|_{\Lambda^-} < \bar s^{-}(\sigma)$,
(4) s_{past} is greater than the threshold value at Λ^+, away from which we propagate our estimates, i.e. $s_{\mathrm{past}}|_{\Lambda^+} > \bar s^{+}(\sigma)$,

then one can propagate regularity and estimates from Λ^+ to Λ^-, and for all N obtain estimates

$$\|U\|_{H^{s_{\mathrm{past}}}} \leq C(\|P_\sigma U\|_{H^{s_{\mathrm{past}}-1}} + \|U\|_{H^{-N}}).$$

With $\bar s^{\pm,*}(\sigma)$ denoting the threshold Sobolev exponents for P_σ^*, the same considerations apply to P_σ^*, i.e., if s^*_{past} is a function on S^*X such that

(1) s^*_{past} is constant near $\Lambda^\pm$,
(2) s^*_{past} is increasing along the H_p-flow on Σ_+, decreasing on Σ_-,
(3) s^*_{past} is less than the threshold exponents at Λ^-, towards which we propagate our estimates, i.e. $s^*_{\mathrm{past}}|_{\Lambda^-} < \bar s^{-,*}(\sigma)$,

(4) s^*_{past} is greater than the threshold value at Λ^+, away from which we propagate our estimates, i.e. $s^*_{\mathrm{past}}|_{\Lambda^+} > \bar{s}^{+,*}(\sigma)$,

then one can propagate regularity and estimates from Λ^+ to Λ^-, and for all N obtain estimates

$$\|U\|_{H^{s^*_{\mathrm{past}}}} \leq C(\|P^*_\sigma U\|_{H^{s^*_{\mathrm{past}}-1}} + \|U\|_{H^{-N}}), \tag{5.2}$$

with analogous results for s^*_{ftr}.

Now, as $\bar{s}^{\pm,*}(\sigma) = -\bar{s}^{\pm}(\sigma) + 1$, if one chooses s_{ftr} as above, then one can take $s^*_{\mathrm{past}} = -s_{\mathrm{ftr}} + 1$: with this choice,

$$(H^{s_{\mathrm{ftr}}})^* = H^{s^*_{\mathrm{past}}-1},\ (H^{s_{\mathrm{ftr}}-1})^* = H^{s^*_{\mathrm{past}}},$$

i.e., the space on the left hand side of (5.1) is dual to the (non-residual) space on the right hand side of (5.2), and (non-residual) the space on the right hand side of (5.1) is dual to the space on the left hand side of (5.2). Taking N sufficiently large such that the inclusions of the spaces on the left hand side of (5.1), resp. (5.2), into H^{-N} are compact, this implies Fredholm properties at once for P_σ and P^*_σ, with a slight change in the spaces as follows. Let

$$\mathcal{Y}^{s_{\mathrm{ftr}}-1} = H^{s_{\mathrm{ftr}}-1},\ \mathcal{X}^{s_{\mathrm{ftr}}} = \{U \in H^{s_{\mathrm{ftr}}} :\ P_\sigma U \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}\}$$

(note that the last statement in the definition of $\mathcal{X}^{s_{\mathrm{ftr}}}$ depends on the principal symbol of P_σ only, which is independent of σ).

Thus, we finally have the following, which follows from Propositions 2.3 and 2.4 of [22] together with the propagation of singularities in variable order Sobolev spaces away from radial points (Proposition A.1 in the appendix).

Proposition 5.2. *The family of maps P_σ enjoys the following properties:*

(1)
$$P_\sigma : \mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1},\ P^*_\sigma : \mathcal{X}^{s^*_{\mathrm{past}}} \to \mathcal{Y}^{s^*_{\mathrm{past}}-1}$$
are Fredholm.

(2) *P_σ is a holomorphic Fredholm family on these spaces in*

$$\mathbb{C}_{s_+,s_-} = \{\sigma \in \mathbb{C} :\ s_+ < \bar{s}^+(\sigma),\ s_- > \bar{s}^-(\sigma)\}, \tag{5.3}$$

with

$$s_{\mathrm{ftr}}|_{\Lambda^\pm} = s_\pm.$$

P^*_σ is antiholomorphic in the same region.

(3) *If P_σ is invertible (or if simply $u \in \mathcal{X}^{s_{\mathrm{ftr}}}$, $f \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}$, $P_\sigma u = f$), and $\mathrm{WF}(f) \cap \Lambda^- = \emptyset$, then*

$$\mathrm{WF}(P_\sigma^{-1} f) \cap \Lambda^- = \emptyset.$$

(4) *If f is $\mathcal{C}^\infty$, then*

$$\mathrm{WF}(P_\sigma^{-1} f) \subset \Lambda^+.$$

For the adjoint, corresponding to propagation in the opposite direction, $f \in \mathcal{C}^\infty$ yields

$$\mathrm{WF}((P^*_\sigma)^{-1} f) \subset \Lambda^-.$$

For the semiclassical problem, with $h^{-1} \equiv |\operatorname{Re}\sigma| \to \infty$, a natural assumption is *non-trapping*, i.e. all semiclassical bicharacteristics in $\Sigma_{\hbar,\pm}$ apart from those in the radial sets are required to tend to L^+ in the forward direction and L^- in the backward direction in Σ_+, while the directions are reversed in Σ_-; here $L^\pm$ denotes the image of $\Lambda^\pm$ in S^*X under the quotient map, and one considers S^*X as the boundary of the radial compactification of the fibers of T^*X. In particular, the non-trapping assumptions on M made in Section 3.2 imply that the operator P_σ is semiclassically non-trapping.

Under this assumption, one has non-trapping semiclassical estimates (analogues of hyperbolic estimates, i.e. with a loss of h relative to elliptic estimates), which, in the non-semi-classical language employed here, corresponds to an understanding of asymptotics as $|\operatorname{Re}\sigma| \to \infty$. The following is proved in the same way as Theorem 2.15 of [22].

Proposition 5.3. *If the non-trapping hypothesis holds, then:*

(1) P_σ^{-1} *has finitely many poles in each strip*

$$a < \operatorname{Im}\sigma < b.$$

(2) *For all a, b, there exists C such that*

$$\left\|P_\sigma^{-1}\right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}_{|\sigma|^{-1}} \to \mathcal{X}^{s_{\mathrm{ftr}}}_{|\sigma|^{-1}}} \leq C\langle \operatorname{Re}\sigma\rangle^{-1}$$

on

$$a < \operatorname{Im}\sigma < b,\ |\operatorname{Re}\sigma| > C.$$

Here the spaces with $|\sigma|^{-1}$ subscripts refer to the variable-order versions of the semiclassical Sobolev spaces discussed in Section 3.7.

6. Conormality of coefficients

In this section we show that the coefficients in the asymptotic expansion which will appear in the sequel are in fact classical conormal distributions with a very explicit singular structure.

6.1. **Conormal and homogeneous distributions.** We begin with some preliminaries on conormal and homogeneous distributions. For Y a connected component of $\{v = 0\}$ in X (such as S_+), let $\mathcal{M}_\partial$ denote the module of first order pseudodifferential operators on $X = \partial M$ with principal symbol vanishing on N^*Y. In particular, $\Psi^0(X) \subset \mathcal{M}_\partial$. Note that vector fields on X tangent to Y lie in $\mathcal{M}_\partial$, and indeed if $A \in \mathcal{M}_\partial$, then because N^*Y is locally defined by $v = 0$, $\eta = 0$, and γ is elliptic on it, $\sigma_1(A) = a_0 v\gamma + \sum a_j\eta_j$, where $a_j \in S^0$. In particular, then, $A = A_0(vD_v) + \sum A_j D_{y_j} + A'$, where $A_j, A' \in \Psi^0(X)$, and so vD_v, D_{y_j}, and I generate $\mathcal{M}_\partial$ as a $\Psi^0(X)$-module.

Below we work with the L^2-based conormal spaces $I^{(s)}(X)$ defined in Section 2.3 above. Recall that $u \in I^{(s)}(X)$ means that $u \in H^s(X)$ and $A_1 \dots A_k u \in H^s(X)$ for all $k \in \mathbb{N}$ and $A_j \in \mathcal{M}_\partial$. Thus $I^{(s)}$ is preserved by elements of $\mathcal{M}_\partial$, while elements of $\Psi^k(X)$ map $I^{(s)}$ to $I^{(s-k)}$. In particular, when restricted to a product neighborhood of Y, elements of $I^{(s)}$ can be considered as $\mathcal{C}^\infty$ functions on Y with values in distributions on $(-\delta,\delta)$ which are conormal to $\{v=0\}$, i.e., $I^{(s)}(N^*Y)=\mathcal{C}^\infty(Y;I^{(s)}(N^*\{0\}))$. We will also use the notation

$$I^{(-\infty)}(N^*Y)\equiv\bigcup_s I^{(s)}(N^*Y).$$

We also recall the standard conormal spaces, defined using the L^∞-based symbol spaces: $a\in S^k(Y\times(-\delta,\delta);\mathbb{R})$ if a is a compactly supported (in the (y,v) variables) and smooth (in all variables) and satisfies the estimates

$$\left|D_y^\alpha D_v^\ell D_\gamma^N a\right|\leq C_{\alpha\ell N}\langle\gamma\rangle^{k-N}.$$

Elements of $I^r(N^*Y)$ are defined as certain oscillatory integrals (which in this case are essentially partial Fourier transforms): $u\in I^r(N^*Y)$ if and only if

$$u=\int e^{\imath v\gamma}a(v,y,\gamma)\,d\gamma \text{ with } a\in S^{r+(n-3)/4}, \text{ modulo } \mathcal{C}^\infty.$$

Thus $a\in S^k$ corresponds to $u\in I^{k-(n-3)/4}(N^*Y)$. Since $a\in S^k$ corresponds to a lying in the weighted L^2 space $\langle\gamma\rangle^{k+1/2+\epsilon}L^2$ for $\epsilon>0$,

$$I^{k-(n-3)/4}(N^*Y)\subset\bigcap_{\epsilon>0}I^{(-k-1/2-\epsilon)}(N^*Y).$$

Note that N^*Y corresponds to $v=0$ in this parameterization, and so the principal symbol is identified with an elliptic multiple of $a|_{v=0}$.

Now, if a is homogeneous outside a compact set in γ (and a is independent of v near $v=0$), one regards it for convenience as a homogeneous function on $Y\times(\mathbb{R}\setminus\{0\})$, and then a basis of such functions of degree κ over $\mathcal{C}^\infty(Y)$ is given by γ^κ times the characteristic function of $(0,\infty)_\gamma$, resp. $(-\infty,0)_\gamma$, which we denote $\gamma_\pm^\kappa$. If κ is not a negative integer, one can go further, and consider the homogeneous distributions $\chi_\pm^\kappa(\gamma)$ on $\mathbb{R}$ (or $Y\times\mathbb{R}$ in our setting) defined by (the analytic continuation in κ, from $\kappa>-1$, when they are locally L^1, of)

$$\chi_\pm^\kappa(\gamma)=\frac{\gamma_\pm^\kappa}{\Gamma(1+\kappa)}.$$

The inverse Fourier transform of these distributions are elliptic multiples of

$$v_{\pm\imath 0}^{-1-\kappa}\equiv(v\pm\imath 0)^{-1-\kappa}$$

(see Section 7.1 of [10]); these are thus a basis for $I^k_{\mathrm{cl}}(N^*Y)/I^{k-1}_{\mathrm{cl}}(N^*Y)$ for $k=\kappa-(n-3)/4$ over $\mathcal{C}^\infty(Y)$. (The "cl" subscript stands for "classical" and refers to conormal distributions whose symbols have polyhomogeneous asymptotic expansions.) For negative integers $\kappa=-k$, one must be more careful in describing a basis, as $\chi_\pm^{-k}$ is then supported at the origin. We instead simply consider directly the inverse Fourier transform of $\psi(\pm\gamma)\gamma^{-k}$ where ψ is a smooth function equal to 0 for $\gamma<1$ and 1 for $\gamma>2$. The result is a k'th antiderivative of $\psi(\pm\gamma)$, whose inverse Fourier transform differs by a smooth function from a multiple of $(v \pm \imath 0)^{-1}$, hence differs by a smooth function from a multiple of

$$v_{\pm \imath 0}^{-1-\kappa} \equiv (v \pm \imath 0)^{-1-\kappa} \log(v \pm \imath 0). \tag{6.1}$$

Note that these are no longer homogeneous distributions. (We also remark that when κ is a negative integer of course we may also write more simply

$$v_{\pm \imath 0}^{-1-\kappa} = v^{-1-\kappa} \left(\log |v| \pm \imath \pi H(-v)\right)$$

with H the Heaviside function; however it is more convenient to stick with the consistent notation offered by the expression as (6.1).)

6.2. **Spaces of solutions.** We now turn to the solution spaces of a class of operators including our P_σ.. We consider a general operator of the form

$$P = vD_v^2 + \alpha D_v + Q,\ Q \in \mathcal{M}_\partial^2,\ \alpha \in \mathcal{C}^\infty(Y); \tag{6.2}$$

since D_v is elliptic on N^*Y, we in particular have $P \in \mathcal{M}_\partial \cdot \Psi^1(\partial M)$. Note in particular that the operator family P_σ defined by (3.6) has the form (6.2) by (3.9), hence the results here apply if $P_\sigma u \in \mathcal{C}^\infty$.

Lemma 6.1. *If $Pu = f \in I^{(s)}(N^*Y)$, $u \in I^{(s)}(N^*Y)$, then*

$$u = g_+ v_{+\imath 0}^{1-\imath\alpha} + g_- v_{-\imath 0}^{1-\imath\alpha} + \tilde{u}, \tag{6.3}$$

with $g_\pm \in \mathcal{C}^\infty(Y)$ (pulled back via a local product decomposition) and $\tilde{u} \in I^{(s+1-\epsilon)}(N^*Y)$ for all $\epsilon > 0$.

Remark 6.2. If $v_{\pm\imath 0}^{1-\imath\alpha} \in I^{(s+1-\epsilon)}$ for all $\epsilon > 0$, then the conclusion is simply $u \in I^{(s+1-\epsilon)}(N^*Y)$. On the other hand, if $v_{\pm\imath 0}^{1-\imath\alpha} \notin I^{(s)}$, then the conclusion is $g_\pm = 0$, and thus again $u \in I^{(s+1-\epsilon)}(N^*Y)$.

If $f \in \mathcal{C}^\infty(\partial M)$, iterative use of the lemma will yield a full expansion of u, provided we replace $g_\pm$ by appropriate functions $\tilde{g}_\pm$ with $P(\tilde{g}_\pm v_{\pm\imath 0}^{1-\imath\alpha}) \in \mathcal{C}^\infty(X)$ (see Lemma 6.4 below).

Proof. We may assume that u is supported in a product neighborhood of Y, identified as $(-\delta, \delta)_v \times Y$, since if $\chi \in \mathcal{C}^\infty(\partial M)$ is compactly supported in such a neighborhood and is identically 1 near Y, then $\mathrm{WF}'([\chi, P]) \cap N^*Y = \emptyset$, so $[\chi, P]u \in \mathcal{C}^\infty(X)$ and thus $P(\chi u) \in I^{(s)}$ as well.

Note that $vD_v^2 = D_v v D_v + \imath D_v$. Thus, if $G \in \Psi^{-1}(X)$ is a parametrix for D_v near N^*Y (where D_v is elliptic), applying G to Pu yields

$$(vD_v + (\imath + \alpha) + GQ)u \in I^{(s+1)}(N^*Y).$$

Since $u \in I^{(s)}$, $Qu \in I^{(s)}$ and thus $GQu \in I^{(s+1)}$, so we have

$$(vD_v + \imath + \alpha)u \in I^{(s+1)} = \mathcal{C}^\infty(Y; I^{(s+1)}(N^*\{0\})).$$

With J a compact interval, let $I_{\mathcal{S}}^{(\ell)}(N^*\{0\})$ denote the sum of elements of $I^{(\ell)}(N^*\{0\})$ supported in J and Schwartz functions on $\mathbb{R}$. Then the Fourier transform on $\mathbb{R}$ maps elements of $I^{(\ell)}_{\mathcal{S}}(N^*\{0\})$ to L^2-based symbols. More precisely, if $S^{(\ell)}$ is the set of smooth functions ϕ on $\mathbb{R}_\gamma$ such that

$$(\gamma D_\gamma)^N \phi \in L^{2,\ell} \equiv \langle\gamma\rangle^{-\ell} L^2$$

for all $N \in \mathbb{N}$, then the Fourier transform is an isomorphism $I^{(\ell)}_{\mathcal{S}}(N^*\{0\}) \to S^{(\ell)}$. (We remark that while our conventions yield a nice correspondence between conormal Sobolev orders and symbol orders as discussed here, they have the unfortunate result that $S^{\ell} \subset S^{\ell'}$ if $\ell \geq \ell'$, contrary to the usual sign convention for symbol spaces.)

Taking the partial Fourier transform, $\widetilde{\mathcal{F}}$, in the interval variable, v, yields

$$(-\gamma D_\gamma + 2\imath + \alpha)\widetilde{\mathcal{F}}u = (-D_\gamma \gamma + \imath + \alpha)\widetilde{\mathcal{F}}u \in \mathcal{C}^\infty(Y; S^{(s+1)}).$$

Now, to analyze the behavior of $\widetilde{\mathcal{F}}u$ at infinity, we conjugate the differential operator by $\gamma^{-2+\imath\alpha}$ on $\mathbb{R}\setminus\{0\}$, where

$$\gamma^{2-\imath\alpha}(-\gamma D_\gamma + 2\imath + \alpha)\gamma^{-2+\imath\alpha} = -\gamma D_\gamma,$$

so one has

$$-\gamma D_\gamma(\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u) = \gamma^{2-\imath\alpha}(-\gamma D_\gamma + 2\imath + \alpha)\widetilde{\mathcal{F}}u \in \mathcal{C}^\infty(Y; S^{(s-1-\operatorname{Im}\alpha)}),$$

and thus

$$D_\gamma(\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u)|_{[1,\infty)} \in \mathcal{C}^\infty(Y; S^{(s-\operatorname{Im}\alpha)}[1,\infty)). \tag{6.4}$$

Note that due to the presence of $\epsilon > 0$ in the statement of the lemma, we may assume that $s - \operatorname{Im}\alpha \neq 1/2$; this simplifies some formulae below (otherwise one would have logarithmic terms).

Now, if $b \in S^{(\ell)}([1,\infty))$, $\ell < 1/2$, then the indefinite integral of b given by

$$c(\gamma) = \int_1^\gamma b(\eta)\, d\eta,$$

satisfies (by Cauchy-Schwarz)

$$\begin{aligned} |c(\gamma)| &\leq \left(\int_1^\gamma |\eta|^{2\ell}|b(\eta)|^2\, d\eta\right)^{1/2} \left(\int_1^\gamma |\eta|^{-2\ell}\, d\eta\right)^{1/2} \\ &\leq C\,\|b\|_{L^{2,\ell}}\left(1 + |\gamma|^{\frac{1}{2}-\ell}\right). \end{aligned}$$

Thus $c \in L^{2,\ell-1-\epsilon}$ for all $\epsilon > 0$, and as $D_\gamma c = b$, $c \in S^{(\ell-1-\epsilon)}$. (Note that constants are in $S^{(\ell-1-\epsilon)}$ since $\ell < 1/2$.)

Returning now to u described by (6.4) above and setting $\ell = s - \operatorname{Im}\alpha$, we see that

$$\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u = \tilde{w} \in S^{(s-\operatorname{Im}\alpha-1-\epsilon)}$$

provided $\ell < 1/2$. On the other hand, if $\ell = s - \operatorname{Im}\alpha > 1/2$, then $S^{(\ell)} \subset L^1$, and if we define the indefinite integral as

$$c(\gamma) = -\int_\gamma^\infty b(\eta)\, d\eta,$$

then, by Cauchy–Schwarz,

$$|c(\gamma)| \leq \left(\int_\gamma^\infty |\eta|^{2\ell}|b(\eta)|^2\,d\eta\right)^{1/2}\left(\int_\gamma^\infty |\eta|^{-2\ell}\,d\eta\right)^{1/2} \leq C\,\|b\|_{L^{2,\ell}}\,|\gamma|^{\frac{1}{2}-\ell},$$

so $c \in S^{(\ell-1-\epsilon)}$. Then, writing

$$\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u - \widetilde{\mathcal{F}}u|_{\gamma=1} = \int_1^\infty \partial_\gamma(\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u) - \int_\gamma^\infty \partial_\gamma(\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u),$$

we deduce that

$$\gamma^{2-\imath\alpha}\widetilde{\mathcal{F}}u|_{\gamma>1} = g_+ + \tilde{w},\ g_+ \in C^\infty(Y),\ \tilde{w} \in S^{(s-\operatorname{Im}\alpha-1-\epsilon)},$$

and thus

$$\widetilde{\mathcal{F}}u|_{\gamma>1} = g_+\gamma^{-2+\imath\alpha} + w_+,\ w_+ \in S^{(s+1-\epsilon)}.$$

A similar calculation applies to $\widetilde{\mathcal{F}}u|_{\gamma<-1}$, yielding

$$\widetilde{\mathcal{F}}u|_{\gamma<-1} = g_-(-\gamma)^{-2+\imath\alpha} + w_-,\ w_- \in S^{(s+1-\epsilon)}.$$

In summary, if ψ_+ is supported in $(1,\infty)$, identically 1 on $[2,\infty)$, and $\psi_-(\gamma) = \psi_+(-\gamma)$, then

$$\widetilde{\mathcal{F}}u = g_+\psi_+\gamma^{-2+\imath\alpha} + g_-\psi_-(-\gamma)^{-2+\imath\alpha} + w,\ w \in S^{(s+1-\epsilon)}.$$

Now, the inverse partial Fourier transform of w is in $I^{(s+1-\epsilon)}$, so it remains to deal with the other terms. Changing these by a compactly supported distribution does not affect their singularities, so we can replace these by the homogeneous distributions $\gamma_\pm^{-2+\imath\alpha}$ for a local description of the inverse partial Fourier transform. But the inverse Fourier transforms of the latter are $v_{\pm\imath 0}^{1-\imath\alpha}$, so we conclude that

$$u = g_+v_{+\imath 0}^{1-\imath\alpha} + g_-v_{-\imath 0}^{1-\imath\alpha} + \tilde{u},\ \tilde{u} \in I^{(s+1-\epsilon)},$$

as claimed. $\square$

Although the following corollary follows directly from the results of [22], we give a proof using Lemma 6.1.

Corollary 6.3. *If $Pu = f \in \mathcal{C}^\infty(X)$, $u \in I^{(s_0)}(N^*Y)$, $s_0 > 3/2 + \operatorname{Im}\alpha$, then $u \in \mathcal{C}^\infty(X)$.*

Proof. Let $\tilde{s}_0 = \sup\{s : u \in I^{(s)}(N^*Y)\}$, so $\tilde{s}_0 > 3/2 + \operatorname{Im}\alpha$ (possibly $\tilde{s}_0 = +\infty$); if $\tilde{s}_0 = +\infty$, then we are done as $\bigcap_{s\in\mathbb{R}} I^{(s)} = \mathcal{C}^\infty(X)$. Thus, $u \in I^{(\tilde{s}_0-\epsilon)}(N^*Y)$ for all $\epsilon > 0$. By Lemma 6.1,

$$u = g_+v_{+\imath 0}^{1-\imath\alpha} + g_-v_{-\imath 0}^{1-\imath\alpha} + \tilde{u},$$

with $g_\pm \in \mathcal{C}^\infty(Y)$ (pulled back via a local product decomposition) and $\tilde{u} \in I^{(\tilde{s}_0+1-\epsilon)}(N^*Y)$ for all $\epsilon > 0$. For all $\epsilon > 0$, $\tilde{u} \in I^{(\tilde{s}_0+1-\epsilon)}(N^*Y)$, which is a subset of $I^{(3/2+\operatorname{Im}\alpha)}(N^*Y)$ for sufficiently small $\epsilon > 0$. On the other hand the sum of the first two terms is not in $I^{(3/2+\operatorname{Im}\alpha)}(N^*Y)$ unless $g_\pm$ vanish. Since $u \in I^{(3/2+\operatorname{Im}\alpha)}(N^*Y)$, $g_\pm$ must vanish, and thus $u = \tilde{u} \in I^{(\tilde{s}_0+1-\epsilon)}(N^*Y)$ for all $\epsilon > 0$, contradicting the definition of $\tilde{s}_0$. Thus, $\tilde{s}_0 = +\infty$. $\square$

Next, under the assumption that α is constant, we show that distributions such as those in the first two terms on the right hand side of the equation (6.3) can be modified to elements of the nullspace of P modulo $\mathcal{C}^\infty(X)$.

Lemma 6.4. *Suppose $\alpha \in \mathbb{C}$ is a constant, $1 - \imath\alpha$ is not an integer, and $g \in \mathcal{C}^\infty(Y)$. Then there exist $u_\pm = g_\pm v_{\pm\imath 0}^{1-\imath\alpha} \in \bigcap_{\epsilon>0} I^{(3/2+\operatorname{Im}\alpha-\epsilon)}(N^*Y)$, with $g_\pm \in C^\infty(X)$ such that $g_\pm|_Y = g$ and $Pu_\pm \in C^\infty(X)$.*

Remark 6.5. If $1 - \imath\alpha$ is an integer, the proof below still proves a slightly different result: logarithmic terms appear. Indeed, if $1-\imath\alpha$ is a nonnegative integer, then logarithmic terms appear from the definition of $v_{\pm\imath 0}^{1-\imath\alpha}$. If it is a negative integer, say, $1 - \imath\alpha = -r \leq -1$, then an additional logarithmic term is incurred at the r-th step in the expansion.

It is more straightforward to state it as follows: $u_\pm$ is a classical conormal distribution of the appropriate order, with principal symbol the same as that of $gv_{\pm\imath 0}^{1-\imath\alpha}$.

Remark 6.6. A similar expansion can be obtained in general, without assuming that α is a constant. This is similar to the treatment of generalized Coulomb type spherical waves in [20].

Proof. We suppose first that $1 - \imath\alpha$ is not an integer. As the indicial roots associated to the ordinary differential operator $vD_v^2 + \alpha D_v$ are 0 and $1 - \imath\alpha$, for $h \in \mathcal{C}^\infty(X)$,

$$Pv^k v_{\pm\imath 0}^{1-\imath\alpha} h = v^k v_{\pm\imath 0}^{-\imath\alpha} w,\ w \in \mathcal{C}^\infty(X), w|_Y = c(k)h|_Y,$$

with $c(0) = 0$, $c(k) \neq 0$ for $k \neq 0$. (We suppress the dependence of $c(k)$ on α.) Correspondingly, given g, consider first $h_{\pm,0} \in \mathcal{C}^\infty(X)$ with $h_{\pm,0}|_Y = g$. Then

$$Pv^0 v_{\pm\imath 0}^{1-\imath\alpha} h_{\pm,0} = v^0 v_{\pm\imath 0}^{-\imath\alpha} w$$

with $w|_Y = 0$, so in fact

$$Pv^0 v_{\pm\imath 0}^{1-\imath\alpha} h_{\pm,0} = v^1 v_{\pm\imath 0}^{-\imath\alpha} \tilde{w}_{\pm,1}$$

for some $\tilde{w}_{\pm,1} \in \mathcal{C}^\infty(X)$. Now, in general, for $k \neq 0$, given $\tilde{w}_{\pm,k} \in \mathcal{C}^\infty(X)$, one can let $h_{\pm,k} = -c(k)^{-1}\tilde{w}_{\pm,k}$, and then

$$Pv^k v_{\pm\imath 0}^{1-\imath\alpha} h_{\pm,k} + v^k v_{\pm\imath 0}^{-\imath\alpha} \tilde{w}_{\pm,k} = v^k v_{\pm\imath 0}^{-\imath\alpha} w_{\pm,k}$$

with $w_{\pm,k}|_Y = 0$, thus the right hand side is of the form $v^{k+1} v_\pm^{-\imath\alpha} \tilde{w}_{\pm,k+1}$. Correspondingly, one can proceed inductively and construct $\tilde{h}_{\pm,k}$ with

$$Pv_{\pm\imath 0}^{1-\imath\alpha} \tilde{h}_{\pm,k} = v^{k+1} v_{\pm\imath 0}^{-\imath\alpha} \tilde{w}_{\pm,k+1}$$

with $\tilde{w}_{\pm,k+1} \in \mathcal{C}^\infty(X)$, e.g. by taking $\tilde{h}_{\pm,k} = \sum_{j=0}^k v^j h_{\pm,j}$. More generally, one can asymptotically sum the series $\sum_{j=0}^\infty v^j h_{\pm,j}$, i.e., construct a function $h_\pm$ which differs from $\sum_{j=0}^k v^j h_{\pm,j}$ by a $\mathcal{C}^\infty$ function vanishing to order $k+1$; then

$$Pv_{\pm\imath 0}^{1-\imath\alpha} h_\pm = v^{k+1} v_{\pm\imath 0}^{-\imath\alpha} W_{\pm,k+1}$$

for every k for some $W_{\pm,k+1} \in \mathcal{C}^\infty(X)$, thus the right hand side is $\mathcal{C}^\infty$, completing the proof.

If $1-\imath\alpha$ is a non-negative integer, then the iterative construction requires including another logarithmic term owing to the logarithmic term in $v_\pm^{1-\imath\alpha}$. If $1-\imath\alpha = r \leq -1$ is a negative integer, then the iterative construction breaks down when finding the coefficient of $v^r v_{\pm\imath 0}^{1-\imath\alpha}$, as in this setting $c(r) = 0$. The proof goes through nearly as stated once we also include terms of the form $v^{r+k} v_{\pm\imath 0}^{1-\imath\alpha} \log(v \pm \imath 0)$ for $k \geq 0$. $\square$

In addition to knowing that we may formally parametrize elements in the approximate nullspace by functions on Y, we will need to know how to formally solve certain inhomogeneous equations with specified conormal right-hand sides. For the following lemma, we assume that Q is a *differential* operator in the module $\mathcal{M}_\partial^2$, although it holds (with a slightly more complicated proof) if Q is pseudodifferential. Note that for our operator P_σ, Q is in fact differential.

Lemma 6.7. *Suppose $\alpha \in \mathbb{C}$ is constant and Q is a* differential *operator in $\mathcal{M}_\partial^2$. Let $h \in \mathcal{C}^\infty(X)$ and let m be a nonnegative integer. If $\imath\alpha$ is not a strictly positive integer, then there exist $g_\pm^0, \dots g_\pm^{m+1} \in \mathcal{C}^\infty(X)$ such that the functions*

$$u_\pm = \sum_{m'=0}^{m+1} g_\pm^{m'} v_{\pm\imath 0}^{1-\imath\alpha} \log(v \pm \imath 0)^{m'}$$

solve

$$P u_\pm = h v_{\pm\imath 0}^{-\imath\alpha} \log(v \pm \imath 0)^m + \tilde{u}$$

with $\tilde{u} \in \mathcal{C}^\infty$ and P as in (6.2)*. If $1-\imath\alpha = -k \leq 0$ is a non-positive integer, then the same statement is true with $u_\pm$ replaced by $u_\pm + w_\pm$, where $w_\pm$ has the form*

$$w_\pm = g_\pm^{m+2} v^{k-1} v_{\pm\imath 0}^{1-\imath\alpha} \log(v \pm \imath 0)^{m+2},$$

with $g_\pm^{m+2}$ a smooth function. (If $1-\imath\alpha$ is a non-negative integer, then there is an additional log term implicit in the formula owing to the definition of $v_{\pm\imath 0}^{1-\imath\alpha}$.)

Proof. The proof is similar to that of Lemma 6.4. Indeed, as the indicial roots of $vD_v^2 + \alpha D_v$ are 0 and $1-\imath\alpha$, for $g \in C^\infty(X)$,

$$P\left(v^k v_{\pm\imath 0}^{1-\imath\alpha} (\log(v \pm \imath 0))^{m'} g\right) = \sum_{\ell=0}^{\min(m',2)} v^k v_{\pm\imath 0}^{-\imath\alpha} (\log(v \pm \imath 0))^{m'-\ell} w^{(\ell)}, \tag{6.5}$$
$$w^{(\ell)} \in \mathcal{C}^\infty(X), \quad w^{(\ell)}|_Y = c(k, m', \ell) g|_Y.$$

We may calculate the $c(k,m',\ell)$ explicitly. If $\imath\alpha$ is not a negative integer, then

$$\begin{aligned} c(k,m',0) &= -k(k+1-\imath\alpha),\\ c(k,m',1) &= -m'(2k+1-\imath\alpha),\\ c(k,m',2) &= -m'(m'-1). \end{aligned}$$

If $\imath\alpha$ is a negative integer, then

$$\begin{aligned} c(k,m',0) &= -k(k+1-\imath\alpha),\\ c(k,m',1) &= -(m'+1)(2k+1-\imath\alpha),\\ c(k,m',2) &= -(m'+1)m'. \end{aligned}$$

In particular, we always have $c(0,m',0)=0$. When $\imath\alpha\neq 1$, then $c(0,m',1)\neq 0$, while for $\imath\alpha=1$, $c(0,m',1)=0$ as well, though $c(0,m',2)\neq 0$. If $\imath\alpha$ is not a positive integer, then $c(k,m',0)\neq 0$ for all k. If $\imath\alpha = r\neq 1$ is a positive integer, then $c(r-1,m',0)=0$ as well.

We start by assuming that $\imath\alpha$ is not a positive integer, so that $c(0,m',1)\neq 0$ and $c(k,m',0)\neq 0$ for all $k\geq 1$. We find the Taylor series for $g_\pm^{m'}$ iteratively; at each step we find the largest remaining term, starting with the leading term of $g_\pm^{m+1}$, then the leading term of $g_\pm^m$, and so on. We then find the first Taylor coefficient of $g_\pm^{m+1}$, then $g_\pm^m$, and continue in this fashion.

We start by finding the leading term in the expansion. For any function $g_{\pm,0}^{m+1}\in C^\infty(X)$, we have

$$\begin{aligned} &Pv^0 v_{\pm\imath 0}^{1-\imath\alpha}(\log(v\pm\imath 0))^{m+1} g_{\pm,0}^{m+1}\\ &\quad= \sum_{\ell=0}^{\min(m+1,2)} v^0 v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^{m+1-\ell} w_{\pm,0,m+1}^{(\ell)}, \end{aligned}$$

with $w_{\pm,0,m+1}^{(\ell)} = c(0,m+1,\ell)g_{\pm,0}^{m+1}$. As $c(0,m+1,0)=0$, and $c(0,m+1,1)\neq 0$, we choose $g_{\pm,0}^{m+1}$ so that

$$g_{\pm,0}^{m+1}|_Y = \frac{1}{c(0,m+1,1)} h|_Y.$$

For a function $g_{\pm,0}^{m'}\in C^\infty$, we have

$$\begin{aligned} &Pv^0 v_{\pm\imath 0}^{1-\imath\alpha}(\log(v\pm\imath 0))^{m'} g_{\pm,0}^{m'}\\ &\quad= \sum_{\ell=0}^{\min(m',2)} v^0 v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^{m'-\ell} w_{\pm,0,m'}^{(\ell)}. \end{aligned}$$

Having now found $g_{\pm,0}^{m'+1}$, we then choose $g_{\pm,0}^{m'}$ so that

$$g_{\pm,0}^{m'}|_Y = -\frac{c(0,m'+1,2)}{c(0,m',1)} g_{\pm,0}^{m'+1}|_Y.$$

Observe that because $v_{\pm\imath 0}^{1-\imath\alpha}$ is in the approximate kernel of P, we may choose $g^0_{\pm,0}|_Y$ freely.

Applying P, all terms other than the leading one cancel at Y and so we have

$$\begin{aligned} &P\sum_{m'=0}^{m+1} v^0 v_{\pm\imath 0}^{1-\imath\alpha}(\log(v\pm\imath 0))^{m'} g^{m'}_{\pm,0} - v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^m h \\ &\quad= \sum_{m'=0}^{m+1} v^1 v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^{m'}\tilde{w}_{\pm,1,m'}. \end{aligned}$$

Now, in general, for $k\neq 0$, given $\tilde{w}_{\pm,1,m'}\in\mathcal{C}^\infty(X)$ for $0\leq m'\leq m+1$, we set

$$g^{m'}_{\pm,k} = -\frac{1}{c(k,m',0)}\left(\tilde{w}_{\pm,k,m'} + c(k,m'+1,1)w^{(1)}_{\pm,k,m'+1} + c(k,m'+2,2)w^{(2)}_{\pm,k,m'+2}\right),$$

where $w^{(\ell)}_{\pm,k,m'}$ are the coefficients in equation (6.5) with applied to $g=g^{m'}_{\pm,k}$. Applying P, we see

$$\begin{aligned} &P\sum_{m'=0}^{m+1} v^k v_{\pm\imath 0}^{1-\imath\alpha}(\log(v\pm\imath 0))^{m'} g^{m'}_{\pm,k} + \sum_{m'=0}^{m+1} v^k v_{\pm\imath 0}^{-\imath\alpha}\tilde{w}_{\pm,k,m'} \\ &\qquad - v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^m h \\ &\quad= \sum_{m'=0}^{m+1} v^k v_{\pm\imath 0}^{-\imath\alpha} w^{(0)}_{\pm,k,m'}, \end{aligned}$$

where $w^{(0)}_{\pm,k,m'}|_Y=0$, and so the right hand side is of the form

$$\sum_{m'=0}^{m+1} v^{k+1} v_{\pm\imath 0}^{-\imath\alpha}\tilde{w}_{\pm,k+1,m'}.$$

We can thus proceed inductively and construct $\tilde{g}^{m'}_{\pm,k}$ with

$$\begin{aligned} &P\sum_{m'=0}^{m+1} v_{\pm\imath 0}^{1-\imath\alpha}(\log(v\pm\imath 0))^{m'}\tilde{g}^{m'}_{\pm,k} - v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^m h \\ &\quad= \sum_{m'=0}^{m+1} v^{k+1} v_{\pm\imath 0}^{-\imath\alpha}\tilde{w}_{\pm,k+1,m'}, \end{aligned}$$

with $\tilde{w}_{\pm,k+1,m'}\in\mathcal{C}^\infty(X)$. (Namely, $\tilde{g}^{m'}_{\pm,k}=\sum_{j=0}^k v^j g^{m'}_{\pm,j}$ works.)

We now asymptotically sum the series $\sum_{j=0}^\infty v^j g^{m'}_{\pm,j}$ to construct a function $g^{m'}_{\pm}$ differing from each $\sum_{j=0}^k v^j g^{m'}_{\pm,j}$ by a smooth function vanishing to order $k+1$, and then

$$P\sum_{m'=0}^{m+1} v_{\pm\imath 0}^{1-\imath\alpha}(\log(v\pm\imath 0))^{m'} g^{m'}_{\pm} - v_{\pm\imath 0}^{-\imath\alpha}(\log(v\pm\imath 0))^m h\in\mathcal{C}^\infty(X),$$

completing the proof.

If $1 - \imath\alpha = -k$ is a non-positive integer, the iteration proceeds nearly as before, but at the expense of an additional log term at the k-th coefficient. (For example, if $k = 0$, then $c(0, m', 1)$ also vanishes and so an additional log term is needed to find the first coefficient.) $\square$

We now combine Lemma 6.1 and Lemma 6.4 to obtain a complete asymptotic expansion of elements of the nullspace of P modulo $\mathcal{C}^\infty(X)$.

Proposition 6.8. *Suppose $\alpha \in \mathbb{C}$ is a constant. If $u \in I^{(-\infty)}(N^*Y)$ and $Pu \in \mathcal{C}^\infty(X)$, then there exist $g_\pm \in \mathcal{C}^\infty(X)$ and $\tilde{u} \in \mathcal{C}^\infty(X)$ such that*

$$u = g_+ v_{+\imath 0}^{1-\imath\alpha} + g_- v_{-\imath 0}^{1-\imath\alpha} + \tilde{u}.$$

See Remark 6.5 if $1 - \imath\alpha$ is an integer.

If instead $Pu \in I^{(\tilde{s})}(N^*Y)$, the same conclusion holds with $\tilde{u} \in \mathcal{C}^\infty(X)$ replaced by $\tilde{u} \in \bigcap_{\delta>0} I^{(\tilde{s}+1-\delta)}(N^*Y)$.

Proof. First suppose $Pu \in \mathcal{C}^\infty(X)$. Let $s_0 = \sup\{s : u \in I^{(s)}(N^*Y)\}$ (the set on the right is non-empty by hypothesis), so either $s_0 = +\infty$, and then $\bigcap_{s\in\mathbb{R}} I^{(s)} = \mathcal{C}^\infty(X)$ shows that the conclusion holds with $g_\pm = 0$, or $s_0 \in \mathbb{R}$ is finite, and then $u \in I^{(s_0-\epsilon)}$ for all $\epsilon > 0$. By Lemma 6.1, there exist $\tilde{g}_\pm \in \mathcal{C}^\infty(Y)$ so that

$$u = \tilde{g}_+ v_{+\imath 0}^{1-\imath\alpha} + \tilde{g}_- v_{-\imath 0}^{1-\imath\alpha} + u',$$

with $u' \in I^{(s_0+1-\delta)}(N^*Y)$ for all $\delta > 0$. Here the first two terms are in $\bigcap_{\delta>0} I^{(3/2+\operatorname{Im}\alpha-\delta)}(N^*Y)$ but not in $I^{(3/2+\operatorname{Im}\alpha)}(N^*Y)$ unless $\tilde{g}_\pm$ vanish; by the assumption that $s_0 < \infty$, we find $3/2 + \operatorname{Im}\alpha = s_0$ and $\tilde{g}_\pm$ cannot both vanish. Let $g_\pm \in C^\infty(X)$, $u_\pm \in \bigcap_{\delta>0} I^{(3/2+\operatorname{Im}\alpha-\delta)}(N^*Y)$ be given by Lemma 6.4 with $\tilde{g}_\pm$ in place of g. Thus, $Pu_\pm \in \mathcal{C}^\infty(X)$, hence $P(u - u_+ - u_-) \in \mathcal{C}^\infty(X)$. Further,

$$u - u_+ - u_- = (g_+ - \tilde{g}_+) v_{+\imath 0}^{1-\imath\alpha} + (g_- - \tilde{g}_-) v_{-\imath 0}^{1-\imath\alpha} + u',$$

and

$$(g_\pm - \tilde{g}_\pm) v_{\pm\imath 0}^{1-\imath\alpha} = O(v) v_{\pm\imath 0}^{1-\imath\alpha} \in \bigcap_{\delta>0} I^{(5/2+\operatorname{Im}\alpha-\delta)}(N^*Y).$$

Thus, $u - u_+ - u_- \in \bigcap_{\delta>0} I^{(5/2+\operatorname{Im}\alpha-\delta)}(N^*Y)$. By Corollary 6.3, $u - u_+ - u_- \in C^\infty(X)$, completing the proof of the proposition in the first case.

If $Pu \in I^{(\tilde{s})}(N^*Y)$ instead, then defining s_0 as above, $s_0 \geq \tilde{s} + 1$ means that we are done, so assume $s_0 < \tilde{s} + 1$. Proceeding as above, we obtain $\tilde{g}_\pm \in \mathcal{C}^\infty(Y)$ so that

$$u = \tilde{g}_+ v_{+\imath 0}^{1-\imath\alpha} + \tilde{g}_- v_{-\imath 0}^{1-\imath\alpha} + u',$$

with $u' \in I^{(\min(s_0,\tilde{s})+1-\delta)}(N^*Y)$ for all $\delta > 0$. Now the first two terms are in $\bigcap_{\delta>0} I^{(3/2+\operatorname{Im}\alpha-\delta)}(N^*Y)$ but not in $I^{(3/2+\operatorname{Im}\alpha)}(N^*Y)$ unless $\tilde{g}_\pm$ vanish; by the assumption that $s_0 < \tilde{s}+1$ (so $(\min(s_0, \tilde{s}) + 1 - \delta) > s_0$ for some $\delta > 0$), we find $3/2 + \operatorname{Im}\alpha = s_0$ and $\tilde{g}_\pm$ cannot both vanish. Proceeding as above,

$P(u-u_+-u_-) \in I^{(\tilde{s})}(N^*Y)$, $u-u_+-u_- \in \bigcap_{\delta>0} I^{(\min(3/2+\operatorname{Im}\alpha,\tilde{s})+1-\delta)}(N^*Y)$. If $\min(3/2+\operatorname{Im}\alpha,\tilde{s}) = \tilde{s}$, we are done, otherwise $3/2+\operatorname{Im}\alpha < \tilde{s}$, $u-u_+-u_- \in \bigcap_{\delta>0} I^{(5/2+\operatorname{Im}\alpha-\delta)}(N^*Y)$ so by [7, Theorem 6.3] $u-u_+-u_- \in \bigcap_{\delta>0} I^{(\tilde{s}+1-\delta)}(N^*Y)$, completing the proof. □

In our setting, where by equation (3.8)

$$\alpha = \sigma - \imath,$$

this gives:

Corollary 6.9. *If $u \in I^{(-\infty)}(N^*Y)$ and $P_\sigma u \in \mathcal{C}^\infty(X)$, then there exist $g_\pm \in \mathcal{C}^\infty(X)$ and $\tilde{u} \in C^\infty(X)$ such that*

$$u = g_+ v_{+\imath 0}^{-\imath\sigma} + g_- v_{-\imath 0}^{-\imath\sigma} + \tilde{u}. \tag{6.6}$$

Again, see Remark 6.5 if $-\imath\sigma$ is an integer.

If instead $P_\sigma u \in I^{(\tilde{s})}(N^*Y)$, the same conclusion holds with $\tilde{u} \in \mathcal{C}^\infty(X)$ replaced by $\tilde{u} \in \bigcap_{\delta>0} I^{(\tilde{s}+1-\delta)}(N^*Y)$.

Note that u as in the corollary lies in $H^{1/2+\operatorname{Im}\sigma-\epsilon}$ for all $\epsilon > 0$, but not in $H^{1/2+\operatorname{Im}\sigma}$ unless $g_\pm|_Y$ vanish. Thus, for s and σ corresponding to the region (5.3), this lies in H^s, the target space of $(P_\sigma)^{-1}_{\mathrm{ftr}}$, as expected—and this containment is sharp insofar as it would fail whenever $g_\pm|_Y$ do not vanish if the inequality in (5.3) is replaced by equality.

Finally, we now use Lemma 6.7 to deduce the structure of solutions to certain inhomogeneous equations with conormal right hand side:

Proposition 6.10. *If $u \in I^{(-\infty)}(N^*Y)$ and*

$$P_\sigma u \in v_{\pm\imath 0}^{-\imath\sigma-1} \log(v \pm \imath 0)^m \mathcal{C}^\infty(X),$$

then there exist $g_\pm^{m'} \in \mathcal{C}^\infty(X)$ (for $m' = 0, \dots m+1$) and $\tilde{u} \in C^\infty(X)$ such that

$$u = \sum_{m'=0}^{m+1} g_\pm^{m'} v_{\pm\imath 0}^{-\imath\sigma} \log(v \pm \imath 0)^{m'} + \tilde{u}. \tag{6.7}$$

See Remark 6.5 and Proposition 6.7 if $-\imath\sigma$ is an integer.

Proof. By Lemma 6.7, we may find a function w of the form (6.7) so that

$$P_\sigma w - P_\sigma u \in \mathcal{C}^\infty(X),$$

with the leading term having the claimed form. As the function w is also conormal, $w-u$ is conormal, and so we may apply Corollary 6.9 to finish the proof. □

7. The connection between P_σ and asymptotically hyperbolic and de Sitter spaces

In this section we clarify the action of P_σ^{-1} on the caps $C_\pm$ and in the equatorial region C_0 as in [22, Sections 3.3 and 4]. Recall that P_σ^{-1} propagates regularity from S_- to S_+; by contrast the behavior at C_- and C_0 is what is studied in detail in [22], with the behavior of P_σ^{-1} at C_0 and C_+ corresponding to the adjoint operator in that paper.

On $C_\pm$ we consider the operators

$$\begin{aligned} L_{\sigma,\pm} &= v^{\frac{1}{2}} v^{\frac{n}{4}\pm\frac{\imath\sigma}{2}} P_{\pm\sigma} v^{-\frac{n}{4}\mp\frac{\imath\sigma}{2}} v^{\frac{1}{2}} \\ &= -\Delta_{k_\pm} + \left(\sigma^2 + \frac{(n-2)^2}{4}\right) + v\mathcal{X}(\pm\sigma) + vV(\pm\sigma), \end{aligned}$$

from (3.10) (note the sign switch in σ relative to (3.10) to keep the behavior for $L_{\sigma,+}$ and $L_{\sigma,-}$ similar in terms of $\operatorname{Im}\sigma > 0$ being the physical half-plane). Here $k_\pm$ are asymptotically hyperbolic metrics, V a smooth potential and $\mathcal{X}$ a vector field tangent to $v = 0$. On C_0 we likewise consider

$$L_{\sigma,0} = \Box_{k_0} - \left(\sigma^2 + \frac{(n-2)^2}{4}\right) + v\mathcal{X}(\sigma) + vV(\sigma)$$

from (3.11), with V, $\mathcal{X}$ as above ($|v| = -v$ being a defining function for $\overline{C_0}$).

Since $L_{\sigma,0}$ is an asymptotically de Sitter operator as in [24], it has a forward solution operator $\mathcal{R}_{C_0}(\sigma)$ propagating towards S_+, i.e. if $f \in \mathcal{C}_c^\infty(C_0)$, $u = \mathcal{R}_{C_0}(\sigma)f$ is the unique solution of $L_{\sigma,0}u = f$ with L_0 vanishing near S_-. On the other hand, $L_{\sigma,\pm}$ are non-self-adjoint perturbations of the asymptotically hyperbolic operator $-\Delta_{k_\pm} + (\sigma^2 + \frac{(n-2)^2}{4})$, as in [14], with the perturbation being non-trapping in the high energy sense. In particular, if we let H_0^2 denote the 0-*Sobolev space* associated to the 0-calculus of [14], then $L_{\sigma,\pm} : H_0^2(\overline{C_\pm}) \to L^2(\overline{C_\pm})$, $\operatorname{Im}\sigma > 0$, is an analytic Fredholm family. A priori we do not have automatic invertibility for such perturbations without appeal to the large parameter behavior, which is only understood from the perspective of the extended operator; we prove this below. Note that if $v\mathcal{X}(\sigma) + vV(\sigma)$ vanishes then the invertibility of L_σ is automatic when $\operatorname{Im}\sigma > 0$, $\operatorname{Im}\sigma^2 \neq 0$ as $\Delta_{k_\pm}$ is self-adjoint.

We begin by recording a result on supports that follows from the proof of Proposition 3.9 of [22] (with the complex absorption hypotheses employed there irrelevant here).

Lemma 7.1. *Suppose* $P_\sigma^{-1} : \mathcal{Y}^{s_{\mathrm{ftr}}-1} \to \mathcal{X}^{s_{\mathrm{ftr}}}$ *is regular at* $\sigma \in \mathbb{C}$ *with* $\operatorname{Im}\sigma > 0$. *If* $f \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ *and* $\operatorname{supp} f \subset \overline{C_+ \cup C_0}$ *then* $\operatorname{supp} P_\sigma^{-1} f \subset \overline{C_+ \cup C_0}$.

Although the proof of Lemma 7.1 is essentially contained in [22, Proposition 3.9], we include a sketch of the proof here.

Proof. The microlocal argument proving the Fredholm estimates in Section 5 in fact yield a microlocal version of the same estimates. In particular, let $\mathbf{t}$

be a global function that is time-like on C_0, with $\mathbf{t}^{-1}(T_0,T_1) \subset C_0$. As we already know $P_\sigma^{-1}f$ is smooth near S_-, we may estimate

$$\|u\|_{\mathcal{X}^{s_{\mathrm{ftr}}}(\mathbf{t}^{-1}(-\infty,T_0))} \leq C\left(|\operatorname{Re}\sigma|\, \|P_\sigma u\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}(\mathbf{t}^{-1}(-\infty,T_1))} + |\operatorname{Re}\sigma|^{-1}\, \|u\|_{H^{-N}(\mathbf{t}^{-1}(-\infty,T_1))}\right).$$

As P_σ is hyperbolic in C_0, energy estimates allow allow us to estimate $\|u\|_{\mathcal{X}^{s_{\mathrm{ftr}}}}$ on $\mathbf{t}^{-1}(-\infty,T_1)$ in terms of the same right hand side. For $|\operatorname{Re}\sigma|$ large enough, we may then absorb the second term on the right side into the left side. In particular, if T_1 is such that $\mathbf{t}^{-1}(-\infty,T_1)$ is disjoint from $\operatorname{supp} f$, then u must vanish on $\mathbf{t}^{-1}(-\infty,T_1)$.

Meromorphic continuation then shows that the same support property holds at all σ so that P_σ^{-1} is regular. □

We now prove the following lemma relating invertibility of P_σ and $L_{\sigma,\pm}$:

Lemma 7.2. *Suppose that $P_\sigma : \mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $P_\sigma^* : \mathcal{X}^{s^*_{\mathrm{past}}} \to \mathcal{Y}^{s^*_{\mathrm{past}}-1}$ are invertible at $\sigma \in \mathbb{C}$ with $\operatorname{Im}\sigma > 0$. Then $L_{\sigma,\pm} : H_0^2(\overline{C_\pm}) \to L^2(\overline{C_\pm})$ is invertible.*

We let $\mathcal{R}_{C_\pm}(\sigma)$ denote the inverse of $L_{\sigma,\pm}$ thus obtained.

Remark 7.3. While we handle the invertibility within our framework, an alternative would be the complex absorption framework used in [21]; the absorption would be placed in $v < -\epsilon$ for some $\epsilon > 0$.

Remark 7.4. We may verify the invertibility hypothesis above by employing Proposition 5.3.

Proof. As already remarked, $L_{\sigma,\pm} : H_0^2(\overline{C_\pm}) \to L^2(\overline{C_\pm})$ is Fredholm, so we only need to show that $\operatorname{Ker} L_{\sigma,\pm}$ and $\operatorname{Ker} L^*_{\sigma,\pm}$ are trivial. By the results of [14], first any element of $\operatorname{Ker} L_{\sigma,\pm}$ is in $H_0^\infty(\overline{C_\pm})$ by elliptic regularity in the 0-calculus, and indeed using the parametrix construction, of [14] they are in $v^{-\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma}{2}}\mathcal{C}^\infty(\overline{C_\pm})$, while any element of $\operatorname{Ker} L^*_{\sigma,\pm}$ is in $v^{-\frac{1}{2}+\frac{n}{4}-\frac{\imath\overline{\sigma}}{2}}\mathcal{C}^\infty(\overline{C_\pm})$. In particular, for any element u_- of the kernel of $L_{\sigma,-}$, we can extend $v^{\frac{1}{2}-\frac{n}{4}-\frac{\imath\overline{\sigma}_0}{2}}u_-$ to an element $\tilde{u}$ of $\mathcal{C}^\infty(X)$. Then $f = P_\sigma\tilde{u}$ is supported in $\overline{C_+\cup C_0}$ by (3.10), hence by Lemma 7.1 $P_\sigma^{-1}f$ is also supported in $\overline{C_+\cup C_0}$, so $u = \tilde{u} - P_\sigma^{-1}f$ solves $P_\sigma u = 0$ and $u|_{C_-} = u_-$. Since $\operatorname{Ker} P_\sigma$ is trivial by assumption, u, and thus u_-, vanish. A similar argument applies to elements of $\operatorname{Ker} L^*_{\sigma,+}$ as $P_\sigma^* : \mathcal{X}^{s^*_{\mathrm{past}}} \to \mathcal{Y}^{s^*_{\mathrm{past}}-1}$ is also invertible; in that case for an element u_+ of the kernel $v^{\frac{1}{2}-\frac{n}{4}+\frac{\imath\overline{\sigma}_0}{2}}u_+$ to an element u of $\mathcal{C}^\infty(X)$ and apply $(P_\sigma^*)^{-1}$ to the result. Finally, for $\operatorname{Ker} L_{\sigma,+}$ and $\operatorname{Ker} L^*_{\sigma,-}$ we switch the direction of propagation for the inverse P_σ^{-1}, i.e. we consider

$$P_\sigma : \mathcal{X}^{s_{\mathrm{past}}} \to \mathcal{Y}^{s_{\mathrm{past}}-1},\ P_\sigma^* : \mathcal{X}^{s^*_{\mathrm{ftr}}} \to \mathcal{Y}^{s^*_{\mathrm{ftr}}-1},$$

and then completely analogous arguments apply as the roles of C_+ and C_- are simply reversed. □

We now make the connection between P_σ^{-1} and the operators on the $C_\pm$ and C_0. Let $-\sigma_0$, be a point with $\operatorname{Im}\sigma_0 < 0$ at which the family P_σ^{-1} is regular. Then by Lemma 7.2 $\mathcal{R}_{C_-}(.)$ is likewise regular at $-\sigma_0$. If $f \in C_c^\infty(C_-) \subset \mathcal{C}^\infty(X)$, then $P_{\sigma_0}^{-1} f$ is smooth on $\overline{C_-}$ by Proposition 5.2. By (3.10), on C_-,

$$u = v^{-\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma_0}{2}}\big(P_{\sigma_0}^{-1}(v^{\frac{n}{4}+\frac{\imath\sigma_0}{2}+\frac{1}{2}}f)\big)|_{C_-}$$

solves $L_{-\sigma_0,-}u = f$, and $u \in v^{-\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma_0}{2}}\mathcal{C}^\infty(\overline{C_-}) \subset L_0^2(\overline{C_-})$ (with $L_0^2(\overline{C_-})$ being the asymptotically hyperbolic L^2 space as described in [14]), with the inclusion holding as $\operatorname{Re}(\imath\sigma_0) = -\operatorname{Im}\sigma_0 > 0$. Thus,

$$v^{-\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma_0}{2}}\big(P_{\sigma_0}^{-1}(v^{\frac{n}{4}+\frac{\imath\sigma_0}{2}+\frac{1}{2}}f)\big)|_{C_-} = \mathcal{R}_{C_-}(-\sigma_0)f,$$

since $\mathcal{R}_{C_-}(-\sigma_0)f$ is the unique L^2 (relative to the asymptotically hyperbolic metric) solution of $L_{-\sigma_0,-}u = f$. By the meromorphy of both sides, the formula is then valid at all σ (regardless of the sign of $\operatorname{Im}\sigma$) at which P_σ^{-1} is regular. Indeed, by the same argument, for $f \in \mathcal{C}^\infty(C_-)$ such that $v^{\frac{n}{4}+\frac{\imath\sigma_0}{2}+\frac{1}{2}}f \in \mathcal{C}^\infty(\overline{C_-})$, and thus has an extension $\tilde{f} = E(v^{\frac{n}{4}+\frac{\imath\sigma_0}{2}+\frac{1}{2}}f)$ to an element of $\mathcal{C}^\infty(X)$, one still has

$$v^{-\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma_0}{2}}\big(P_{\sigma_0}^{-1}E(v^{\frac{n}{4}+\frac{\imath\sigma_0}{2}+\frac{1}{2}}f)\big)|_{C_-} = \mathcal{R}_{C_-}(-\sigma_0)f.$$

Recall that if $f \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $\operatorname{supp} f \subset \overline{C_+ \cup C_0}$ then $\operatorname{supp} P_\sigma^{-1} f \subset \overline{C_+ \cup C_0}$ by Lemma 7.1. Turning to the region C_0, the Carleman-type estimates in [24, Proposition 5.3] (see also [22, Section 4]) imply that $P_{\sigma_0}^{-1} f$ must vanish in a neighborhood of S_-. In particular, $P_{\sigma_0}^{-1}$ (applied to such f, with the result restricted to C_0) must be a conjugate of the forward fundamental solution of the operator in equation (3.11) (applied to $f|_{C_0}$), denoted $\mathcal{R}_{C_0}(\sigma)$ above. Indeed, again a simple generalization shows the same conclusion when one merely has $f \in \dot{\mathcal{C}}^\infty(\overline{C_0} \cup \overline{C_+})$, with the dot denoting infinite order vanishing at the boundary of this set, namely S_-.

Finally, if $f \in C_c^\infty(C_+)$, or indeed $f \in \dot{\mathcal{C}}^\infty(C_+)$ then the above discussion implies that $P_{\sigma_0}^{-1} f$ vanishes in $\overline{C}_-$ and C_0. Moreover, the expansion of Corollary 6.9 implies that in fact $P_{\sigma_0}^{-1}$ is a conjugate of $\mathcal{R}_{C_+}(\sigma)$.

In particular, the above discussion proves the following proposition:

Proposition 7.5. *Let $\varpi_\pm = 1/2 \pm n/4 \pm \imath\sigma/2$. If P_σ^{-1} is regular at σ, then it has the following "block structure" (here the rows and columns correspond to support in C_+, C_0, and C_-):*

$$\begin{pmatrix} |v|^{\varpi_-}\mathcal{R}_{C_+}(\sigma)|v|^{\varpi_+} & * & * \\ 0 & |v|^{\varpi_-}\mathcal{R}_{C_0}(\sigma)|v|^{\varpi_+} & * \\ 0 & 0 & |v|^{\varpi_-}\mathcal{R}_{C_-}(-\sigma)|v|^{\varpi_+} \end{pmatrix}$$

in the strong sense that if P_σ^{-1} is applied to a $\mathcal{C}^\infty$ function on X, the restriction of the result to C_- is given by the lower right block,

$$|v|^{\frac{1}{2}-\frac{n}{4}-\frac{\imath\sigma}{2}}\mathcal{R}_{C_-}(-\sigma)|v|^{\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma}{2}},$$

if P_σ^{-1} is applied to a $\mathcal{C}^\infty$ function supported in $\overline{C_0 \cup C_+}$, the restriction of the result to C_0 is given by $|v|^{\frac{1}{2}-\frac{n}{4}-\frac{\imath\sigma}{2}}\mathcal{R}_{C_0}(\sigma)|v|^{\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma}{2}}$ (and this result vanishes in $\overline{C_-}$), while finally if P_σ^{-1} is applied to a $\mathcal{C}^\infty$ function supported in $\overline{C_+}$, the restriction of the result to C_+ is given by $|v|^{\frac{1}{2}-\frac{n}{4}-\frac{\imath\sigma}{2}}\mathcal{R}_{C_+}(\sigma)|v|^{\frac{1}{2}+\frac{n}{4}+\frac{\imath\sigma}{2}}$ (and the result vanishes in $\overline{C_-} \cup C_0$).

By our non-trapping assumption on the null-geodesics of g, $-\Delta_{k_\pm} + v\mathcal{X}(\sigma) + vV(\sigma) + \sigma^2 + (n-2)^2/4$ is a semi-classically non-trapping operator and thus the following proposition (which follows from, e.g., the work of Vasy [22]) applies.

Proposition 7.6 (cf. [21, Theorem 4.7])**.** *Consider the operators*

$$L_{\sigma,\pm} = -\Delta_{k_\pm} + \left(\sigma^2 + \frac{(n-2)^2}{4}\right) + v\mathcal{X}(\pm\sigma) + vV(\pm\sigma),$$

with $k_\pm$ asymptotically hyperbolic metrics, V a smooth potential and $\mathcal{X}$ a vector field tangent to $v = 0$. If $L_{\sigma,\pm}$ is semiclassically non-trapping, then it has a meromorphic inverse $\mathcal{R}_{C_\pm}(\sigma)$ with finite rank poles, is holomorphic for $\operatorname{Im}\sigma \gg 0$, and has only finitely many poles in each strip $C_1 \le \operatorname{Im}\sigma \le C_2$. Moreover, non-trapping estimates hold in each strip $\operatorname{Im}\sigma > -C$ for large $\operatorname{Re}\sigma$:

$$\left\|\mathcal{R}_{C_\pm}(\sigma)f\right\|_{H^s_{|\sigma|^{-1}}} \le C\left\|f\right\|_{H^{s-1}_{|\sigma|^{-1}}}$$

Moreover, if $L_{\sigma,\pm}$ has no L^2 kernel (with respect to the metric $k_\pm$) then all poles σ_0 of $\mathcal{R}_{C_\pm}(\sigma)$ have $\operatorname{Im}\sigma_0 \le 0$.

Proof. The bounded σ properties were already explained above. The high energy estimates then follow from those for P_σ^{-1}. Since P_σ has index zero, its invertibility amounts to having a trivial kernel. Since an element of $\operatorname{Ker} P_\sigma$ restricts to a $\mathcal{C}^\infty$ function on $\overline{C_-} \cup C_0$, thus $v^{\frac{n}{4}-\frac{1}{2}+\imath\frac{\sigma}{2}}$ times the restriction to C_- is an element of $v^{\frac{n}{4}-\frac{1}{2}+\imath\frac{\sigma}{2}}\mathcal{C}^\infty(\overline{C_-})$, in view of the asymptotically hyperbolic metric on C_- this gives an element of L^2 if $\operatorname{Im}\sigma < 0$. Thus under the assumption of no L^2 “eigenvalues” all poles σ_0 of $\mathcal{R}_{C_-}(\sigma)$ indeed have $\operatorname{Im}\sigma_0 \le 0$. On the other hand, by Section 6, an element of $\operatorname{Ker} P_\sigma$ whose support is disjoint from C_- is supported in $\overline{C_+}$, and restricted to C_+ it has the asymptotic form $v^{-\imath\sigma}\mathcal{C}^\infty(\overline{C_+})$, and thus $v^{\frac{n}{4}-\frac{1}{2}+\imath\frac{\sigma}{2}}$ times the restriction to C_+ is an element of $v^{\frac{n}{4}-\frac{1}{2}-\imath\frac{\sigma}{2}}\mathcal{C}^\infty(\overline{C_-})$, and thus is in L^2 if $\operatorname{Im}\sigma > 0$, under the assumption of no L^2 “eigenvalues” all poles σ_0 of $\mathcal{R}_{C_+}(\sigma)$ indeed have $\operatorname{Im}\sigma_0 \ge 0$. □

Remark 7.7. Proposition 7.5 implies that the poles of $\mathcal{R}_{C_+}(\sigma)$ yield poles of P_σ^{-1}. A partial converse is true as well. If σ_0 is a pole of P_σ^{-1} so that the corresponding resonant dual state has support intersecting $X \setminus \overline{C}_-$, then either σ_0 is a pole of $\mathcal{R}_{C_+}(\sigma)$ or the corresponding resonant state is supported at S_+ (see [22, Remark 4.6] for more details). Such poles may occur only for σ_0 a pure imaginary negative integer. In other words, the relevant poles of P_σ^{-1} are either poles of $\mathcal{R}_{C_+}(\sigma)$ or have state supported at S_+ (and hence are differentiated delta functions in v). We remark that such states occur in even-dimensional Minkowski space, where $-\imath$ is a pole of P_σ^{-1} in 2- and 4-dimensions.

8. Structure of the poles of P_σ^{-1}

While the results in the previous section fully address the structure of nullspace of P_σ, knowledge of nullspace alone is clearly not sufficient to deal with the structure of the poles of P_σ^{-1}. Even for a spectral family of the form

$$(P_0 - \sigma\,\mathrm{I})^{-1},$$

with P_0 as in (6.2), the poles may of course be multiple owing to generalized eigenspaces; thus knowing that the nullspace of P_0 has a particular conormal form v^γ would in general permit the range of the polar part of the resolvent to have log terms. Here the situation is further complicated by the fact that our family P_σ is not of the form $P_0 - \sigma\,\mathrm{I}$ but rather has nontrivial dependence on σ, so that we cannot even employ the usual machinery of Jordan decomposition. A careful analysis of the log terms will, however, be essential in order to see that excess log terms in our asymptotic expansion (1.1) do not spoil the restriction of the rescaled solution to the front face of the radiation field blowup, which we know a priori must be smooth if we impose an additional support hypothesis in $s = v/\rho$ (cf. Section 3.8). In this section we demonstrate (among other things) that the top-order terms with $\log\rho$ are balanced by terms containing $\log v$ in such a way as to permit the solution to be smooth across the front face. (In light of the smoothness of the solution across the front face, we expect all such log terms to be balanced in this manner, but we consider only the top-order terms, i.e., the terms affecting the radiation field, here.) We should emphasize that these log terms are typically *not* vanishing, and are still a relevant part of the expansion away from the interior of the front face. In particular, we prove the following proposition, which is an extension of Corollary 6.9:

Proposition 8.1. *Let σ_0 be a pole of order k of the operator family*

$$P_\sigma^{-1} : \mathcal{Y}^{s_{\mathrm{ftr}}-1} \to \mathcal{X}^{s_{\mathrm{ftr}}},$$

and let

$$(\sigma-\sigma_0)^{-k} A_k + (\sigma-\sigma_0)^{-k+1} A_{k-1} + \ldots + (\sigma-\sigma_0)^{-1} A_1 + A_0$$

denote the Laurent expansion near σ_0, with A_0 (locally) holomorphic. If $f \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ vanishes in a neighborhood of $\overline{C}_-$, there are smooth functions $\phi_{\pm,1}, \ldots \phi_{\pm,k}$ so that for $0 \le \ell \le k-1$, $A_{k-\ell} f$ has an asymptotic expansion *near* S_+ :

$$A_{k-\ell}f = v_{+\imath 0}^{-\imath\sigma_0}\left[\sum_{j=0}^{\ell}\frac{(-\imath)^j}{j!}(\log(v+\imath 0))^j\phi_{+,k-(\ell-j)}\right] + v_{-\imath 0}^{-\imath\sigma_0}\left[\sum_{j=0}^{\ell}\frac{(-\imath)^j}{j!}(\log(v-\imath 0))^j\phi_{-,k-(\ell-j)}\right] + O\left(v^{-\imath\sigma_0+1}(\log v)^{\ell}\right).$$

(Although we use the notation $O(v^{\gamma}(\log v)^{\kappa})$ here, the term in fact has a polyhomogeneous expansion with index sets shifted from the "base" ones.)

If $-\imath\sigma_0$ is a non-negative integer, then there are smooth functions $\phi_1, \dots, \phi_k$ so that $A_{k-\ell}f$ has a similar expansion in terms of the distributions $v_+^{-\imath\sigma_0} = H(v)v^{-\imath\sigma_0}$:

$$A_{k-\ell}f = v_+^{-\imath\sigma_0}\sum_{j=0}^{\ell}\frac{(-\imath)^j}{j!}(\log|v|)^j\phi_{k-(\ell-j)} + O(v^{-\imath\sigma_0+1}(\log|v|)^{\ell}).$$

Remark 8.2. This proposition serves two purposes. The first is to show that Laurent coefficients have asymptotic expansions at $v = 0$, while the second is to show that the leading terms in this expansion have a specific form. This form is later used to show that the terms of the form $\log\rho$ cancel at the radiation field face so that the radiation field may be defined.

The additional logarithmic terms occurring at imaginary integers in Proposition 6.7 would in general disrupt the form of this expansion, but we use the support of the states to conclude that in fact it has the desired form.

One could also write the entire expansion in terms of $H(v)v^{-\imath\sigma_0}$ even if $-\imath\sigma_0$ is not a positive integer. To do this, we would have to include derivatives of delta functions if $-\imath\sigma_0$ is a negative integer. (As noted in Remark 7.7 these occur even in the case of even dimensional Minkowski space.)

The proof requires the following lemma:

Lemma 8.3. *If f vanishes in a neighborhood of $\overline{C}_-$ then $A_{k-\ell}f$ is supported in $\overline{C}_+$ for $\ell = 0, 1, \dots, k-1$.*

Remark 8.4. This lemma implies that there are two types of "resonant" states. If the state is given by $\phi\langle\psi, \cdot\rangle$, then either ϕ is supported in $\overline{C}_+$ or ψ is supported in $\overline{C}_-$. See [22, Section 4.9], especially Remark 4.6, for more details.

Proof. Near a pole σ_0 of P_σ^{-1}, we may write

$$P_\sigma = P_0 + (\sigma-\sigma_0)P_1 + (\sigma-\sigma_0)^2P_2, \tag{8.1}$$

where $P_0 = P_{\sigma_0}$, $P_1 = D_v + E$, and P_2 is a smooth function. Here $E \in \mathcal{M}_\partial$ is a first order differential operator characteristic on N^*S_+. The proof relies on the following relationships between P_i and A_j, which holds because $P_\sigma P_\sigma^{-1} = \mathrm{I}$:

$$\begin{aligned} &P_0 A_k = 0 \\ &P_1 A_k + P_0 A_{k-1} = 0 \\ &P_2 A_{k-i} + P_1 A_{k-(i-1)} + P_0 A_{k-(i-2)} = 0, \quad i = 0, \dots, k-3 \end{aligned} \tag{8.2}$$

We first observe that $A_{k-\ell} f$ vanishes near $\overline{C}_-$. Indeed, for $\ell = 0$ this follows from the Cauchy integral formula applied to $(\sigma - \sigma_0)^{k-1} P_\sigma^{-1} f$ and Proposition 7.5, while for $\ell > 0$, it follows inductively from Proposition 7.5, (8.2), and the Cauchy integral formula applied to

$$(\sigma - \sigma_0)^{k-\ell-1} P_\sigma^{-1} f.$$

To observe that $A_{k-\ell} f$ vanishes in C_0, we again proceed inductively. For $\ell = 0$, as $P_0 A_k f = 0$, the proof of Proposition 7.5 implies that it vanishes in a neighborhood of S_- and hence in all of C_0. (The Proposition does not apply as stated only because we are not at a regular point of P_σ^{-1}, but the Carleman and energy estimates—the latter being used in the proof of Lemma 7.1—employed nonetheless apply here as well.) If $\ell > 0$, the relationship (8.2) implies that $P_0 A_{k-\ell} f$ vanishes in $C_0 \cup \overline{C}_-$ and so $A_{k-\ell} f$ also vanishes in $C_0 \cup \overline{C}_-$. □

Proof of Proposition 8.1. We rely on the structure of P_σ near S_+. Indeed, recall from above that $P_\sigma = D_v(vD_v + \sigma) + Q$, where $Q \in M_\partial^2$ (in the notation of Section 6) is a differential operator. We rely on the form (8.1) of P_σ near a pole σ_0 of P_σ^{-1} as well as the relationships (8.2) between P_i and A_j.

We start by assuming that $-\imath\sigma_0$ is not an integer and proceed by induction on ℓ. As f vanishes near $\overline{C}_-$, Lemma 8.3 implies that $A_k f$ is supported in $\overline{C}_+$, while Proposition 5.2 (or, indeed, elliptic regularity) implies it is smooth away from the radial set Λ^+. We may thus apply a theorem of Haber–Vasy [7, Theorem 6.3] to conclude that in fact $A_k f \in I^{(-\infty)}(\Lambda^+) = I^{(-\infty)}(N^* S_+)$. In particular, then, Corollary 6.9 implies that there are smooth functions $\phi_{\pm,k}$ and ψ so that

$$A_k f = v_{+\imath 0}^{-\imath\sigma_0} \phi_{+,k} + v_{-\imath 0}^{-\imath\sigma_0} \phi_{-,k} + \psi.$$

By Lemma 8.3, $A_k f$ is supported in $\overline{C}_+$, so ψ vanishes to infinite order at S_+ and may be absorbed into the other terms, i.e.,

$$A_k f = v_{+\imath 0}^{-\imath\sigma_0} \phi_{+,k} + v_{-\imath 0}^{-\imath\sigma_0} \phi_{-,k}.$$

Now suppose that the statement is true for $0 \le \ell' \le \ell - 1$. As $P_0 A_{k-\ell} f = -P_1 A_{k-\ell+1} f - P_2 A_{k-\ell+2} f$, we have

$$\begin{aligned} P_0 A_{k-\ell} f = \sum_{\pm} \sum_{j=0}^{\ell-1} v_{\pm \imath 0}^{-\imath\sigma_0 - 1} (\log(v \pm \imath 0))^j \frac{(-\imath)^j}{j!} \left[\sigma_0 \phi_{\pm, k-(\ell-1-j)} + \phi_{\pm, k-(\ell-2-j)}\right] \\ + O(v^{-\imath\sigma_0} (\log v)^\ell), \end{aligned}$$

where the $O(v^{-\imath\sigma_0}(\log v)^\ell)$ in fact has an asymptotic expansion of a similar form. Observe that the right hand side is an element of $I^{(-\infty)}(N^*S_+)$, so again Haber–Vasy [7] implies that $A_{k-\ell}f \in I^{(-\infty)}(N^*S_+)$. Proposition 6.10 then implies that $A_{k-\ell}f$ has a similar expansion, say (suppressing dependence of coefficients on k, ℓ)

$$A_{k-\ell}f = \sum_{\pm}\sum_{j=0}^{\ell} v_{\pm\imath 0}^{-\imath\sigma_0}(\log(v \pm \imath 0))^j a_{\pm,j}$$

To determine the leading coefficients in the expansion, we calculate

$$\begin{aligned} P_0 A_{k-\ell}f = -\sum_{\pm}\sum_{j=0}^{\ell-1} v_{\pm\imath 0}^{-\imath\sigma_0-1}(\log(v\pm\imath 0))^j (j+1)\left[(-\imath\sigma_0)a_{\pm,j+1} + (j+2)a_{\pm,j+2}\right] \\ + O(v^{-\imath\sigma_0}(\log v)^\ell), \end{aligned}$$

where again the last term has an expansion. We now simply equate coefficients, starting with the largest one. If $j = \ell - 1$, we must have

$$\imath\sigma_0 \ell a_{\pm,\ell} = \frac{(-\imath)^{\ell-1}}{(\ell-1)!}\sigma_0\phi_{\pm,k},$$

i.e.,

$$a_{\pm,\ell} = \frac{(-\imath)^\ell}{\ell!}\phi_{\pm,k}.$$

Now for $j < \ell - 1$, we have

$$\begin{aligned} \imath\sigma_0(j+1)a_{\pm,j+1} - (j+2)(j+1)\frac{(-\imath)^{j+2}}{(j+2)!}\phi_{\pm,k-(\ell-2-j)} \\ = \frac{(-\imath)^j}{j!}\left(\sigma_0\phi_{\pm,k-(\ell-1-j)} + \phi_{\pm,k-(\ell-2-j)}\right), \end{aligned}$$

i.e.,

$$a_{\pm,j+1} = \frac{(-\imath)^{j+1}}{(j+1)!}\phi_{\pm,k-(\ell-1-j)}.$$

This determines $a_{\pm,2}, \dots, a_{\pm,\ell}$, while $a_{\pm,1}$ are given by Corollary 6.9 and are denoted $\phi_{\pm,k-\ell}$.

We now consider when $-\imath\sigma_0$ is an integer, in which case additional logarithmic terms appear in our application of Proposition 6.10. If $-\imath\sigma_0 < 0$, these additional logarithms are not in the leading order terms and so the results above still hold. For $-\imath\sigma_0 \geq 0$ an integer, however, we must be a bit more careful and rely on Lemma 8.3 as follows.

Let us assume for now that $-\imath\sigma_0 \neq 0$. Indeed, we again proceed inductively. Consider first $A_k f$. The same arguments as above imply that $A_k f$ has an expansion of the form

$$A_k f = v_{+\imath 0}^{-\imath\sigma_0}\phi_+ + v_{-\imath 0}^{-\imath\sigma_0}\phi_- + \phi.$$

As $A_k f$ is supported in $\overline{C}_+$ and $v_{\pm\imath 0}^{-\imath\sigma_0} = v^{-\imath\sigma_0}\log(v \pm \imath 0)$ in this case, given ϕ_+, the behavior of ϕ_- and ϕ at S_+ is determined by the support condition.

Indeed, we must have $\phi_- = -\phi_+$ and $\phi = -2\pi\imath v^{-\imath\sigma_0}\phi_+$. In other words, there is a smooth function ϕ_k so that

$$A_k f = v^{-\imath\sigma_0} H(v)\phi_k$$

with H denoting the Heaviside function.

Now suppose that the statement holds for $A_{k-\ell'}f$ for $0 \le \ell' \le \ell - 1$. Then $P_0 A_{k-\ell} f$ must satisfy

$$P_0 A_{k-\ell} f = -\sum_{j=0}^{\ell-1} v^{-\imath\sigma_0 - 1} H(v)(\log|v|)^j \frac{(-\imath)^j}{j!}\left(\sigma_0\phi_{k-(\ell-1-j)} + \phi_{k-(\ell-2-j)}\right) + O(v^{-\imath\sigma_0}(\log|v|)^\ell),$$

where again the $O(v^{-\imath\sigma_0}(\log|v|)^\ell)$ term has an expansion of a similar form. The theorem of Haber–Vasy and Lemma 8.3 then imply that $A_{k-\ell}f$ has an expansion

$$A_{k-\ell} f = \sum_{j=0}^{\ell} H(v) v^{-\imath\sigma_0}(\log|v|)^j a_j.$$

Applying P_0 and equating coefficients finishes the proof in this case.

Finally, if $-\imath\sigma_0 = 0$, the same argument as in the case of $-\imath\sigma_0 > 0$ still works, but differentiating the $j = 0$ term yields a $\delta(v)$ term. This term is no problem, as we still simply solve for its coefficient. This process yields an identical result. □

9. An asymptotic expansion

In this section we detail the iteration scheme used to obtain a preliminary asymptotic expansion for (smooth) solutions w of $\Box_g w \in \dot{\mathcal{C}}^\infty(M)$ that vanish in a neighborhood of $\overline{C_-}$.

We start with a tempered solution w of $\Box_g w = f \in \dot{\mathcal{C}}^\infty(M)$ vanishing in a neighborhood of $\overline{C_-}$. We immediately replace w by χw, where χ is a cut-off function supported near the boundary and vanishing identically near $\overline{C_-}$. By choosing χ appropriately, we guarantee that $\chi w = w$ near the boundary and that the replacement is supported in a collar neighborhood of the boundary and still solves an equation of the same form.

Because w is tempered, we know that $w \in \rho^\gamma H_b^{s_0}(M) = H_b^{s_0,\gamma}(M)$ for some s_0 and γ. *We decrease s_0 so that $\gamma + s_0 < 1/2$.* Corollary 4.6 and our non-trapping hypothesis then imply that w has module regularity at $\Lambda^+ = N^* S_+$ relative to $H_b^{s_0,\gamma}(M)$.

As before, write

$$L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2} \in \mathrm{Diff}_b^2(M),$$

so that setting

$$u = \rho^{-(n-2)/2} w$$

we have

$$Lu = g \in \dot{\mathcal{C}}^\infty(M),$$

with u, g vanishing near $\overline{C_-}$. Now let $N(L)$ denote the normal operator of L and set $E = L - N(L)$; E thus measures the failure of L to be dilation-invariant in ρ. Thus,

$$E \in \rho \operatorname{Diff}_b^2(M).$$

By the form of G^{-1} given by equation (3.3)(3.3), we note that the coefficient of D_v^2 in E is of the form $O(\rho^2)+O(\rho v)$ and hence $Ew \in \rho^{\gamma+1} H_b^{s_0-1}(M)$.

At this juncture, we discuss the mapping properties of the Mellin conjugate of E. To begin, we let R_σ be the family of operators satisfying

$$\mathcal{M} \circ E = R_\sigma \circ \mathcal{M};$$

thus R_σ is an operator on meromorphic families in σ in which ρD_ρ is replaced by σ and multiplication by ρ translates the imaginary part. Since, as remarked above, the coefficient of D_v^2 in E is of the form $O(\rho^2) + O(\rho v)$, i.e., is a sum of terms having better decay either in the sense of v or ρ than the rest of the operator, we have the following result on the mapping properties of R_σ (cf. Section 2.3 above for notation):

Lemma 9.1. *For each ν, k, ℓ, s the operator family R_σ enjoys the following mapping properties:*

(1) *R_σ enlarges the region of holomorphy at the cost of regularity at Λ^+:*

$$\begin{aligned} R_\sigma :& \mathcal{H}(\mathbb{C}_\nu) \cap \langle\sigma\rangle^{-k} L^\infty L^2(\mathbb{R}; I^{(s)}(\Lambda^+)) \\ &\to \mathcal{H}(\mathbb{C}_{\nu+1}) \cap \langle\sigma\rangle^{-k+2} L^\infty L^2(\mathbb{R}; I^{(s-1)}(\Lambda^+)) \\ &\quad + \mathcal{H}(\mathbb{C}_{\nu+2}) \cap \langle\sigma\rangle^{-k+2} L^\infty L^2(\mathbb{R}; I^{(s-2)}(\Lambda^+)) \end{aligned} \tag{9.1}$$

(2) *If f_σ vanishes near $\overline{C_-}$ for $\operatorname{Im}\sigma \geq -\nu$, then $R_\sigma f_\sigma$ also vanishes near $\overline{C_-}$ for $\operatorname{Im}\sigma \geq -\nu - 1$.*

(3) *If*

$$\phi \in \mathcal{H}(\mathbb{C}_\nu) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))$$

then

$$\begin{aligned} &R_\sigma \Big((\sigma - \sigma_0)^{-k-1} v_{\pm\imath 0}^{-\imath\sigma_0} (\log(v \pm \imath 0))^k \phi \Big) \\ &\quad = (\sigma - (\sigma_0 - \imath))^{-k-1} v_{\pm\imath 0}^{-\imath\sigma_0 - 1} \sum_{j=0}^k (\log(v \pm \imath 0))^j \tilde{\phi}_{j,1} \\ &\qquad + (\sigma - (\sigma_0 - 2\imath))^{-k-1} v_{\pm\imath 0}^{-\imath\sigma_0 - 2} \sum_{j=0}^k (\log(v \pm \imath 0))^j \tilde{\phi}_{j,2}, \end{aligned}$$

where $\tilde{\phi}_{j,i}$ enjoy the same same properties and are holomorphic on $\operatorname{Im}\sigma \geq -\nu - 1$.

(4) *If*

$$\phi \in \mathcal{H}(\mathbb{C}_\nu) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))$$

then

$$R_\sigma\left(v_{\pm\imath 0}^{-\imath\sigma}\phi\right) \in v_{\pm\imath 0}^{-\imath\sigma-1}\mathcal{H}(\mathbb{C}_{\nu+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+)) \\ + v_{\pm\imath 0}^{\imath\sigma-2}\mathcal{H}(\mathbb{C}_{\nu+2}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))$$

The proof is a simple application of the properties of the Mellin transform discussed in Section 2.3. Note that in the first term the Sobolev order has decreased by 1 arising from the action of the $O(\rho)vD_v^2$ term in $\Box_g$ (rather than by 2 as would be the effect of $O(\rho)D_v^2$ term). In the second term, we see the action of $O(\rho^2)D_v^2$ terms, which give a family holomorphic in an even larger strip, at the cost of further worsening of Sobolev regularity. We also lose at high frequency owing to the $(\rho D_\rho)^2$ error term in the rescaled $\Box_g$, which Mellin transforms to an $O(\sigma^2)$. (We further note that a sharper result holds, keeping precise accounts of tradeoffs between σ powers and Sobolev orders in the boundary, but this refinement will not be needed for our argument.)

We now apply the Mellin transform to the identity $Lu = g$, splitting up $L = N(L) + E$ to obtain

$$P_\sigma \tilde{u}_\sigma = \tilde{g}_\sigma - R_\sigma \tilde{u}_\sigma, \tag{9.2}$$

where, as above, $P_\sigma = \widehat{N}(L)$. As $g \in \dot{\mathcal{C}}^\infty(M)$, we have

$$\tilde{g}_\sigma \in \mathcal{H}(\mathbb{C}_C) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s)}(\Lambda^+)) \text{ for all } C, s.$$

Because $w = \rho^{(n-2)/2}u \in H_b^{s_0,\gamma}(M)$, Lemma 2.3 implies that

$$\tilde{u}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0}) \cap \langle\sigma\rangle^{\max(0,-s_0)} L^\infty L^2(\mathbb{R}; I^{(s_0)}(\Lambda^+)), \tag{9.3}$$

where

$$\varsigma_0 = \gamma - (n-2)/2.$$

(Recall that we have already reduced s_0 so that $s_0 + \gamma < 1/2$ and Corollary 4.6 applies.)

As u vanishes near $\overline{C_-}$, $\tilde{u}_\sigma$ also vanishes there. Thus, in the notation of Section 5, the right-hand-side of equation (9.2) is in $\mathcal{Y}^{s_{\text{ftr}}-1}$ and $\tilde{u}_\sigma \in \mathcal{X}^{s_{\text{ftr}}}$. Here we may choose s_{ftr} to be constant on the singular support of $\tilde{u}_\sigma$ as $\tilde{u}_\sigma$ is smooth near $\overline{C_-}$; in fact, we may take it to be constant except in a small neighborhood of $\overline{C_-}$ where $\tilde{u}_\sigma$ vanishes. We take s_{ftr} equal to s_0 outside this neighborhood.

Because w is conormal with respect to $N^*S_+ = \{\rho = v = 0\}$, Lemma 2.3 implies that

$$\tilde{u}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(-\infty)}(\Lambda^+)).$$

Thus, by interpolation with equation (9.3),

$$\tilde{u}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s_0-0)}(\Lambda^+)).$$

By Lemma 9.1, then,

$$R_\sigma \tilde{u}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s_0-1-0)}(\Lambda^+)) \\ + \mathcal{H}(\mathbb{C}_{\varsigma_0+2}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s_0-2-0)}(\Lambda^+)), \tag{9.4}$$

and hence $P_\sigma \tilde{u}_\sigma$ lies in this space as well.

Because $P_\sigma \tilde{u}_\sigma$ is now known to be holomorphic in a larger half-plane, we can now invert P_σ to obtain meromorphy of $\tilde{u}_\sigma$ on this larger space: by Propositions 5.2 and 5.3, P_σ is Fredholm as a map

$$\mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1}$$

and P_σ^{-1} has finitely many poles in any horizontal strip $\operatorname{Im} z \in [a, b]$, and satisfies polynomial growth estimates as $|\operatorname{Re} z| \to \infty$.

Here we recall from Section 5 that given any ς', in order for P_σ to be Fredholm for $\sigma \in \mathbb{C}_{\varsigma'}$, the (constant) value $s(S_+)$ assumed by the variable Sobolev order s_{ftr} near S_+ must satisfy $s(S_+) < 1/2 - \varsigma'$. In other words, as one enlarges the domain of meromorphy for $\tilde{u}_\sigma$, one must relax control of the derivatives.

We then conclude that $\tilde{u}_\sigma$ is the sum of two terms arising from the two terms on the right side of equation (9.4). For any $\delta > 0$, the first term is meromorphic in $\mathbb{C}_{\varsigma_0+1}$ with values in

$$\langle\sigma\rangle^{-\infty} L^2(\mathbb{R}, H^{\min(s_0, 1/2-(\varsigma_0+1)-\delta)}) \tag{9.5}$$

with finitely many poles in this strip arising from the poles of P_σ^{-1}. (Note the improvement in the Sobolev orders: by applying P_σ^{-1} we win back the derivative we lost from applying R_σ, but only up to the threshold value.) Likewise, the second term is meromorphic in $\mathbb{C}_{\varsigma_0+2}$ with values in

$$\langle\sigma\rangle^{-\infty} L^2(\mathbb{R}, H^{\min(s_0-1, 1/2-(\varsigma_0+2)-\delta)}), \tag{9.6}$$

again with finitely many poles in the strip. (Here and below we are ignoring the distinction between $\mathcal{X}^{s_{\mathrm{ftr}}}$ and H^s as $\tilde{u}_\sigma$ is trivial by hypothesis on the set where the regularity in the variable-order Sobolev space differs from H^s.)

We now refine the description of the terms in equations (9.5) and (9.6) in two steps using what we know about the conormal structure of solutions to inhomogeneous equations involving P_σ. To begin, since P_σ maps the terms in question to conormal spaces, they must in fact lie in the conormal spaces

$$\langle\sigma\rangle^{-\infty} L^2(\mathbb{R}, I^{(\min(s_0-0, 1/2-(\varsigma_0+1)-0))}), \text{ resp.,} \\ \langle\sigma\rangle^{-\infty} L^2(\mathbb{R}, I^{(\min(s_0-1-0, 1/2-(\varsigma_0+2)-0))}).$$

This improvement follows by Proposition 4.1 (propagation of singularities away from the radial points) and the first case of Theorem 6.3 of [7].

Remark 9.2. Here Theorem 6.3 of [7] is applied pointwise in σ. The result there is not stated in terms of bounds (just as a membership in the claimed set), but, just as in the case of Proposition 4.4 here, estimates can be recovered from the statement of Theorem 6.3 by the closed graph theorem or alternatively recovered from examination of the proof (which proceeds via such estimates).

Finally, since we have now established conormality, we may use Corollary 6.9 to determine the detailed structure of the conormal singularities. We find that for any $\delta' > 0$, the two terms in question lie in

$$\begin{aligned}&\langle\sigma\rangle^{-\infty}L^2(\mathbb{R}, I^{(s_0-\delta')}) + v_{+\imath 0}^{-\imath\sigma}\langle\sigma\rangle^{-\infty}L^2(\mathbb{R}, \mathcal{C}^\infty) + v_{-\imath 0}^{-\imath\sigma}\langle\sigma\rangle^{-\infty}L^2(\mathbb{R}, \mathcal{C}^\infty), \text{ resp.,}\\ &\langle\sigma\rangle^{-\infty}L^2(\mathbb{R}, I^{(s_0-1-\delta')}) + v_{+\imath 0}^{-\imath\sigma}\langle\sigma\rangle^{-\infty}L^2(\mathbb{R}, \mathcal{C}^\infty) + v_{-\imath 0}^{-\imath\sigma}\langle\sigma\rangle^{-\infty}L^2(\mathbb{R}, \mathcal{C}^\infty).\end{aligned}$$

Again, Corollary 6.9 does not state the desired estimates explicitly, but these follow either directly from the proofs or by an application of the closed graph theorem.

Consequently, we have now established

$$\begin{aligned}\tilde{u}_\sigma \in\ &\mathcal{H}(\mathbb{C}_{\varsigma_0+1})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(s_0-0)}(\Lambda^+)) + \mathcal{H}(\mathbb{C}_{\varsigma_0+2})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(s_0-1-0)}(\Lambda^+))\\ &+ v_{+\imath 0}^{-\imath\sigma}\mathcal{H}(\mathbb{C}_{\varsigma_0+1})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+)) + v_{-\imath 0}^{-\imath\sigma}\mathcal{H}(\mathbb{C}_{\varsigma_0+1})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\\ &+ \sum_{\operatorname{Im}\sigma_j > -(\varsigma_0+2)} (\sigma-\sigma_j)^{-m_j} a_j,\end{aligned}\tag{9.7}$$

where

$$a_j \in \mathcal{H}(\mathbb{C}_{\varsigma_0+1})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\operatorname{Im}\sigma_j+1/2-0)}(\Lambda^+)).$$

(Note that we have dropped terms $v_{\pm\imath 0}^{-\imath\sigma}\mathcal{H}(\mathbb{C}_{\varsigma_0+2})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))$ by absorbing them in the terms of the same form, holomorphic in the smaller half-space; also recall that $I^{(\infty)} = \mathcal{C}^\infty$.) As the a_j are given by the polar parts of P_σ^{-1} at value σ_j lying in a strip in $\mathbb{C}$, the coefficients of the polar part of the sum are described by Proposition 8.1:

$$\begin{aligned}a_j = &\sum_{\kappa=0}^{m_j-1}(\sigma-\sigma_j)^\kappa \sum_{\ell=0}^{\kappa}\Big(v_{+\imath 0}^{-\imath\sigma_j}(\log(v+\imath 0))^\ell a_{j\kappa\ell+} + v_{-\imath 0}^{-\imath\sigma_j}(\log(v-\imath 0))^\ell a_{j\kappa\ell-}\Big)\\ &+ O((\sigma-\sigma_j)^{m_j}),\end{aligned}\tag{9.8}$$

with

$$a_{j\kappa\ell\pm} = \frac{(-\imath)^\ell}{\ell!}\phi_{j,m_j-(\kappa-\ell),\pm}$$

and $\phi_{j,\alpha,\pm} \in \mathcal{C}^\infty(X)$. (If $-\imath\sigma_j$ is a positive integer, then the term is of the form in Proposition 8.1.) We may further arrange that $a_{j\kappa\ell\pm}$ are holomorphic and rapidly decaying in strips. Also, $\operatorname{supp} a_j \cap \overline{C_-} = \emptyset$ by Proposition 7.5.

Finally, we remark that

$$
\begin{aligned}
(9.9) \qquad & v_{+\imath 0}^{-\imath\sigma}\left(\mathcal{H}(\mathbb{C}_{\varsigma_0+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\right) \\
& + v_{-\imath 0}^{-\imath\sigma}\left(\mathcal{H}(\mathbb{C}_{\varsigma_0+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\right) \\
& \subset \mathcal{H}(\mathbb{C}_{\varsigma_0+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(1/2+\operatorname{Im}\sigma-0)}(\Lambda^+)) \\
& \subset \mathcal{H}(\mathbb{C}_{\varsigma_0+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(1/2-(\varsigma_0+1)-0)}(\Lambda^+)),
\end{aligned}
$$

so that we have established

$$
\begin{aligned}
(9.10) \qquad \tilde{u}_\sigma \in & \mathcal{H}(\mathbb{C}_{\varsigma_0+1}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s_0-0,1/2-(\varsigma_0+1)-0))}(\Lambda^+)) \\
& + \mathcal{H}(\mathbb{C}_{\varsigma_0+2}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s_0-1-0)}(\Lambda^+)) \\
& + \sum_{\operatorname{Im}\sigma_j > -(\varsigma_0+2)} (\sigma-\sigma_j)^{-m_j} a_j
\end{aligned}
$$

with a_j given by equation (9.8). (Through careful accounting it is possible to keep track of the $v_{\pm\imath 0}^{-\imath\sigma}$ terms. Indeed, these terms contribute both to the radiation field and to other terms in the expansion after the blow-up, but become rapidly decaying along the front face after taking the Mellin transform.)

Now we iterate this argument: by equation (9.9), Lemma 9.1, and Lemma 6.4,

$$
\begin{aligned}
R_\sigma \tilde{u}_\sigma \in\ & \mathcal{H}(\mathbb{C}_{\varsigma_0+2}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s_0-1-0,1/2-(\varsigma_0+2)-0))}(\Lambda^+)) \\
& + \mathcal{H}(\mathbb{C}_{\varsigma_0+3}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s_0-2-0,1/2-(\varsigma_0+3)-0))}(\Lambda^+)) \\
& + \mathcal{H}(\mathbb{C}_{\varsigma_0+4}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s_0-3-0)}(\Lambda^+)) \\
& + \sum_{\operatorname{Im}\sigma_j > -(\varsigma_0+2)} (\sigma-(\sigma_j-\imath))^{-m_j} b_j' \\
& + \sum_{\operatorname{Im}\sigma_j > -(\varsigma_0+2)} (\sigma-(\sigma_j-2\imath))^{-m_j} b_j''
\end{aligned}
$$

where the polar parts lie in the spaces

$$
\begin{aligned}
b_j' &\in \mathcal{H}(\mathbb{C}_{\varsigma_0+2}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\operatorname{Im}\sigma_j-1/2-0)}(\Lambda^+)) \\
b_j'' &\in \mathcal{H}(\mathbb{C}_{\varsigma_0+2}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\operatorname{Im}\sigma_j-3/2-0)}(\Lambda^+)),
\end{aligned}
$$

and have the forms

$$
\begin{aligned}
b_j' = \sum_{\kappa=0}^{m_j}\sum_{\ell=0}^{\kappa} (\sigma-(\sigma_j-\imath))^\kappa \Big(& v_{+\imath 0}^{-\imath\sigma_j-1}(\log(v+\imath 0))^\ell b'_{j\kappa\ell+} \\
& + v_{-\imath 0}^{-\imath\sigma_j-1}(\log(v-\imath 0))^\ell b'_{j\kappa\ell-}\Big) \\
& + O((\sigma-(\sigma_j-\imath))^{m_j})
\end{aligned}
$$

and

$$\begin{aligned}b''_j = \sum_{\kappa=0}^{m_j-1}\sum_{\ell=0}^{\kappa}(\sigma-(\sigma_j-2\imath))^\kappa\Big(v_{+\imath 0}^{-\imath\sigma_j-2}(\log(v+\imath 0))^\ell b''_{j\kappa\ell+}\\ + v_{-\imath 0}^{-\imath\sigma_j-2}(\log(v-\imath 0))^\ell b''_{j\kappa\ell-}\Big)\\ + O((\sigma-(\sigma_j-2\imath))^{m_j}).\end{aligned}$$

Moreover, the $b'_{j\kappa\ell\pm}$ and $b''_{j\kappa\ell\pm}$ are smooth and supported away from $\overline{C_-}$.

Again inverting P_σ and employing Proposition 8.1 and Corollary 6.9 yields

$$\begin{aligned}\tilde{u}_\sigma \in\ & \mathcal{H}(\mathbb{C}_{\varsigma_0+2})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\min(s_0-0,1/2-(\varsigma_0+1)-0))}(\Lambda^+))\\ &+\mathcal{H}(\mathbb{C}_{\varsigma_0+3})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\min(s_0-1-0,1/2-(\varsigma_0+2)-0))}(\Lambda^+))\\ &+\mathcal{H}(\mathbb{C}_{\varsigma_0+4})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(s_0-2)}(\Lambda^+))\\ &+v_{+\imath 0}^{-\imath\sigma}\mathcal{H}(\mathbb{C}_{\varsigma_0+2})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\\ &+v_{-\imath 0}^{-\imath\sigma}\mathcal{H}(\mathbb{C}_{\varsigma_0+2})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\\ &+\sum_{\operatorname{Im}\sigma_j>-(\varsigma_0+4)}(\sigma-\sigma_j)^{-m_j}a_j\\ &+\sum_{\operatorname{Im}\sigma_j>-(\varsigma_0+2)}(\sigma-(\sigma_j-\imath))^{-\tilde{m}_j}\tilde{a}_{j1}\\ &+\sum_{\operatorname{Im}\sigma_j>-(\varsigma_0+2)}(\sigma-(\sigma_j-2\imath))^{-\tilde{m}_j}\tilde{a}_{j2},\end{aligned}$$

where again the coefficients of the poles have expansions as in equation (9.8) (although the expansion for $\tilde{a}_{j2}$ begins at $v^{-\imath\sigma_j-1}$) and support away from $\overline{C_-}$. Here we may have $\tilde{m}_j > m_J$ if there are integer coincidences among the poles of P_σ^{-1}, i.e., if σ_j and $\sigma_j-\imath$ or $\sigma_j-2\imath$ are both poles. As before we may use the inclusions of $v_\pm^{-\imath\sigma}$ in conormal spaces to rewrite this (for purposes of the next step of our iteration) as

$$\begin{aligned}\tilde{u}_\sigma \in\ & \mathcal{H}(\mathbb{C}_{\varsigma_0+2})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\min(s_0-0,1/2-(\varsigma_0+2)-0))}(\Lambda^+))\\ &+\mathcal{H}(\mathbb{C}_{\varsigma_0+3})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(\min(s_0-1-0,1/2-(\varsigma_0+2)-0))}(\Lambda^+))\\ &+\mathcal{H}(\mathbb{C}_{\varsigma_0+4})\cap\langle\sigma\rangle^{-\infty}L^\infty L^2(\mathbb{R}; I^{(s_0-2)}(\Lambda^+))\\ &+\sum_{\operatorname{Im}\sigma_j>-(\varsigma_0+4)}(\sigma-\sigma_j)^{-m_j}a_j\\ &+\sum_{\operatorname{Im}\sigma_j>-(\varsigma_0+2)}(\sigma-(\sigma_j-\imath))^{-\tilde{m}_j}\tilde{a}_{j1}\\ &+\sum_{\operatorname{Im}\sigma_j>-(\varsigma_0+2)}(\sigma-(\sigma_j-2\imath))^{-\tilde{m}_j}\tilde{a}_{j2},\end{aligned}$$

Iterating in this fashion, we obtain after N such steps (and slightly weakening our Sobolev exponents over the foregoing for the sake of simplicity):

(9.11)

$$\begin{aligned}
\tilde{u}_\sigma \in\ & \mathcal{H}(\mathbb{C}_{\varsigma_0+N}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s_0-0,1/2-(\varsigma_0+N-1)-0))}(\Lambda^+)) \\
& + \cdots + \mathcal{H}(\mathbb{C}_{\varsigma_0+2N}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s_0-N-0,1/2-(\varsigma_0+2N-1)-0))}(\Lambda^+)) \\
& + v_{+\imath 0}^{-\imath\sigma} \Big(\mathcal{H}(\mathbb{C}_{\varsigma_0+N}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\Big) \\
& + v_{-\imath 0}^{-\imath\sigma} \Big(\mathcal{H}(\mathbb{C}_{\varsigma_0+N}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\Big) \\
& + \sum_{\operatorname{Im}\sigma_j > -(\varsigma_0+2N)} (\sigma-\sigma_j)^{-m_j} a_j \\
& + \sum_{\ell=1}^{N} \sum_{\operatorname{Im}\sigma_j > -(\varsigma_0+2N)} (\sigma-(\sigma_j-\imath\ell))^{-\tilde{m}_j} \tilde{a}_{j\ell}.
\end{aligned}$$

Here again $\tilde{m}_j$ may exceed m_j in the case of integer coincidences among poles of P_σ^{-1}. Moreover, while a_j is described by equation (9.8), we also have

$$\begin{aligned}
\tilde{a}_{j\ell} = \sum_{\kappa=0}^{\tilde{m}_j-1} (\sigma-(\sigma_j-\imath\ell))^\kappa \sum_{k=0}^{\ell-1} \sum_{p=0}^{P(j,\ell,\kappa,k)} & \Big(v_{+\imath 0}^{-\imath\sigma_j-k} (\log(v+\imath 0))^p a_{j\ell\kappa k p+} \\
& + v_{-\imath 0}^{-\imath\sigma_j-k} (\log(v-\imath 0))^p a_{j\ell\kappa k p-}\Big) \\
& + O((\sigma-(\sigma_j-\imath\ell))^{\tilde{m}_j}).
\end{aligned}$$

As the inverse Mellin transform of $(\sigma-\sigma_0)^{-m}$ is

$$\frac{\imath^m}{(m-1)!} \rho^{\imath\sigma_0} (\log\rho)^{m-1},$$

under inverse Mellin transform with a contour deformation to the line $\mathbb{R} - \imath(\varsigma_0+N)$, equation (9.8) shows that the poles in the sum $\sum(\sigma-\sigma_j)^{-m_j} a_j$ yield the residues

$$\sum_j \sum_{\kappa=0}^{m_J} \sum_{\ell=0}^{\kappa} \frac{\imath^{m_j-\kappa}(-\imath)^\ell}{(m_j-\kappa-1)!\ell!} \rho^{\imath\sigma_j} v^{-\imath\sigma_j} (\log\rho)^{m-\kappa-1} (\log v)^\ell \phi_{j,m-(\kappa-\ell)},$$

i.e., the main terms in our asymptotic expansion. (Strictly speaking, this is an expansion in powers of $v_{\pm\imath 0}$ rather than in powers of v; however, we are primarily concerned with asymptotics in the regime $v/\rho \gg 0$. Likewise, we have chosen to write $\phi_\bullet = \phi_{-,\bullet} + \phi_{+,\bullet}$, but could carry along both terms separately if desired.) Rearranging this sum yields

$$\sum_j \sum_{k=0}^{m_j-1} \frac{\imath^{-k+1}}{k!} \rho^{\imath\sigma_j} v^{-\imath\sigma_j} (\log v - \log\rho)^k \phi_{j,k+1},$$

i.e., the only logarithmic terms in this sum are powers of $\log v - \log \rho$.

The terms in the last sum in equation (9.11) become

$$\sum_j \sum_{\ell=1}^{N} \sum_{\kappa+\alpha<\tilde{m}_{j\ell}} a_{j\ell\kappa\alpha} \rho^{\imath\sigma_j+\ell} v^{-\imath\sigma_j-\ell+1} |\log\rho|^\kappa |\log v|^\alpha.$$

The other, "remainder" terms in equation (9.11) lie in

$$\begin{aligned}
&\sum_{j=0}^{N} \mathcal{H}(\mathbb{C}_{\varsigma_0+N+j}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\min(s_0-j-0,1/2-(\varsigma_0+N+j-1)-0))}(\Lambda^+)) \\
&\quad + v_{+\imath 0}^{-\imath\sigma} \left(\mathcal{H}(\mathbb{C}_{\varsigma_0+N}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\right) \\
&\quad + v_{-\imath 0}^{-\imath\sigma} \left(\mathcal{H}(\mathbb{C}_{\varsigma_0+N}) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(\infty)}(\Lambda^+))\right).
\end{aligned} \tag{9.12}$$

Thus by Lemma 2.3, after applying the inverse Mellin transform the terms in the first sum of equation (9.12) become

$$O\left(\rho^{\varsigma_0+N+j-0} v^{\min(s_0,1/2-(\varsigma_0+N-1))-1/2-j-0}\right).$$

In particular, the power of v appearing is at least one larger than the power of ρ^{-1}. The last two terms of equation (9.12) Mellin transform to Schwartz functions of v/ρ.

Thus, returning to the solution w of $\Box_g w = f \in \dot{\mathcal{C}}^\infty(M)$, and taking N large to simplify the remainder, we find that near $\{\rho = v = 0\}$, w has an asymptotic expansion of the form

$$\begin{aligned}
w =& \rho^{(n-2)/2} \sum_j \sum_{\kappa\le m_j} \rho^{\imath\sigma_j} v^{-\imath\sigma_j} (\log v - \log\rho)^\kappa a_{j\kappa} \\
&+ \rho^{(n-2)/2} \sum_j \sum_{\ell=1}^{N} \sum_{\kappa+\alpha<\tilde{m}_{j\ell}} \tilde{a}_{j\ell\kappa\alpha} \rho^{\imath\sigma_j+\ell} v^{-\imath\sigma_j-\ell+1} |\log\rho|^\kappa |\log v|^\alpha + w'
\end{aligned} \tag{9.13}$$

with

$$w' \in \sum_{j=0}^{N} \rho^{(n-2)/2+\varsigma_0+N+j-0} v^{-\varsigma_0-N-j+1-0} L^\infty,$$

where σ_j are the poles of the meromorphic inverse P_σ^{-1}, and the coefficients are the corresponding resonance states. Here $v^{-\imath\sigma_j} a_{j\kappa}$ (and its counterpart in the second sum) is understood to mean a sum of the two $v_{\pm\imath 0}^{-\imath\sigma_j}$ terms (which we write out fully below).

Remark 9.3. Note that the only log terms in the $\rho^{\imath\sigma_j} v^{-\imath\sigma_j}$ term occur as powers of $\log v - \log\rho$. Because $\log v - \log\rho = \log s$ in the radiation field blow-up, this will imply that $\rho^{-(n-2)/2} w$ has a restriction to the front face of the radiation field blowup.

10. The asymptotics of the radiation field

We now lift the expansion (9.13) to the radiation field blowup. We thus introduce the "radiation field" coordinates $\rho, y, s = v/\rho$; note that these constitute a coordinate system on the blown up space described in Section 3.8, and note that ∂_s is well-defined as a vector field on the fibers of ff. In these coordinates, then, homogeneity yields the expansion

$$\sum_j \sum_{\alpha+\kappa<m_j} (\log s)^\alpha \big(a'_{j\kappa\alpha,+} s_{+\imath 0}^{-\imath\sigma_j} + a'_{j\kappa\alpha,-} s_{-\imath 0}^{-\imath\sigma_j}\big)$$
$$+ \sum_j \sum_{\ell=1}^{N} \sum_{\kappa+\alpha<\tilde{m}_{j\ell}} \rho^\ell |\log\rho|^\kappa \left(\log\rho + \log s\right)^\alpha \left(a_{j\ell\kappa\alpha,+} s_{+\imath 0}^{-\imath\sigma_j} + a_{j\ell\kappa\alpha,-} s_{-\imath 0}^{-\imath\sigma_j}\right)$$
$$+ u'$$

for $u = \rho^{-\frac{n-2}{2}} w$. Consequently, restricting terms of the expansion to $\rho = 0$ yields an expansion

$$\sum_j \big(a_{jk,+} s_{+\imath 0}^{-\imath\sigma_j} + a_{jk,-} s_{-\imath 0}^{-\imath\sigma_j} + \tilde{\tilde{u}}_{jk}\big)$$

with a remainder term u'. Notice that the presence of $\log\rho$ factors in the ρ^0 ($\ell = 0$) terms would prevent the restriction of u to the front face of the blow-up, but in Section 9 we showed that in fact (at top-order) those terms possessing a logarithmic factor cancel. We can now define the radiation field as in Section 3.8:

Definition 10.1. If w is a solution of $\Box_g w = f$, $f \in \dot{\mathcal{C}}^\infty(M)$, w vanishing near $\overline{C_-}$, we define the (forward) *radiation field* of w by

$$\mathcal{R}_+[w](s, y) = \partial_s u(0, s, y), \qquad u = \rho^{-\frac{n-2}{2}} w.$$

The rest of Theorem 1.1 follows immediately. As identified in Section 7, the exponents σ_j are the poles of $\mathcal{R}_{C_+}(\sigma)$, i.e., the resonances of the asymptotically hyperbolic problem on the cap C_+, while the terms supported at S_+ do not contribute to the expansion as $s \to \infty$.

Remark 10.2. While it may seem that the coefficients in the expansion are singular at $s = 0$, this is an artifact of the basis chosen. The b-regularity established in Section 4 (see, in particular, Remark 4.7) implies that the solution is conormal to the front face of the radiation field blow-up and hence the coefficients may be taken to be smooth.

10.1. **Asymptotically Minkowski space.** We now consider the special case of asymptotically Minkowski space (i.e., "normally very short range" perturbations of Minkowski space). Here we are assuming that the metric takes the form (3.1) modulo

$$O(\rho^2)\frac{d\rho^2}{\rho^4} + O(\rho)\frac{d\rho d(v, y)}{\rho^3} + O(\rho)\left(\frac{d(v, y)}{\rho}\right)^2.$$

Then the induced metric on C_+ (which is diffeomorphic to a ball) is the metric on $(n-1)$-dimensional hyperbolic space; P_σ is a conjugate of the spectral family on hyperbolic space. (See Section 5 of [22].) In particular, the relevant poles of P_σ^{-1} (i.e., those of $L_{\sigma,+}^{-1}$ from Section 7) are given by the poles of the meromorphic expansion of $\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \frac{(n-2)^2}{4}\right)^{-1}$. These poles can be calculated explicitly: when n is even (and hence the spatial dimension is odd), there are no poles, while if n is odd, the poles are given by $\sigma_j = -\imath\frac{n-2}{2} - \imath j$ for $j \in \mathbb{N}$. In particular, $\mathcal{R}_+[w]$ has an asymptotic expansion of the following form:

$$\mathcal{R}_+[w](s,\omega) \sim \begin{cases} O(s^{-\infty}) & n \text{ even} \\ \sum_{j=0}^{\infty} \sum_{\kappa \le j} s^{-\frac{n}{2}-j}(\log s)^{\kappa} a_{j\kappa} & n \text{ odd.} \end{cases}$$

(Recall that one differentiates the restriction of u in s to obtain $\mathcal{R}_+$.) In the special case when the metric is in fact *exactly* Minkowski in a neighborhood of C_+ in M, we remark that the whole iterative apparatus of Section 9 can be dispensed with, in favor of a single application of P_σ^{-1} to the Mellin-transformed inhomogeneity, with the result that the the log terms in the expansion do not appear in that case.

The stability of P_σ^{-1} under perturbations implies that for small "normally short range" perturbations of Minkowski space, the radiation field still decays. In this setting, however, poles of P_σ^{-1} that are not poles of $L_{\sigma,+}^{-1}$ (and hence do not affect the decay of the radiation field) may become relevant under perturbations. As discussed earlier, such poles must occur at purely imaginary negative integers and the corresponding states must be supported exactly at S_+. Such a state occurs even in 4-dimensional Minkowski space at $\sigma = -\imath$. Under small "normally short range" perturbations, then, the first pole occurs close to $\sigma = -\imath$ and so we conclude that the radiation field is $O(s^{-2+\epsilon})$ as $s \to \infty$.

Appendix A. Variable order Sobolev spaces

First recall that (uniform) symbols $a \in S^m_{\rho,\delta}$ on $\mathbb{R}^n \times \mathbb{R}^n$ of type (ρ,δ) of order $r \in \mathbb{R}$ are $\mathcal{C}^\infty$ functions on $\mathbb{R}^n_z \times \mathbb{R}^n_\zeta$ such that

$$|D_z^\alpha D_\zeta^\beta a| \le C\langle\zeta\rangle^{r+\delta|\alpha|-\rho|\beta|}.$$

For various applications, the natural type is $\rho = 1-\delta$, $\delta \in [0,1/2)$, with $\delta = 0$ corresponding to the standard symbol class. We assume these restrictions from now on; for us the relevant regime will be $\delta > 0$ but arbitrarily small. Note that $S^{-\infty} = \bigcap_r S^r_{1-\delta,\delta}$ is independent of δ. There is a symbol calculus within this class $S^r_{1-\delta,\delta}$, which works modulo $S^{r-1+2\delta}_{1-\delta,\delta}$; the principal symbol of the composition of two operators is the product of the two principal symbols in this sense. Further, one has the full symbol expansion of the composition modulo $\Psi^{-\infty}$; namely if $(Au)(z) = (2\pi)^{-n}\int e^{i(z-z')\cdot\zeta} a(z,\zeta)u(z')\,dz'$ is the left quantization of $a \in S^r_{1-\delta,\delta}$, and B is the left quantization of $b \in S^{r'}_{1-\delta,\delta}$ then AB is the left quantization of

$$c \sim \sum_\alpha \frac{\imath^{|\alpha|}}{\alpha!} D^\alpha_\zeta a D^\alpha_z b.$$

As usual these can be transferred to manifolds by local coordinates, allowing the addition of globally $\mathcal{C}^\infty$ kernels as well.

We can now turn to variable order operators. Suppose that s is a real-valued function on $S^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{S}^{n-1} = \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+$, which we assume is constant outside a compact set since we are interested only in transferring the result to manifolds via local coordinates – one could assume instead uniform bounds on derivatives on $\mathbb{R}^n \times \mathbb{S}^{n-1}$. On $\mathbb{R}^n \times \mathbb{R}^n$, we say that a is a (variable order) symbol of order s, written $a \in S^s_{1-\delta,\delta}$, $\delta \in (0, 1/2)$ if

$$a = \langle\zeta\rangle^s a_0, \qquad a_0 \in S^0_{1-\delta,\delta}(T^*X). \tag{A.1}$$

So $S^s_{1-\delta,\delta} \subset S^{s_0}_{1-\delta,\delta}$ with $s_0 = \sup s$. Thus, one can quantize these symbols, with the result, $\Psi^s_{1-\delta,\delta}$ being a subset of $\Psi^{s_0}_{1-\delta,\delta}$. One calls the equivalence class of a in $S^s_{1-\delta,\delta}/S^{s-1+2\delta}_{1-\delta,\delta}$ the principal symbol of the left quantization A of a. We could of course just as well used another choice of quantization such as right- or Weyl-quantization. Note, though, that the condition $\delta > 0$ is crucial for making the different choices of quantizations equivalent since the right reduction formula is

$$\sim \sum \frac{(-\imath)^{|\alpha|}}{\alpha!} D^\alpha_z D^\alpha_\zeta a,$$

and the derivatives falling on the exponent of $\langle\zeta\rangle$ give logarithmic terms, which do not have the full $S_{1,0}$ type gain.

The full asymptotic expansion for composition shows that if s, s' are real valued functions on $S^*\mathbb{R}^n$ then

$$A \in \Psi^s_{1-\delta,\delta},\ B \in \Psi^{s'}_{1-\delta,\delta} \Longrightarrow AB \in \Psi^{s+s'}_{1-\delta,\delta},$$

and modulo $\Psi^{s+s'-1+2\delta}_{1-\delta,\delta}$ it is given by a quantization of the product of the principal symbols; again $\delta > 0$ is important. The commutator $[A, B]$ is then in $\Psi^{s+s'-1+2\delta}_{1-\delta,\delta}$, and its principal symbol (modulo $S^{s+s'-2+4\delta}_{1-\delta,\delta}$) is $\frac{1}{\imath}\{a, b\}$, where $\{.,.\}$ is the Poisson bracket, and a, b are the respective principal symbols. Defining $a \in S^s_{1-\delta,\delta}$ to be elliptic if there exists $c, R > 0$ such that $|a| \geq c\langle\zeta\rangle^s$ for $\langle\zeta\rangle \geq R$, i.e. if a_0 is elliptic in (A.1) in the analogous standard sense, the (microlocal) elliptic parametrix construction works, i.e. if $A \in \Psi^s_{1-\delta,\delta}$ has elliptic principal symbol then there is $G \in \Psi^{-s}_{1-\delta,\delta}$ such that $GA - \mathrm{I}, AG - \mathrm{I} \in \Psi^{-\infty}$. We can transfer these operators to manifolds X via localization and adding $\mathcal{C}^\infty$ Schwartz kernels to the space; here we may assume that X is compact. In this manner, for s

a real-valued function on $S^*X = (T^*X \setminus o)/\mathbb{R}^+$ with $s_0 = \sup s$, we define $\Psi^s_{1-\delta,\delta}(\tilde{X}) \subset \Psi^{s_0}_{1-\delta,\delta}(X)$. The principal symbol of $A \in \Psi^s_{1-\delta,\delta}(X)$ is a well-defined element of $S^s_{1-\delta,\delta}(T^*X)/S^{s-1+2\delta}_{1-\delta,\delta}(T^*X)$.

We can now define Sobolev spaces: fix $A \in \Psi^s(X)$ elliptic, $s_1 = \inf s$. We write

$$H^s = \{U \in H^{s_1} : \ AU \in L^2\}, \ \|U\|^2_{H^s} = \|U\|^2_{H^{s_1}} + \|AU\|^2_{L^2};$$

this is a Hilbert space and all the standard mapping properties of ps.d.o's apply. Different elliptic choices $A, B \in \Psi^s_{1-\delta,\delta}$ defining H^s give the same space, since if G is a parametrix for A, then $BU = BGAU + EU$, where $E \in \Psi^{-\infty}$, so $BG \in \Psi^0_{1-\delta,\delta}$, $AU \in L^2$ shows $BU \in L^2$ by the standard L^2-boundedness of $\Psi^0_{1-\delta,\delta}$, and also shows the equivalence of the norms. Further, if $s, s' \in \mathcal{C}^\infty(S^*X)$ and B is order s then

$$B : H^{s'} \to H^{s'-s}$$

is continuous; taking Λ_s elliptic of order s, then this is equivalent to

$$\Lambda^{s'-s} B \Lambda^{-s} : L^2 \to L^2$$

bounded, but the left hand side is in $\Psi^0_{1-\delta,\delta}$, so this is again the standard L^2 boundedness.

Since the elliptic parametrix construction works, elliptic estimates hold without conditions on s in this setting. In our considerations, near the radial sets s will be taken constant, so the previous results apply microlocally there. However, one needs new real principal type estimates; these hold if s is non-increasing along the direction of the H_p-flow in which we want to propagate the estimates.

Proposition A.1. *Suppose that $P \in \Psi^m(X)$ has real principal symbol. Suppose that $s \in \mathcal{C}^\infty(S^*X)$ is non-increasing along H_p on a neighborhood O of $q \in S^*X$. Let $B, G, R \in \Psi^0$, with the property that* $\mathrm{WF}'(B) \subset \mathrm{Ell}(G)$ *and such that if $\alpha \in \mathrm{WF}'(B) \cap \Sigma$ then the backward (null-)bicharacteristic of p from α reaches* $\mathrm{Ell}(R)$ *while remaining in* $\mathrm{Ell}(G) \cap O$. *Then for all N there is $C > 0$ such that*

$$\|BU\|_{H^s} \leq C(\|GPU\|_{H^{s-m+1}} + \|RU\|_{H^s} + \|U\|_{H^{-N}}).$$

A similar result holds if s is non-decreasing along H_p and "backward" is replaced by "forward."

Related results appear in [19], but there the weights arise from the base space X, and logarithmic weights are used as well, which would require some definiteness of the derivative of s along H_p that we do not have here.

Proof. As the result states nothing about radial points, one may assume that H_p is non-radial on O. This then reduces to a microlocal result, namely that there is a neighborhood of a point q in which the analogous property holds. This can be proved by a positive commutator estimate as in [12].

Let $|\xi|$ be a positive homogeneous degree 1 elliptic function on T^*X; since we are working microlocally, we may take $|\xi|$ to be the function $|\zeta|$ in local coordinates. With $\mathsf{H}_{p,m} = |\xi|^{-m+1}\mathsf{H}_p$ denoting the rescaled Hamilton vector field, which is homogeneous of degree zero, thus a vector field on S^*X, one can introduce local coordinates $q_1, \ldots, q_{2n-1}$ on S^*X centered at α such that $\mathsf{H}_{p,m} = \frac{\partial}{\partial q_1}$; one writes $q' = (q_2, \ldots, q_{2n-1})$. Then one fixes $t_2 < t_1 < 0 < t_0$ and a neighborhood U of 0 in $\mathbb{R}^{2n-2}_{q'}$ such that $[t_2, t_0]_{q_1} \times \overline{U_{q'}} \subset O$ and such that one has a priori regularity near $[t_2, t_1]_{q_1} \times \overline{U_{q'}}$, i.e. R in the notation of the proposition is elliptic there. For $r \in [0, 1]$ (the regularization parameter) one considers

$$a_r = |\xi|^{s-(m-1)/2}\chi(q_1)\phi(q')\psi_r(|\xi|),$$

where $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^{2n-2})$ is supported in U,

$$\psi_r(t) = (1 + rt)^{-1},$$

and

$$\chi(q_1) = \chi_0(q_1)\chi_1(q_1),$$

with $\chi_0(t) = e^{-\digamma/(t_0-t)}$, $t < t_0$, $\chi_0(t) = 0$ for $t \geq t_0$ and $\chi_1(t) \equiv 1$ near $[t_1, \infty)$, 0 near $(-\infty, t_2]$; here $\digamma > 0$ will be taken sufficiently large. Taking $\delta \in (0, 1/2)$ arbitrary (i.e. δ can be very small), ψ_r reduces the order of a_r for $r > 0$, so $a_r \in S^{s-(m-1)/2-1}_{1-\delta,\delta}$ for $r > 0$, and for $r \in [0, 1]$, a_r is uniformly bounded in $S^{s-(m-1)/2}_{1-\delta,\delta}$, converging to a_0 in $S^{s-(m-1)/2+\epsilon}_{1-\delta,\delta}$ for $\epsilon > 0$. Then as $\mathsf{H}_{p,m} q_1 = 1$ and $\psi_r' = r\psi_r^2$,

$$\begin{aligned}
\mathsf{H}_p a_r^2 = {} & 2|\xi|^{2s}\phi(q')^2\psi_r(|\xi|)^2\chi_1(q_1)\chi_0(q') \\
& \times \Big(\chi_0'(q_1)\chi_1(q_1) + \chi_0(q_1)\chi_1'(q_1) \\
& \quad + (s - (m-1)/2 + r|\xi|\psi_r)|\xi|^{-1}(\mathsf{H}_{p,m}|\xi|)\chi_0(q_1)\chi_1(q_1) \\
& \quad + (\log|\xi|)(\mathsf{H}_{p,m}s)\chi_0(q_1)\chi_1(q_1)\Big).
\end{aligned}$$

Now, $\chi_0' \leq 0$, giving rise to the main "good" term, while the χ_1' term is supported in $(t_2, t_1)_{q_1} \times U_{q'}$, where we have a priori regularity and estimates. Further, by making $\digamma$ large, taking into account that $r|\xi|\psi_r$ is bounded, we can dominate the $|\xi|^{-1}\mathsf{H}_{p,m}|\xi|$ term since χ_0 can be bounded by a small multiple of χ_0' for $\digamma > 0$ large, and $\mathsf{H}_{p,m}s \leq 0$, i.e. has the same sign as the χ_0' term. The imaginary (or skew-adjoint in the non-scalar setting) part of the subprincipal symbol also gives a contribution that can be dealt with as the $|\xi|^{-1}\mathsf{H}_{p,m}|\xi|$ term. Thus, taking A_r to have principal symbol a_r and family wave front set $\mathrm{WF}'(\{A_r\}) = \mathrm{esssupp}\, a$ (for instance a quantization of a_r), B_r have principal symbol

$$b_r = |\xi|^s\phi(q')\chi_1(q_1)\sqrt{\chi_0(q')\chi_0'(q')}\psi_r(\xi),$$

and similar WF' one obtains an estimate of the desired kind, and by estimating the χ_1' term (which is the only one having the wrong sign) by the

R term, first by obtaining an estimate for $r > 0$ and then letting $r \to 0$ to obtain the result of the desired form. Corresponding to the symbol class, this can give $1/2 - \delta$ order of improvement (i.e. allows $-N = s - 1/2 + \delta$) for all $\delta > 0$; iterating gives the stated result. $\square$

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Department of Mathematics, Northwestern University
Current address: Department of Mathematics, Texas A&M University

Department of Mathematics, Stanford University

Department of Mathematics, Northwestern University
E-mail address: dbaskin@math.northwestern.edu
E-mail address: andras@math.stanford.edu
E-mail address: jwunsch@math.northwestern.edu