ASYMPTOTICS OF THE RADIATION FIELD FOR THE MASSLESS DIRAC–COULOMB SYSTEM

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ABSTRACT. We consider the long-time behavior of solutions to the massless Dirac equation coupled to a Coulomb potential. For nice enough initial data, we find a joint asymptotic expansion for solutions near the null and future infinities and characterize explicitly the decay rates seen in the expansion.

This paper can be viewed as a successor to previous work on asymptotic expansions for the radiation field [BVW15, BVW18, BM19]. The key new elements are propagation estimates near the singularity of the potential, building on work of the first author with Wunsch [BW20] and an explicit calculation with hypergeometric functions to determine the rates of decay.

1. INTRODUCTION

We consider the long-time asymptotics of solutions of the massless Dirac–Coulomb equation on $\mathbb{R} \times \mathbb{R}^3$. We show that if the initial data (or source term for the inhomogeneous equation) is sufficiently nice, the solution has a complete asymptotic expansion near null and timelike infinities. The decay rates of the terms in the asymptotic expansion are analogous to resonances and we compute them explicitly. One way to capture the leading order part of these asymptotics is through the Friedlander radiation field, which is a rescaled restriction of the solution to null infinity $\mathcal{I}^+$. We find, as in previous work in related settings [BVW15, BVW18, BM19], that the asymptotics of the radiation field as the lapse function $t - r = s \to \infty$ are given by the resonance poles of a Dirac-type operator (in the sense that its square is principally the Laplacian) on hyperbolic space.

The following theorem is the main result of this paper. Notation involving the Dirac equation will be explained in Section 2. Let η denote the (mostly plus) Minkowski metric on $\mathbb{R} \times \mathbb{R}^3$, whose coordinates are $t = x^0$, x^1, x^2, x^3 and γ^α denote the Dirac matrices, which satisfy $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = -2\eta^{\alpha\beta}\,\mathrm{Id}_4$. We let $r = r(x) = \left((x^1)^2 + (x^2)^2 + (x^3)^2\right)^{1/2}$ denote radius in the spatial coordinates.

Theorem 1. *For $\mathbf{Z} \in \mathbb{R}$, $|\mathbf{Z}| < 1/2$, consider the massless Dirac–Coulomb operator in $\mathbb{R} \times \mathbb{R}^3$:*

$$\not{\partial}_{\mathbf{Z}/r} = \gamma^0 \left(\partial_t + \frac{i\mathbf{Z}}{r}\right) + \sum_{j=1}^{3} \gamma^j \partial_j.$$

Let ψ be the forward solution of $\not{\partial}_{\mathbf{Z}/r}\psi = f$ with $f \in C_c^\infty$. Then its Friedlander radiation field, given in terms of $s = t - r$ and $\theta \in \mathbb{S}^2$ by

$$\mathcal{R}_+[\psi](s, \theta) = \lim_{r \to \infty} r^{1+i\mathbf{Z}} \psi(s + r, r\theta),$$

is a smooth function on $\mathbb{R}_s \times \mathbb{S}^2_\theta$.

Moreover, the radiation field admits an asymptotic expansion as $s \to +\infty$:

$$\mathcal{R}_+[\psi](s,\theta) \sim s^{i\mathbf{Z}} \sum_{j,k=1}^{\infty} s^{-j-\sqrt{k^2-\mathbf{Z}^2}} a_{jk}(\theta),$$

where a_{jk} are smooth functions on $\mathbb{S}^2$.

In fact, we prove the following somewhat stronger theorem, showing that in fact ψ is polyhomogeneous on a compactification $[M;S_+]$ of $\mathbb{R}\times\mathbb{R}^3$ which we describe in detail below (the factor of i in the index set is due to our convention described below in Section 3.4):

Theorem 2. *If ψ is the forward solution of $i\eth_{\mathbf{Z}/r}\psi = g$ with $g \in C_c^\infty$, then ψ is polyhomogeneous on $[M;S_+]$ with index sets*

$$\begin{cases} \emptyset & \text{at } C_- \cup C_0, \\ \{(i-\mathbf{Z}+ik,0) : k = 0,1,2,\dots\} & \text{at } \mathcal{I}^+, \\ \{(i(1+j+\sqrt{k^2-\mathbf{Z}^2}),0) : j,k = 1,2,\dots\} & \text{at } C_+. \end{cases}$$

In particular, the leading order behavior near the light cone is given by

$$\frac{1}{(t+r)^{1+i\mathbf{Z}}(t-r)^{1+\sqrt{1-\mathbf{Z}^2}}} \text{ as } t,r \to +\infty.$$

The apparent difference between the leading order behavior of $\mathcal{R}_+[\psi]$ as $s \to \infty$ and ψ as $t,r \to \infty$ is an consequence of our definition of the radiation field below in equation (4). Indeed, as explained there, $\mathcal{R}_+[\psi]$ is $\rho^{-1-i\mathbf{Z}}\psi$ restricted to null infinity. The expansion exponents of $\mathcal{R}_+[\psi]$ at C_+ are therefore shifted in comparison to that of ψ itself by $1+i\mathbf{Z}$.

A significant novelty of this paper is the *explicit* characterization of the decay rates of solutions off the light cone. We compute these estimates by finding the "resonant states" associated to a family of operators at infinity. The solutions are given in terms of hypergeometric functions and we characterize the exponents as poles of the inverse of this operator family.

Though we do not state it explicitly, one can also see that the Friedlander radiation field in this context carries a particular polarization; the forward radiation field always lies in the $+1$-eigenspace of the Dirac matrix corresponding to Clifford multiplication by ∂_r. (The backward radiation field lies in the other eigenspace.) This can be seen either through a modification of the argument given in Section 2.3 or explicitly in terms of the hypergeometric functions of Section 8.

The proof, described in steps at the end of this section, follows the same outline used to study the radiation field in other settings [BVW15, BVW18, BM19]; arguments near the singularity of the potential are modeled on the proof of the diffractive propagation theorem for the Dirac–Coulomb system [BW20]. The methods used in this paper apply in higher dimensions (and, indeed, for conic Dirac operators), but we specialize to the case of three spatial dimensions for reasons of clarity and physical interest. See Section 1.3 for a discussion of the higher-dimensional case.

The Dirac–Coulomb equation provides a model for spin-$\frac{1}{2}$ particles in the presence of a point charge $\mathbf{Z}$. In the massive setting, much of the literature about the system and its related operators is focused on the characterization of the point spectrum. In contrast, the massless case has purely continuous spectrum[1] and so this description is insufficient to characterize the asymptotic behavior of solutions of the time-dependent equation. Darwin [Dar28] used separation of variables to characterize the generalized eigenfunctions of the massive Hamiltonian; a similar approach applies to the massless case. In principle, one could derive our theorem by a careful analysis of the special functions involved but it would be delicate and our methods apply more generally. Indeed, our methods also treat certain perturbations of the equation; specifically, one could add a compactly supported first order term (compact support implying also support away from the singularity $r = 0$) or even a compactly supported leading order term which does not change the large scale structure of the characteristic set and in particular does not introduce trapping.

We further remark that the restriction that $|\mathbf{Z}| < 1/2$ owes to our repeated use of the Hardy inequality in the propagation arguments. We conjecture that the theorem holds (though the proof would require considerably more care in the construction of the commutants) for the entire range of charges for which the Hamiltonian is essentially self-adjoint, i.e., $|\mathbf{Z}| < \sqrt{3}/2$. That the results of Section 8 hold for this range can be viewed as partial evidence for this view.

Interest in the massless Dirac–Coulomb system as an evolution equation has also arisen in the community surrounding dispersive equations. That work has largely focused on proving dispersive and Strichartz estimates for solutions both via separation of variables and by treating the components as solutions of systems of coupled wave equations. Notable here is the work of D'Ancona and collaborators [DF07, BDF11, CD13], the work of Cacciafesta–Séré [CS16], and the work of Erdoğan–Green–Toprak [EGT19]. In many respects this paper is complementary to that work, as we are also concerned with the global decay of solutions of the equation, but use very different methods.

A natural follow-up question to this work is whether similar results hold for smooth potentials decaying like $1/r$ at infinity (i.e., potentials of critical decay). Although the methods employed here do not directly apply to treat more general potentials, we expect that they can be combined with an adaptation of Vasy's second microlocalization [Vas21a, Vas21b] to obtain related results. We expect to treat this setting in future work.

1.1. **Notation.** Norms without subscript decorations are always the relevant L^2 norm. We use the notation $s - 0$ to denote $s - \epsilon$ for all $\epsilon > 0$. Our convention is that the natural numbers include zero, i.e.,

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

Some further notation is as follows:

- M is the radially compactified spacetime with the singularity of the potential blown up; mf is the boundary at spacetime infinity. Section 1.2.
- $L^2(\mathbb{R} \times \mathbb{R}^4)$ is the standard L^2 space on $\mathbb{R}^4$ for the Euclidean measure $dtdz$, while $L^2(M)$ is the weighted L^2 space based on the measure that is "b" at space time infinity. Section 3.3.1.
- The Sobolev spaces we use most on the bulk are $H_{\mathrm{b}}^{k,m,l}(M) = H_{\mathrm{b}}^{k,m,l}$, $k \in \{0, 1\}$. Here k is the standard Sobolev regularity order, m is the b-Sobolev regularity order, and l describes a weight. Section 3.3.

[1]The essential self-adjointness of the Hamiltonian implies that the spectrum is entirely real; separation of variables and a simple ODE analysis shows that there are no L^2 solutions and hence no eigenvalues.

- For distributions u, $\mathrm{WF}_{\mathrm{b}}^{1,m,\ell} u$ and $\mathrm{WF}_{\mathrm{b}}^{0,m,\ell} u$ are the notions of wavefront sets associated to $H^{1,m,l}$ and $H^{0,m,l}$. Section 3.3.
- The functions $s = s_{\mathrm{ftr}}, s^* = s_{\mathrm{past}}$ on the sphere at infinity are the regularity functions on the boundary used to analyze the Mellin transformed normal operators. Section 4.3.
- $\mathcal{X}^s$ and $\mathcal{Y}^s$ and their semiclassical versions $\mathcal{X}_h^s$ and $\mathcal{Y}_h^s$ are variable order Sobolev spaces of distributions on the sphere at infinity. Section 4.3, same for the dual spaces s^*.

The space $\Psi_{\mathrm{b}}^m(M)$ is the space of b-pseudodifferential operators of order m on M (Section 3.2). Below we often let

$$\{A_1, \dots, A_n\}\Psi_{\mathrm{b}}^m(M)$$

denote the (right) module of operators generated by $A_1, \dots, A_n \in \mathrm{Diff}^*(\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ over $\Psi_{\mathrm{b}}^m(M)$.

1.2. **Outline of proof.** The proofs of Theorems 1 and 2 follow the same general contours of analogous results for the scalar wave equation on asymptotically Minkowski spaces [BVW15, BVW18] and on cones [BM19]. In particular, the analysis is somewhat round-about and has five major steps, which we describe now.

Set-up. We define a compactification M of $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ to a manifold with corners; this has the effect of "resolving" the singularity of the potential and making the use of Melrose's b-calculus more natural. We let ρ be a boundary defining function for the main face mf of the boundary (at infinity). In the region of greatest interest, one can take $\rho = t^{-1}$. We then consider the equation

$$i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g,$$

but then rescale and conjugate to rewrite it as

$$\mathcal{L}u = f,$$

where

$$\mathcal{L} \equiv \rho^{-2-i\mathbf{Z}}\gamma^0 i\partial\!\!\!/_{\mathbf{Z}/r}\rho^{1+i\mathbf{Z}},$$
$$u = \rho^{-1-i\mathbf{Z}}\psi \in C^{-\infty}(M), \quad f = \rho^{-2-i\mathbf{Z}}g \in C_c^\infty(M^\circ).$$

The rescaling is helpful because $\mathcal{L}$ is then a "b-differential operator" in the parlance of Melrose [Mel93], and it enables the use of the b-pseudodifferential calculus to obtain microlocal estimates on u near mf.

Propagation of b*-regularity.* We prove the propagation of b-regularity, i.e., microlocalized conormal regularity with respect to the main face mf starting at the backwards null cone, where by hypothesis the solution is trivial (as the solution is zero for t sufficiently negative). The aim is to propagate this regularity until we reach the intersection S_+ of the forward null cone with the boundary of M, where the relevant bicharacteristic flow has radial points and so one needs subtler estimates. Most of this step is essentially contained in prior work [BVW15, BVW18], though the propagation through the singularity of the potential is new; we adapt the argument used by the first author to prove a diffractive theorem for the Dirac–Coulomb system [BW20] here. As the argument is so similar, we leave many of the proofs in this step to an appendix.

Fredholm estimates. We then use a strategy developed by Vasy [Vas13] to show that we may set up a global Fredholm problem on mf for the family of "reduced normal operators" $\widehat{\mathcal{L}}_\sigma$. This is the family of operators given by freezing coefficients at $\rho = 0$ and then conjugating $\mathcal{L}$ by the Mellin transform in ρ. To obtain a Fredholm problem, we view $\widehat{\mathcal{L}}_\sigma$ as acting on spaces with varying degrees of regularity, with more regularity mandated at the backward end of the flow lines than the forward end. The family $\widehat{\mathcal{L}}_\sigma^{-1}$ then only has finitely many poles in any given horizontal strip in $\mathbb{C}$ and satisfies polynomial estimates as $|\operatorname{Re}\sigma| \to \infty$. To propagate the estimates near the singularity of the potential on mf, we rely on a semiclassical version of the diffractive theorem in the bulk.

Asymptotic expansions. This portion of the argument is essentially identical to the setting of asymptotically Minkowski spaces [BVW15, BVW18]. To begin the asymptotic development of u (and therefore ψ) near mf, one cuts off near mf and takes the Mellin transform to obtain a σ-dependent family of equations of the form

$$\widehat{\mathcal{L}}_\sigma \widetilde{u_\sigma} = \widetilde{f}_\sigma,$$

where $\widetilde{u_\sigma}$ is known to be analytic in a half-plane $\operatorname{Im}\sigma \geq \varsigma_0$ by the propagation of b-regularity. Inverting $\widehat{\mathcal{L}}_\sigma$, one then obtains the global meromorphy of $\widetilde{u_\sigma}$. Applying the inverse Mellin transform turns the poles $\widetilde{u_\sigma}$ into terms in an asymptotic expansion, with poles at $\sigma = z$ of degree k becoming a term $\rho^{iz}(\log\rho)^{k-1}$. The coefficients in this expansion are functions on mf that become worse in their regularity at S_+ as $\operatorname{Im} z$ decreases (so as we gain more decay in ρ). To obtain the full expansion as in Theorems 1 and 2, one then tests via shifts of the scaling vector field at S_+.

Identification of the exponents. Having established the polyhomogeneity of the solution, we then explicitly identify the poles σ of $\widehat{\mathcal{L}}_\sigma^{-1}$. By changing coordinates, we find an expression for $\widehat{\mathcal{L}}_\sigma$ in which the ODEs obtained after separating variables become a family of hypergeometric equations, which we solve explicitly near past and future infinity.

1.3. **Other dimensions.** We now briefly describe how the theorem changes in $(n+1)$-dimensions for $n > 3$. Our approach applies nearly verbatim as long as the Hardy inequality applies, i.e., for $|\mathbf{Z}| < \frac{n-2}{2}$. In this setting we write the Dirac–Coulomb operator as

$$\partial\!\!\!/_{\mathbf{Z}/r} = \gamma^0\left((\partial_t + i\frac{\mathbf{Z}}{r}\right) + \gamma^r\left(\partial_r + \frac{n-1}{2r}\right) + \frac{1}{r}D_S.$$

Here the spinor bundle on $\mathbb{R}\times\mathbb{R}^n$ is trivial and 2^{k+1}-dimensional, where $k = \lfloor (n-1)/2 \rfloor$. The spinor bundle on $\mathbb{S}^{n-1}$ is similarly trivial and 2^k-dimensional. In fact, over the sphere, we can identify the spinor bundle on $\mathbb{R}\times\mathbb{R}^n$ with two copies of the spinor bundle on $\mathbb{S}^{n-1}$. The operator D_S then enjoys the property that $\gamma^r D_S$ acts as the spherical Dirac operator $\partial\!\!\!/_{\mathbb{S}^{n-1}}$ on one copy and $-\partial\!\!\!/_{\mathbb{S}^{n-1}}$ on the other. The eigenvalues of $\gamma^r D_S$ (and hence D_S) are then given by (see, e.g., Bär [B9̈6])

$$\pm\left(\frac{n-1}{2} + k\right), \quad k \in \mathbb{N}_0.$$

To prove the analogous theorem in higher dimensions, we then consider

$$\mathcal{L} = \rho^{-1-\frac{n-1}{2}-i\mathbf{Z}\gamma^0} i\partial\!\!\!/_{\mathbf{Z}/r}\rho^{\frac{n-1}{2}+i\mathbf{Z}},$$

and proceed as in the three-dimensional case. Although the propagation results for finite t and $n > 3$ are not in the literature, the same proof as in $n = 3$ applies and yield Theorem 2 in $(n+1)$-dimensions, $n > 3$, with the following index sets:

$$\begin{cases} \emptyset & \text{at } C_- \cup C_0, \\ \left\{ \left(i \left(\frac{n-1}{2} + i\mathbf{Z} + k \right), 0 \right) : k = 0, 1, 2, \dots \right\} & \text{at } \mathcal{I}^+, \\ \left\{ \left(i \left(\frac{n+1}{2} + j + \sqrt{\left(\frac{n-1}{2} + k \right)^2 - \mathbf{Z}^2} \right), 0 \right) : j, k = 0, 1, 2, \dots \right\} & \text{at } C_+. \end{cases}$$

In $(2+1)$-dimensions, however, our basic approach fails due to the failure of the Hardy inequality. Indeed, the Dirac–Coulomb system fails to be essentially self-adjoint for $\mathbf{Z} \neq 0$. We conjecture that a similar theorem could hold with $|\mathbf{Z}| < 1/2$ for the distinguished self-adjoint extension found by Schmincke [Sch72] (see also Wüst [W7̈5, W7̈7], Nenciu [Nen76] and Klaus–Wüst [KW79]). The failure of the Hardy inequality in two dimensions, however, suggests that the propagation arguments require significantly more care. The authors believe that they could be made to work by appealing to the "very basic" operators of Melrose–Vasy–Wunsch [MVW08, Section 10] but this discussion would take us too far afield.

1.4. **Structure of the paper.** Section 2 fixes our notation and conventions for the massless Dirac–Coulomb system. In Section 3 we introduce the relevant compactifications of our space time and recall some facts about Melrose's b-calculus and its relationship with standard differential operators. Collected in Section 4 are a number of results and definitions used in the main analysis.

The remainder of the paper is devoted to the proof of the theorems: Section 5 describes the needed results in the bulk of the spacetime, while Section 6 is devoted to the proof that the operator on the boundary is Fredholm with finitely many poles in any strip. In Section 7 we show the polyhomogeneity of the solution. Finally, in Section 8 we find the exponents explicitly.

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2. The Dirac–Coulomb equation

2.1. **Notation.** We use coordinates x^α, $\alpha = 0, \dots, 3$ on $\mathbb{R} \times \mathbb{R}^3$. When referring to spatial coordinates (i.e., indices 1, 2, 3) we use Latin rather than Greek subscripts and superscripts. When convenient we use $t = x^0$ and spatial polar coordinates $r \in (0, \infty)$, $\theta \in \mathbb{S}^2$. (In Section 3.1 we describe coordinate systems that are better adapted to use "near infinity".)

The Dirac operator on $\mathbb{R} \times \mathbb{R}^3$ is given by

$$\partial\!\!\!/ = \gamma^\alpha \partial_\alpha,$$

where γ^α are the 4×4 Dirac matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

and

$$\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix},$$

and σ_j are the 2×2 Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The γ^α satisfy the anticommutation relation

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = -2\eta^{\alpha\beta}\,\mathrm{Id}_4,$$

where $\eta^{\alpha\beta}$ are the components of the Minkowski metric, i.e.,

$$\eta^{\alpha\beta} = \begin{cases} -1 & \alpha = \beta = 0 \\ 1 & \alpha = \beta \in \{1,2,3\} \\ 0 & \alpha \neq \beta \end{cases}.$$

Given a charge $\mathbf{Z} \in \mathbb{R}$, we couple the Dirac operator to a Coulomb electric potential of charge $\mathbf{Z}$ via the "minimal coupling" convention:

$$\partial\!\!\!/_{\mathbf{Z}/r} = \gamma^0 \left(\partial_t + i\frac{\mathbf{Z}}{r} \right) + \gamma^j \partial_j.$$

The operator $\partial\!\!\!/_{\mathbf{Z}/r}$ is not symmetric with respect to the standard flat inner product on functions $\phi, \psi \colon \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{C}^4$ given by $\langle \phi, \psi \rangle = \int \langle \phi, \psi \rangle_{\mathbb{C}^4} dt dx^1 dx^2 dx^3$. (Here $\langle \cdot, \cdot' \rangle_{\mathbb{C}^4}$ is the pointwise hermitian inner product on $\mathbb{C}^4$.) Indeed, $\gamma_0\partial_0$ is antisymmetric while the $\gamma_j\partial_j$ are symmetric. On the other hand, by the anticommutation properties of the Dirac matrices, the operators $i\gamma_0\partial\!\!\!/_{\mathbf{Z}/r}$ and $i\partial\!\!\!/_{\mathbf{Z}/r}\gamma_0$ are easily checked to be symmetric with respect to this inner product.

We employ several other notational conventions. In keeping with physics notation (see, e.g., Rose [Ros61]), we write

$$\beta = \gamma^0,$$

and let α_j be defined by

$$\gamma^j = \beta\alpha_j,$$

i.e.,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

In spherical coordinates, we require the radial versions of the various matrices and so we set

$$\sigma_r = \sum_{j=1}^{3} \frac{x_j}{r}\sigma_j, \quad \alpha_r = \sum_{j=1}^{3} \frac{x_j}{r}\alpha_j, \quad \gamma^r = \sum_{j=1}^{3} \frac{x_j}{r}\gamma^j. \tag{1}$$

2.2. **Separation of variables.** In this section we use the convention that a boldface letter (such as $\boldsymbol{\sigma}$ or $\mathbf{r}$) denotes the associated 3-vector of matrices or operators (such as $(\sigma_1, \sigma_2, \sigma_3)$ or $\frac{1}{r}(x^1, x^2, x^3)$). We also set

$$\mathbf{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

We let

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

denote the orbital angular momentum operators, where, as is standard,

$$\mathbf{p} = \begin{pmatrix} \frac{1}{i}\partial_{x^1} \\ \frac{1}{i}\partial_{x^2} \\ \frac{1}{i}\partial_{x^3} \end{pmatrix}.$$

We then let

$$\mathbf{J} = \mathbf{L} + \frac{1}{2}\mathbf{\Sigma}$$

denote the total angular momentum operators (so orbital angular momentum and spin together). We now introduce Dirac's K operator and set

$$K = \beta(1 + \mathbf{\Sigma} \cdot \mathbf{L}).$$

The remarkable property of K is the following lemma found in many physics texts (e.g., Rose [Ros61, Section 12]).

Lemma 3. *The following operators are mutually commuting:*

$$\partial\!\!\!/_{\mathbf{Z}/r}, J^2 = \mathbf{J} \cdot \mathbf{J}, J_3, K.$$

Moreover,

$$[\beta, K] = 0.$$

In Section 8 below, we consider the action of a rescaling of $\partial\!\!\!/_{\mathbf{Z}/r}$ on the common eigenfunctions of the remaining operators in the lemma. These eigenfunctions are described blockwise by two component spinor spherical harmonics. Following, e.g., Szmytkowski [Szm07], for $\theta \in \mathbb{S}^2$ we set

$$\Omega_{\kappa\mu}(\theta) = \begin{pmatrix} \operatorname{sgn}(-\kappa)\left(\frac{\kappa+1/2-\mu}{2\kappa+1}\right)^{1/2} Y_{l,\mu-1/2}(\theta) \\ \left(\frac{\kappa+1/2+\mu}{2\kappa+1}\right)^{1/2} Y_{l,\mu+1/2}(\theta) \end{pmatrix},$$

where

$$\begin{aligned} &\kappa \in \mathbb{Z} \setminus \{0\}, \\ &\mu \in \{-|\kappa| + 1/2, \dots, |\kappa| - 1/2\}, \\ &l = \left|\kappa + \frac{1}{2}\right| - \frac{1}{2}, \end{aligned}$$

and $Y_{l,m}$ are the standard spherical harmonics. The eigenvectors of K are given by the span of

$$\begin{pmatrix} \Omega_{\kappa\mu} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \Omega_{-\kappa\mu'} \end{pmatrix}, \quad \mu, \mu' \in \{-|\kappa| + 1/2, \dots, |\kappa| - 1/2\},$$

where both of these are understood to be 4-vectors, and the eigenvalue of K on this eigenspace is $-\kappa$.

We further observe that

$$\alpha_r \begin{pmatrix} a\Omega_{\kappa\mu} \\ b\Omega_{-\kappa\mu'} \end{pmatrix} = \begin{pmatrix} -b\Omega_{\kappa\mu'} \\ -a\Omega_{-\kappa\mu} \end{pmatrix}.$$

The spherical Laplacian Δ_θ and K are related by

$$I_4\Delta_\theta = K^2 - \beta K,$$

so that Δ_θ commutes with K and β, where I_4 is the 4×4 identity matrix.

Writing

$$i\gamma^0 \partial\!\!\!/_{\mathbf{Z}/r} = i\partial_t - \mathcal{B},$$

i.e.,

$$\mathcal{B} = \sum_{j=1}^{3} \frac{1}{i}\alpha_j\partial_j + \frac{\mathbf{Z}}{r}. \tag{2}$$

In polar coordinates, we then have

$$\mathcal{B} = -i\alpha_r\left(\partial_r + \frac{1}{r} - \frac{1}{r}\beta K\right) + \frac{\mathbf{Z}}{r}. \tag{3}$$

We recall for $|\mathbf{Z}| < \sqrt{3}/2$, the operator $\mathcal{B}$ is essentially self-adjoint on $\mathbb{R}^3$ with domain H^1. Kato in his book [Kat66] established this result for $|\mathbf{Z}| < 1/2$ and Weidmann [Wei71] later extended it to $|\mathbf{Z}| < \sqrt{3}/2$. Beyond this range it is no longer essentially self-adjoint. Previous work [BW20] provided another proof of this fact based on the structure of the indicial operator of $\mathcal{B}$.

2.3. **The radiation field.** As our definition of the radiation field differs slightly from Friedlander's [Fri80], we briefly recall its definition and construction.

Given a solution ψ of $\partial\!\!\!/_{\mathbf{Z}/r}\psi = f$ with f smooth and compactly supported, we define the function

$$\varphi(s, \theta, \rho) = \rho^{-(1+i\mathbf{Z})}\psi\left(s + \frac{1}{\rho}, \frac{1}{\rho}\theta\right),$$

i.e., a rescaling of ψ written in terms of the coordinates $\rho = 1/r$, $s = t - r$, and $\theta \in \mathbb{S}^2$, with $\mathbb{S}^2$ identified with the unit sphere in $\mathbb{R}^3$ so that $r\theta = (x^1, x^2, x^3)$.

Because ψ is a solution of $\partial\!\!\!/^2_{\mathbf{Z}/r}\psi = 0$ near infinity, φ is a solution of

$$\rho^{-(1+i\mathbf{Z})}\left(2(1+i\mathbf{Z})\rho\partial_s - 2\rho^2\partial_\rho\partial_s + \rho^2\Delta_\theta - \rho^2(\rho\partial_\rho)^2 + \rho^3\partial_\rho - \mathbf{Z}^2\rho^2 + i\alpha_r\mathbf{Z}\rho^2\right)\rho^{1+i\mathbf{Z}}\varphi = 0$$

near $\rho = 0$. Rewriting this equation yields

$$\rho^2\left(-2\partial_\rho\partial_s + \Delta_\theta - (\rho\partial_\rho + 1 + i\mathbf{Z})^2 + (\rho\partial_\rho + 1 + i\mathbf{Z}) - \mathbf{Z}^2 + i\alpha_r\mathbf{Z}\right)\varphi = 0.$$

In other words, φ is the solution of a hyperbolic equation that is non-degenerate near $\rho = 0$. If ψ vanishes identically for $s \le s_0$ (as is the case for the forward solution,) then the argument of Friedlander [Fri80, Section 1] shows that φ may be smoothly extended across $\rho = 0$. In particular, φ and its derivatives may be restricted to $\rho = 0$.

If ψ is the forward solution of $i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g$, with g smooth and compactly supported, we may therefore define the (forward) *radiation field* of ψ by

$$\mathcal{R}_+[\psi](s, \theta) = \varphi(s, \theta, 0).$$

Note that our definition differs from Friedlander's original construction in two important ways. First, in our construction we have conjugated by $\rho^{1+i\mathbf{Z}}$ rather than ρ to account for the additional oscillations introduced by the potential; this modification is required to ensure that the initial-value formulation of the radiation field is a translation representation of the evolution semigroup. Indeed, if $U(t)\psi_0$ is the solution operator associated to the problem

$$\not{\partial}_{\mathbf{Z}/r}\psi = 0, \quad \psi(0,x) = \psi_0(x),$$

and $\mathcal{R}_+(\psi_0)(s,\theta) = \mathcal{R}_+[\psi](s,\theta)$, then

$$\mathcal{R}_+(U(T)\psi_0)(s,\theta) = \mathcal{R}_+(\psi_0)(s+T,\theta),$$

i.e., the radiation field intertwines wave evolution and translation.

The second important difference is in the normalization of the radiation field; Friedlander's construction includes a derivative to ensure that the L^2 norm of the radiation field of a solution of the wave equation is bounded by the energy of the initial data (indeed, in that setting it is an isometry). As the Dirac–Coulomb system is first order, no derivative is warranted; it is straightforward to see that

$$\|\mathcal{R}_+(\psi_0)\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \leq \|\psi_0\|_{L^2(\mathbb{R}^3\setminus\{0\})}.$$

The question of whether the map taking initial data to the radiation field is an isometry is essentially a question of local energy decay and is left to future work.

In Section 3.1 below, we realize the radiation field as a rescaled restriction of ψ to one face in a compactification of our spacetime.

3. b-geometry and the b-calculus

3.1. **Compactifications.** As the operator $\not{\partial}_{\mathbf{Z}/r}$ is singular at the spatial origin, most of the analysis to follow takes place on a compactification M of $\mathbb{R}\times(\mathbb{R}^3\setminus\{0\})$. In particular, we treat $\mathbb{R}^3\setminus\{0\}$ as a conic manifold and compactify as in previous work [BM19, Section 3]. Roughly speaking, we resolve the singularity at the origin and consider the radial compactification at infinity.

For clarity, we first discuss the setting where the underlying spatial manifold is a half-line. We compactify $\mathbb{R}_t\times(0,\infty)_r$ by stereographic projection to a (closed) quarter-sphere $\mathbb{S}^2_{++}$ as depicted in Figure 1. The map $\mathbb{R}_t\times(0,\infty)_r\to\mathbb{S}^2\subset\mathbb{R}^3$ given by

$$(t,r)\mapsto\frac{(t,r,1)}{\sqrt{1+t^2+r^2}}$$

sends $\mathbb{R}\times(0,\infty)$ to the interior of the quarter-sphere given by

$$\mathbb{S}^2_{++} = \{(z_1,z_2,z_3)\in\mathbb{S}^2\subset\mathbb{R}^3 \mid z_2\geq 0, z_3\geq 0\}.$$

Here $\mathbb{S}^2_{++}$ is a manifold with corners and has two boundary hypersurfaces defined by $z_2=0$ and $z_3=0$, respectively. We let cf (or "conic face") denote the hypersurface defined by

$$z_2 = \frac{r}{\sqrt{1+r^2+t^2}} = 0,$$

while we use mf (or "main face") to denote the boundary hypersurface defined by

$$z_3 = \frac{1}{\sqrt{1+r^2+t^2}} = 0.$$

The above construction defines a smooth structure on the compactification of $\mathbb{R}\times(0,\infty)$. We thereby obtain a compactification M of $\mathbb{R}\times(\mathbb{R}^3\setminus\{0\})$ is then given by

$$M = \mathbb{S}^2_{++}\times\mathbb{S}^2_\theta,$$

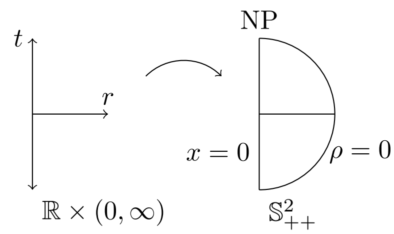

FIGURE 1. The compactification of $\mathbb{R} \times (0, \infty)$ to $\mathbb{S}^2_{++}$

where we use polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}^2$ on $\mathbb{R}^3 \setminus \{0\}$ and identify the interior of first factor $\mathbb{S}^2_{++}$ with $t \in \mathbb{R}$ and $r \in (0, \infty)$ via the above construction.

Away from mf, we use the coordinates (t, r, θ). Near the north pole (given by $(z_1, z_2, z_3) = (1, 0, 0)$), in lieu of the boundary defining functions z_2 and z_3, it is convenient to use the (homogeneous) functions

$$\rho = \frac{1}{t}, \quad x = \frac{r}{t}.$$

Similarly, near the south pole (given by $t < 0$, $z_2 = 0$, $z_3 = 0$), we use

$$\rho = \frac{1}{|t|}, \quad x = \frac{r}{|t|}.$$

In the region between the two poles, (i.e., where $|x| > 1$), we choose our boundary-defining function ρ so that it agrees with the above definition in both polar regions and is homogeneous of degree -1 in the scaling $(t, z) \mapsto (ct, cz)$ near mf. We extend x to be smooth and strictly greater than 1 in this region.

In our discussion of the radiation field, the smooth submanifolds

$$S_\pm = \{\rho = 0, x = 1, \pm t > 0\} = \{(z_1 = \pm 1/\sqrt{2}, z_2 = 1/\sqrt{2}, 0)\} \times \mathbb{S}^2_\theta \subset \mathrm{mf}$$

play a crucial role. These are defined by the functions ρ and $v = 1 - x$. Lightlike geodesics on the interior all have limits at $S_\pm$ in the future/past time directions.

The complement of $S_\pm$ in mf consists of three open components. We denote by C_0 the region in mf where $x > 1$, while the region where $x < 1$ has two components which we denote by $C_\pm$ according to whether $\pm t > 0$ nearby.

The submanifold S_+ plays an additional role; in order to identify the forward radiation field $\mathcal{R}_+$, we *blow up* S_+ in M by replacing it with its inward pointing spherical normal bundle, and in doing so we introduce a boundary hypersurface $\mathcal{I}^+$ which can be thought of as future null infinity and which will serve as the domain of the radiation field. Though we elide detailed background on radial blow-ups, we briefly describe the upshot here.[2] This construction introduces a manifold with corners $[M; S_+]$ and a "blow-down" map

$$\beta_{bd} \colon [M; S_+] \longrightarrow M$$

such that $\mathcal{I}^+ = \beta_{bd}^{-1}(S_+)$ is a boundary hypersurface of $[M; S_+]$ and "cylindrical coordinates"

$$\rho_{\mathcal{I}^+} = (\rho^2 + (1 - x)^2)^{1/2}, \ \phi_{\mathcal{I}^+} = (\rho/\rho_{\mathcal{I}^+}, 1 - x/\rho_{\mathcal{I}^+}), \ \theta \in \mathbb{S}^2_\theta$$

give a smooth parametrization of a neighborhood of $\mathcal{I}^+ = \{\rho_{\mathcal{I}^+} = 0\} \cap \{t > 0\}$. The map β_{bd} is a diffeomorphism from $[M; S_+] \setminus \mathcal{I}^+$ to $M \setminus S_+$, i.e. the construction only "modifies"

[2]For more information about the blow-up construction, we refer the reader to Melrose's book [Mel93].

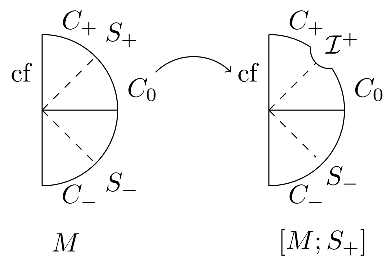

FIGURE 2. A schematic view of the forward radiation field blow-up. The lapse function s increases along $\mathcal{I}^+$ toward C_+.

M near S_+. The structure of this manifold with corners depends only on the submanifold S_+ and not on the particular choice of defining functions ρ and v, and in our setting, this is equivalent to blowing up the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \in \mathbb{S}^2_{++}$ and then taking the product with $\mathbb{S}^2$. See Figure 2.

The new space $[M; S_+]$ has four boundary hypersurfaces: the closure of the lifts of the interiors of C_+ and $C_0 \cup C_-$ by the blow-down map, the lift of cf, and a new boundary hypersurface $\mathcal{I}^+$ introduced by blow up. By construction, $\mathcal{I}^+$ is naturally a fiber bundle over S_+ with fibers diffeomorphic to intervals. Indeed, given $v = 1 - x$ and ρ, the fibers of the interior of $\mathcal{I}^+$ in $[M; S_+]$ can be identified with the cylinder $\mathbb{R}_s \times \mathbb{S}^2$ by the coordinate $s = v/\rho$.

A simple computation (and the observation that, for fixed s, $(1 + s\rho)^{1+i\mathbf{Z}} \to 1$ as $\rho \to 0$) shows that for solutions of $\not{\partial}_{\mathbf{Z}/r}\psi = f$ with smooth, compactly supported f, the Friedlander radiation field defined above agrees with the restriction

$$\mathcal{R}_+[\psi](s, \theta) = \rho^{-1-i\mathbf{Z}}\psi|_{\mathcal{I}^+}. \tag{4}$$

3.2. **The pseudodifferential b-calculus.** We describe now the homogeneous version of the b-calculus, which we use in two separate cases. We primarily use it in our discussion of propagation on the compactification M of bulk spacetime to a manifold with corners. We later use it briefly in our discussion of the non-semiclassical aspects of the operator $\widehat{\mathcal{L}}_\sigma$ on mf. In the discussion below, we therefore describe the b-calculus on a manifold with corners X, though the explicit examples are given only for M.

(Recall, briefly that a smooth manifold with corners X of dimension m is locally diffeomorphic to $\mathbb{R}^k_+ \times \mathbb{R}^{m-k}$ and that ∂X is the union of the boundary hypersurfaces $\{H_1, \dots, H_l\}$, which are themselves manifolds with corners. Given a particular boundary hypersurface H there exists a boundary defining function $\rho_H \colon X \longrightarrow \mathbb{R}_+$, meaning ρ_H is smooth, non-negative, that $\{\rho_H = 0\} = H$, and $d\rho_H$ is non-vanishing on H.)

To begin, recall the space of b-*vector fields*, $\mathcal{V}_{\mathrm{b}}(X)$, that is, vector fields on X which are tangent to the boundary. These are exactly those vector fields $V \in C^\infty(X; TX)$, defined and smooth on the whole of X, which over ∂X point along the boundary, i.e. which satisfy $V|_{\partial X} \in C^\infty(\partial X, T(\partial X))$. On M these can be described easily enough; in a neighborhood of the codimension 2 corner $\mathrm{mf} \cap \mathrm{cf}$ they are generated over $C^\infty(M)$ by the vector fields $\rho\partial_\rho, x\partial_x$, and ∂_θ, where here and below we abuse notation slightly by allowing $\theta \in \mathbb{S}^2$ to denote local coordinates on $\mathbb{S}^2$ (which can be accomplished locally on the sphere by dropping on of the three components of θ). Thus still near the corner,

$$V = a(\rho, x, \theta)\rho\partial_\rho + b(\rho, x, \theta)x\partial_x + \sum_{k\in\{1,2,3\}} c_k(\rho, x, \theta)\partial_{\theta_k}.$$

where the θ_k are whichever components of θ define local coordinates on $\mathbb{S}^2$, and where smoothness on M in this neighborhood simply means the a, b, c_k are smooth functions on $[0, 1)_\rho \times [0, 1)_x \times \mathbb{S}^2$. Near mf but away from mf $\cap$ cf, they can be written in terms of by $\rho\partial_\rho$ and the remaining coordinate vector fields (such as ∂_x, ∂_θ), while near cf away from mf, it suffices to use $r\partial_r$, ∂_t, and ∂_θ, always with coefficients which are smooth up to the boundary. It is straightforward to check that $\mathcal{V}_b$ is a Lie algebra, meaning for $V, W \in \mathcal{V}_b$, the commutator $[V, W] \in \mathcal{V}_b$; its universal enveloping algebra over $C^\infty(M)$ is (by definition) the algebra of b-differential operators and is denoted $\mathrm{Diff}^*_b(X)$. Near the codimension two corner mf $\cap$ cf, an operator $A \in \mathrm{Diff}^m_b(M)$ has the form

$$A = \sum_{|\alpha|+j+k\le m} a_{jk\alpha}(\rho, x, \theta)(\rho D_\rho)^j (xD_x)^k D^\alpha_\theta, \tag{5}$$

where the coefficients $a_{jk\alpha} \in C^\infty(M)$.

The b-pseudodifferential operators $\Psi^*_b(X)$ are the microlocalization of this algebra and formally consist of properly supported operators of the form (for $\Psi^*_b(M)$ near mf $\cap$ cf)

$$a(\rho, x, \theta, \rho D_\rho, xD_x, D_\theta),$$

where $a(\rho, x, \theta, \sigma, \xi, \eta)$ is a Kohn–Nirenberg symbol.

The space $\mathcal{V}_b(X)$ is additionally the space of sections of the b-tangent bundle ${}^bTX$, which is a smooth vector bundle over X with the feature that for a given boundary hypersurface H of X with boundary defining function ρ_H, the b-vector field $\rho_H\partial_{\rho_H}$ defines a *non-vanishing* section of ${}^bTX$ at H. Its dual bundle is denoted ${}^bT^*X$. On M near mf $\cap$ cf it is locally spanned over $C^\infty(M)$ by $d\rho/\rho$, dx/x, and $d\theta$ and we can write parametrize points in ${}^bT^*M$ by writing

We may thus regard the symbols of operators in $\Psi^*_b(X)$ as symbols on ${}^bT^*X$, and the principal symbol map, denoted σ_b, maps the classical subalgebra of Ψ^m_b to homogeneous functions of order m on ${}^bT^*X$.[3] In the particular case of b-differential operators on M, if A is given as above (5), we have

$$\sigma_b(A) = \sum_{|\alpha|+j+k=m} a_{jk\alpha}(\rho, x, \theta)\sigma^j\xi^k\eta^\alpha,$$

where σ, ξ, and η are the canonical fiber coordinates on ${}^bT^*M$ defined by specifying that the canonical one-form is given by

$$\sigma\frac{d\rho}{\rho} + \xi\frac{dx}{x} + \eta \cdot d\theta.$$

Up to this point all the operators we have discussed are *scalar*, meaning they act on functions. To include operators on vector valued functions one simply assumes that the coefficients $a_{jk\alpha}$ above lie in $C^\infty(M; Mat_N)$ meaning they are smooth functions values in $N \times N$ matrices, (for us typically $N = 4$). It will be clear from context below whether we are considering scalar or non-scalar operators.

[3]Recall that we can identify homogeneous functions on ${}^bT^*X$ of a given order with smooth functions on ${}^bS^*X$. In an abuse of notation, we often view $\sigma_b(A)$ as a smooth function on ${}^bS^*X$.

As in previous work [BW20], it is also convenient to identify a subalgebra of $\Psi_{\mathrm{b}}^*(X)$ essential for a commutator argument in Section 5. To do this, recall first that there is an action of $SO(3)$ on the spatial variables; given $R \in SO(3)$, then $R \cdot (t, x^1, x^2, x^3) = (t, R(x^1, x^2, x^3))$. This induces a left action on functions, $Rf = f \circ R^{-1}$ and thus allows us to make the following definition.

Definition 4. We say that a scalar pseudodifferential operator $A \in \Psi_{\mathrm{b}}^m$ is *invariant* if it is invariant with respect to the action of $SO(3)$ on functions, i.e. if for all $R \in SO(3)$ and all $f : C^\infty(M)$, $A(Rf) = RAf$.

Any scalar symbol invariant under the lifted action of $SO(3)$ on ${}^{\mathrm{b}}T^*M$ can be quantized to an invariant operator. Invariant operators commute with the angular operators Δ_θ and K [BW20, Lemma 4].

Accompanying the principal symbol map (which describes the leading order behavior of elements of $\Psi_{\mathrm{b}}^*(X)$ in terms of the filtration), there is another collection of maps measuring the leading order behavior of the operators at each boundary hypersurface. In our setting, we need only the map in each section: for $X = M$, we use the map associated to mf, while $X = \mathrm{mf}$ has only one boundary hypersurface. Together with the principal symbol, these maps measure the obstruction to compactness of b-operators. We need this notion below only in the case of b-differential operators, where it is simple to describe. This extra operator-valued symbol is the operator given by freezing the coefficients of powers of b-vector fields at the relevant boundary hypersurface. In the case of $X = M$, if A is given by

$$\sum_{|\alpha|+j+k \le m} a_{jk\alpha}(\rho, x, \theta)(\rho D_\rho)^j (x D_x)^k D_\theta^\alpha,$$

we then define the normal (or indicial) operator

$$N(A) = \sum_{|\alpha|+j+k \le m} a_{jk\alpha}(0, x, \theta)(\rho D_\rho)^j (x D_x)^k D_\theta^\alpha.$$

Recall that N is a homomorphism and its conjugation by the Mellin transform (described below in Section 3.4) yields the reduced normal operator (also called the indicial family)

$$\widehat{N}(A) = \sum_{|\alpha|+j+k \le m} a_{jk\alpha}(0, x, \theta)\sigma^j (x D_x)^k D_\theta^\alpha.$$

We then define the *boundary spectrum of* A (in the case of $X = M$; for $X = \mathrm{mf}$ we must change $C^\infty(\mathrm{mf})$ to $C^\infty(\mathbb{S}^2)$ below):

$$\operatorname{spec}_{\mathrm{b}}(A) = \{\sigma \in \mathbb{C} \mid \widehat{N}(A) \text{ is not invertible on } C^\infty(\mathrm{mf})\}.$$

This set plays two important roles in our context: it is a key ingredient in the identification of the domain of the essentially self-adjoint Hamiltonian $\mathcal{B}$ and, more centrally, it determines, through its relationship with polyhomogeneity (described below), the exponents seen in the asymptotic expansions of Theorem 2.

Further associated to an operator $A \in \Psi_{\mathrm{b}}^m(X)$ is its microsupport

$$\mathrm{WF}'_{\mathrm{b}}(A) \subset {}^{\mathrm{b}}S^*M.$$

The microsupport is a closed subset and is the essential support of the total symbol, just as in the usual pseudodifferential calculus. It obeys the usual microlocality property

$$\mathrm{WF}'_{\mathrm{b}}(AB) \subset \mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}'_{\mathrm{b}}(B).$$

We also use the notion of b-ellipticity at a point, which is equivalent (in the classical subalgebra) to the invertibility of the principal symbol. We postpone our discussion of the b-wavefront set of distributions to later as we require a variant of it in our estimates.

We also require a semiclassical version of the b-calculus on the boundary hypersurface mf. We use $\Psi^*_{\mathrm{b},h}(\mathrm{mf})$ to denote this space and refer the reader to previous work [BM19] and especially to the excellent paper of Gannot–Wunsch [GW18, Section 3] for more details. Analogues of the constructions above exist for the semiclassical calculus as well and are typically decorated with an h.

3.3. **Interaction with differential operators.** The proofs of the propagation estimates near the singularity of the potential rely on the understanding of the interaction between differential operators and the b-calculus.

3.3.1. *The homogeneous version.* We let $L^2(M)$ denote the space of square integrable functions with respect to a *mixed* b-*metric density*, that is, a density with the metric-induced behavior near cf with the b-induced behavior near mf. Concretely, if $\mu_{euc} = dx^0dx^1dx^2dx^2$ denotes the Euclidean density, then setting

$$\mu = \rho^4\mu_{euc},$$

we set

$$L^2(M) = L^2(M,\mu) = L^2(\mathbb{R}\times\mathbb{R}^3;\rho^4\mu_{euc}) = \rho^{-2}L^2(\mathbb{R}\times\mathbb{R}^3), \tag{6}$$

where $L^2(\mathbb{R}\times\mathbb{R}^3)$ denotes the standard L^2 space. Near $\mathrm{mf}\cap\mathrm{cf}$, the density μ is given by

$$\frac{d\rho}{\rho}\, x^2\, dx\, d\theta.$$

As the (standard) Sobolev space H^1 is the domain of the various operators we consider below, we use this space as the basis for the Sobolev spaces on M. We will use the space $H_{\mathrm{b}}^{1,0,0}(M)$ of distributions which are H^1 near the pole and H^1_{b} at mf. More precisely, $H_{\mathrm{b}}^{1,0,0}(M)$ is (by definition) equal to the standard b-Sobolev space $H^1_{\mathrm{b}}(\overline{\mathbb{R}^4})$ where $\overline{\mathbb{B}^4} = \overline{\mathbb{R}^4}$ is the radial compactification and the b refers to the behavior at $\partial\overline{\mathbb{B}^4}$. The identification of $H^1_{\mathrm{b}}(\overline{\mathbb{R}^4})$ with a space of distributions on M is realized by pullback via the map $M \longrightarrow \overline{\mathbb{R}^4}$ which collapses the $\mathbb{S}^2_\theta$ factors over $\{x=0\}$. For ψ supported near $\mathrm{mf}\cap\mathrm{cf}$, the $H_{\mathrm{b}}^{1,0,0}$ norm can be taken to be

$$\|\psi\|^2_{H_{\mathrm{b}}^{1,0,0}} = \int \left(|\rho\partial_\rho\psi|^2 + |\partial_x\psi|^2 + \left|\frac{1}{x}\nabla_\theta\psi\right|^2 + |\psi|^2\right)\frac{d\rho}{\rho}\, x^2\, dx\, d\theta.$$

For $m \geq 0$, we then let $H_{\mathrm{b}}^{1,m,0}(M)$ denote the Sobolev space of order m relative to $H_{\mathrm{b}}^{1,0,0}(M)$, i.e., fixing $A \in \Psi^m_{\mathrm{b}}(M)$ elliptic and invertible, we have $w \in H_{\mathrm{b}}^{1,m,0}(M)$ if $w \in H_{\mathrm{b}}^{1,0,0}(M)$ and $Aw \in H_{\mathrm{b}}^{1,0,0}(M)$. (This is independent of the choice of A.) In particular, the $H_{\mathrm{b}}^{1,m,0}$ norm near $\mathrm{mf}\cap\mathrm{cf}$ is given by

$$\|\psi\|^2_{H_{\mathrm{b}}^{1,m,0}} = \int \left(|\rho\partial_\rho A\psi|^2 + |\partial_x A\psi|^2 + \left|\frac{1}{x}\nabla_\theta A\psi\right|^2 + |A\psi|^2\right)\frac{d\rho}{\rho} x^2\, dx\, d\theta.$$

In the present manuscript we need only $m \geq 0$. Finally, let $H_{\mathrm{b}}^{1,m,l}(M) = \rho^\ell H_{\mathrm{b}}^{1,m,0}(M)$ denote the corresponding weighted spaces. A brief calculation in the region near the north pole ($x < c < 1$) with $x = r/t$ and $\rho = 1/t$ shows that membership in $H^1(\mathbb{R} \times \mathbb{R}^3)$ is equivalent to membership in $H_{\mathrm{b}}^{1,0,1}(M) \cap \rho^2 L^2(M)$.

Note that away from the singularity, $H_{\mathrm{b}}^{1,m,l}$ regularity is equivalent to b-regularity, meaning for $\chi(x)$ supported near $x = 0$,

$$(1 - \chi(x)) H_{\mathrm{b}}^{1,m,l} \subset H_{\mathrm{b}}^{m+1,l}(M).$$

where $H_{\mathrm{b}}^{m',l}(M)$ is the "standard" b$-$Sobolev space. To avoid excessive notation below we avoid the notation $H_{\mathrm{b}}^{m',l}(M)$.

We now describe our microlocal characterization of regularity, the wavefront set. We define the notion only in the bulk M. Although it would be natural to define the analogous notions (both homogeneous and semiclassical) on mf, our propagation estimates are stated explicitly in terms of operators in that region and so we omit those definitions here.

On the bulk M, we use $\mathrm{WF}_{\mathrm{b}}^{1,m,\ell}$ to describe a failure to lie in the space $H_{\mathrm{b}}^{1,m,\ell}$, while we use $\mathrm{WF}_{\mathrm{b}}^{0,m,\ell}$ to describe the "standard" b-wavefront set with respect to the underlying space $L^2(M)$ (with the metric volume form μ above.) In this direction we let $H_{\mathrm{b}}^{0,m,\ell}$ denote the b-spaces with respect to $L^2(M)$, meaning $u \in H_{\mathrm{b}}^{0,m,\ell}$ if an only if for all $A \in \Psi_{\mathrm{b}}^m(M)$, $\rho^{-\ell} A u \in L^2(M)$.

Definition 5. Suppose $u \in H_{\mathrm{b}}^{1,s,r}(M)$ for some s, r, and suppose $m, \ell \in \mathbb{R}$. We say $q \in {}^{\mathrm{b}}S^*M$ is *not* in $\mathrm{WF}_{\mathrm{b}}^{1,m,\ell}(u)$ if there is some $A \in \rho^{-\ell}\Psi_{\mathrm{b}}^m(M)$ that is elliptic at q and so that $Au \in H_{\mathrm{b}}^{1,m,\ell}$.

Similarly, for $u \in H_{\mathrm{b}}^{0,s,r}$ and $m, \ell \in \mathbb{R}$, we say that $q \in {}^{\mathrm{b}}S^*M$ is *not* in $\mathrm{WF}_{\mathrm{b}}^{0,m,\ell}(u)$ if there is some $A \in \rho^{-\ell}\Psi_{\mathrm{b}}^m(M)$ elliptic at q with $Au \in L^2(M, \frac{d\rho}{\rho}\, x^2 dx\, d\theta)$.

Although the definition of $\mathrm{WF}_{\mathrm{b}}^{0,m,\ell}(u)$ is nearly the same as that of the "standard" b-wavefront set, we keep the 0 in the notation as a reminder that the L^2 space is equipped with the rescaled metric density.

Throughout the arguments in Section 5 we rely on the Hardy inequality, which allows us to estimate the 0-th order terms near the singularity of the potential.

Lemma 6. *If $u \in H^1(\mathbb{R}^n)$ with $n \geq 3$, then*

$$\frac{(n-2)^2}{4} \int \frac{|u|^2}{r^2}\, dz \leq \int |\nabla u|^2 \, dz.$$

We use this inequality repeatedly in $\mathbb{R}^3$ and its analogues for distributions defined on mf near $x = 0$ and for distributions defined on M near mf, where it reads in both cases

$$\left\| x^{-1} u \right\| \leq 2 \left\| \partial_x u \right\|.$$

The following lemma is essentially from [MVW08, Lemma 8.6] (and similar to [Vas08, Lemma 2.8]) will be used in the commutator computations below.

Lemma 7. *If $A \in \Psi_{\mathrm{b}}^m$ with principal symbol a, then*

$$\left[\frac{1}{x}, A\right] = C_L \frac{1}{x} = \frac{1}{x} C_R,$$

where $C_\bullet \in \Psi_{\mathrm{b}}^{m-1}$ with

$$\sigma_{\mathrm{b}}(C_\bullet) = \frac{1}{i} \partial_\xi a.$$

Moreover,

$$[D_x, A] = B + C_L D_x,$$

with C_L as above and

$$B \in \Psi_{\mathrm{b}}^{m}, \quad \sigma_{\mathrm{b}}(B) = \frac{1}{i}\partial_\xi a.$$

Proof. The proof uses standard tools from the b-calculus, and therefore we sketch only the main steps. We discuss only the proof of the first statement, i.e. for the commutators with $1/x$, as the statement for D_x follows exactly as in the references given.

Writing $[x^{-1}, A] = x^{-1}[A, x]x^{-1}$ and using the fact that $x \in \Psi_{\mathrm{b}}^{0}(M)$ we obtain $[A, x] \in \Psi_{\mathrm{b}}^{m-1}(M)$. Using that $\sigma_{\mathrm{b},m-1}(i[A, x]) = \{A, x\}$, the Poisson bracket, and that in the coordinates $(\rho, x, \theta, \sigma, \xi, \eta)$ above the Hamilton vector field of a symbol a is

$$(\partial_\sigma a)\rho\partial_\rho - (\rho\partial_\rho a)\partial_\sigma + (\partial_\xi a)x\partial_x - (x\partial_x a)\partial_\xi + \sum_k (\partial_{\eta_k} a)\partial_{\theta_k} - (\partial_{\theta_k} a)\partial_{\eta_k}$$

we see that $\sigma_{\mathrm{b},m-1}(i[A, x]) = x\partial_\xi a$. It is now a standard fact from the b-calculus that both $C_L = x^{-1}[A, x]$ and $C_R = [A, x]x^{-1}$ lie in Ψ_{b}^{m-1} and have $\sigma_{\mathrm{b},m-1}(C_\bullet) = x^{-1}\sigma_{\mathrm{b},m-1}([A, x])$. □

It is also convenient to know we can microlocalize our estimates. The following lemma is essentially in previous work of the first author with Wunsch [BW20, Lemma 9, Lemma 12].

Lemma 8. *If $A, G \in \Psi_{\mathrm{b}}^{s}$ with $\mathrm{WF}_{\mathrm{b}}'(A) \subset \operatorname{ell} G$, then for all u with*

$$\mathrm{WF}_{\mathrm{b}}^{1,s,\ell}\, u \cap \mathrm{WF}_{\mathrm{b}}'\, G = \emptyset,$$

we may bound

$$\|Au\|_{H_{\mathrm{b}}^{1,0,\ell}} \leq C \left(\|Gu\|_{H_{\mathrm{b}}^{1,0,\ell}} + \|u\|_{H_{\mathrm{b}}^{1,0,\ell}} \right). \tag{7}$$

In particular, if $A \in \Psi_{\mathrm{b}}^{0}$ then

$$\|Au\|_{H_{\mathrm{b}}^{1,0,\ell}} \leq C \|u\|_{H_{\mathrm{b}}^{1,0,\ell}}. \tag{8}$$

The proof is identical to that in the referenced paper. The boundedness statement in (8) follows from the commutator formulas in Lemma 7. Once boundedness is established, the small calculus elliptic parametrix used to prove (7) is also valid in on M. (Note that there is no improvement in the weight ℓ.)

3.3.2. *The semiclassical version.* As mf blows down to a sphere $\mathbb{S}^3$, we can appeal to the standard notion of differential operators on mf. We denote by $H_h^1(\mathrm{mf})$ the lift of the semiclassical Sobolev space $H_h^1(\mathbb{S}^3)$ to mf via the blow-down map. For $u \in H_h^1(\mathrm{mf})$, in particular $u \in L^2(\mathbb{S}^3)$, and the $H_h^1(\mathrm{mf})$ norm controls the L^2 norms of $h\partial_x u$, and $\frac{h}{x}\partial_\theta u$.

The classical analogues of the following lemmas can be found in prior work [BW20].

Lemma 9. *If $A \in \Psi_{\mathrm{b},h}^{0}$ then*

$$\|Av\|_{H_h^{\pm 1}} \leq C \|v\|_{H_h^{\pm 1}}.$$

Lemma 10. *Suppose $A, G \in \Psi_{\mathrm{b},h}^{0}$ are supported near $x = 0$ and $\mathrm{WF}_{\mathrm{b},h}'(A) \subset \mathrm{ell}_{\mathrm{b},h}(G)$. For any k, N, there is a constant C so that*

$$\|Au\|_{H_h^{\pm 1}} \leq C \|Gu\|_{H_h^{\pm 1}} + Ch^k \|\chi u\|_{H_h^{\pm 1}} + Ch^k \|(1-\chi)u\|_{H_h^{-N}},$$

where $\chi \in C^\infty$ is identically 1 on the support of G.

Proof. The result follows from a standard elliptic parametrix construction: we can find $B \in \Psi^0_{\mathrm{b},h}$ and $R \in \Psi^{-\infty}_{\mathrm{b},h}$ so that on the microsupport of A, we have

$$\mathrm{Id} = BG + h^k R.$$

and so

$$\|Au\|_{H_h^{\pm 1}} \le \|ABGu\|_{H_h^{\pm 1}} + h^k \|ARu\|_{H_h^{\pm 1}}.$$

Lemma 9 bounds the first term; to bound the second term we insert cutoff functions. Near $x = 0$ the term is bounded by $\|\chi u\|_{H_h^{\pm 1}}$ while away from $x = 0$, we exploit that R is an operator of order $-\infty$. □

As in the bulk setting, we repeatedly use the Hardy inequality on mf, where it reads

$$\left\|hx^{-1}u\right\| \le 2\|h\partial_x u\|.$$

Clearly the h factors out from both sides, but this phrasing of the inequality emphasizes that h/x will be estimated as a semiclassical operator of order 1.

We also need to understand the commutators of $1/x$ and hD_x with semiclassical b-pseudodifferential operators.

Lemma 11. *If $A \in \Psi^0_{\mathrm{b},h}$, then*

$$\left[\frac{1}{x}, A\right] = \frac{h}{x}C_R = C_L\frac{h}{x},$$

with

$$C_L, C_R \in \Psi^{-1}_{\mathrm{b},h}, \quad \sigma_{\mathrm{b},h}(C_L) = \sigma_{\mathrm{b},h}(C_R) = \frac{1}{i}\partial_\xi \sigma_{\mathrm{b},h}(A).$$

Moreover,

$$\frac{1}{h}[hD_x, A] = B + ChD_x,$$

with

$$B \in \Psi^0_{\mathrm{b},h}, \qquad C \in \Psi^{-1}_{\mathrm{b},h},$$
$$\sigma_{\mathrm{b},h}(B) = \frac{1}{i}\partial_x a, \qquad \sigma_{\mathrm{b},h}(C) = \frac{1}{i}\partial_\xi a.$$

3.4. **The Mellin transform and polyhomogeneity.** Just as the Fourier transform is a key element in the study of translation-invariant operators, we consider here the Mellin transform, its analogue for dilation-invariant operators. For our purposes, we need only the Mellin transform associated to the single boundary hypersurface mf. Suppose u is a distribution on M localized near mf (which is defined by the function ρ). The Mellin transform of u associated to mf is defined by

$$\widetilde{u_\sigma} \equiv \mathcal{M}_{\mathrm{mf}} u(\sigma, y) = \int_0^\infty \chi(\rho)u(\rho, y)\rho^{-i\sigma}\frac{d\rho}{\rho},$$

where y denote the remaining coordinates near mf and χ is a smooth compactly supported function localizing near $\rho = 0$. The Mellin transform has many rich properties analogous to those enjoyed by the Fourier transform and many of its mapping properties can be deduced from those of the Fourier transform by a change of variables.

The Mellin transform is particularly helpful in the study of asymptotic expansions in powers of ρ (the boundary defining function for the hypersurface mf) and $\log \rho$. For simplicity, we first discuss the case where our manifold has only a single boundary hypersurface, i.e., when we have a manifold with boundary X. In particular, we recall from Melrose [Mel93, Section 5.10] the definition of a polyhomogeneous conormal distribution in this setting. If u is a distribution on a manifold with boundary X, we write

$$u \in \mathcal{A}^{E}_{\mathrm{phg}}(M) \quad (u \text{ is polyhomogeneous with index set } E)$$

if u is conormal to ∂X and

$$u \sim \sum_{(z,k)\in E} \rho^{iz} (\log \rho)^{k} a_{z,k},$$

where the $a_{z,k}$ are smooth functions on ∂X. Here the expansion should be interpreted as an asymptotic series as $\rho \to 0$ and E is an *index set* and must satisfy[4]

- $E \subset \mathbb{C} \times \{0, 1, 2, \dots\}$,
- E is discrete,
- if $(z_j, k_j) \in E$ with $|(z_j, k_j)| \to \infty$, then $\operatorname{Im} z_j \to -\infty$,
- if $(z, k) \in E$, then $(z, l) \in E$ for all $l = 0, 1, \dots, k-1$, and
- if $(z, k) \in E$, then $(z - ij, k) \in E$ for all $j = 1, 2, \dots$.

With these conventions, the functions that are smooth up to ∂X are polyhomogeneous with index set

$$\mathcal{E}_0 = \{(-ik, 0) \mid k = 0, 1, 2, \dots\}.$$

The distributions in $\mathcal{A}^{E}_{\mathrm{phg}}(X)$ can be characterized by the Mellin transform, in which case the Mellin transform is meromorphic with appropriate decay estimates in σ and $(z, k) \in E$ if the Mellin transform has a pole of order $k+1$ at z. Polyhomogeneous distributions can also be characterized by testing with radial vector fields. In the case of a manifold with boundary X, let R denote the radial vector field ρD_ρ. Then $u \in \mathcal{A}^{E}_{\mathrm{phg}}(X)$ if for all A, there is some γ_A with $\gamma_A \to +\infty$ as $A \to +\infty$ so that

$$\left(\prod_{(z,k)\in E, \operatorname{Im} z > -A} (R - z) \right) u \in \rho^{\gamma_A} H^{\infty}_{\mathrm{b}}(X).$$

Here $H^{\infty}_{\mathrm{b}}(X)$ is the standard b-Sobolev space of order ∞ and indicates iterated regularity under the application of arbitrarily many b-vector fields.

Our main theorem concerns joint polyhomogeneity jointly at $\mathcal{I}^+$ and C_+ in $[M; S_+]$, which is a manifold with codimension 2 corners. In this case one wants a polyhomogeneous distribution to have compatible expansions at the two faces. The index sets seen in the expansions at the two faces are typically different and so we use the notation $\mathcal{E} = (E_1, E_2)$ to denote the pair of index sets and $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}([M; S_+])$ to denote this space of distributions.

To test for polyhomogeneity at multiple boundary hypersurfaces, it suffices to test *individually* at each one with uniform estimates at the other. The following lemma is due independently to Melrose [Mel96, Chapter 4] and Mazzeo [Eco93, Appendix].

Lemma 12 (Mazzeo, Melrose)**.** *Suppose H_ℓ, ℓ are boundary hypersurfaces of a manifold with corners X and suppose ρ_ℓ defines H_ℓ. Let R_ℓ denote $\rho_\ell D_{\rho_\ell}$, the radial vector field at the*

[4] As in the first author's previous works [BVW15, BVW18, BM19], we adopt the convention of Melrose's unpublished book [Mel96] rather than the other reference [Mel93].

ℓ-th boundary hypersurface. Suppose that for each ℓ, there exists a γ', and for all A there is a γ_A with $\lim_{A\to\infty}\gamma_A = +\infty$ such that

$$\left(\prod_{(z,k)\in E_\ell, \operatorname{Im} z > -A} (R_\ell - z)\right) u \in \rho_\ell^{\gamma_A}\rho^{\gamma'} H_{\mathrm{b}}^{\infty}(X), \tag{9}$$

where $\rho^{\gamma'}$ denotes the multiproduct of the defining functions for H_j, $j \neq \ell$. Then $u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M)$, where $\mathcal{E} = (E_1, E_2, \dots)$.

In other words, one can test for polyhomogeneity with radial vector fields as in the setting of a manifold with boundary provided that the remainder improves the decay at the hypersurface in question at no cost to the growth or decay at the other boundary hypersurfaces.

4. Analytic preliminaries

4.1. **Related wave equations and domains.** We introduce two first order operators related to the Dirac operator via a conjugation

$$\mathcal{L} = i\rho^{-2-i\mathbf{Z}}\gamma^0 \partial\!\!\!/_{\frac{\mathbf{Z}}{r}}\rho^{1+i\mathbf{Z}},$$
$$\widetilde{\mathcal{L}} = i\rho^{-2-i\mathbf{Z}}\partial\!\!\!/_{\frac{\mathbf{Z}}{r}}\gamma^0\rho^{1+i\mathbf{Z}},$$

as well as the conjugated second order operator

$$P = -\rho^{-3-i\mathbf{Z}}\partial\!\!\!/^2_{\frac{\mathbf{Z}}{r}}\rho^{1+i\mathbf{Z}}.$$

To uncover the relationship between these three operators, we introduce three additional "l-based" families of operators

$$\mathcal{L}_\ell = \rho^{-\ell}\mathcal{L}\rho^\ell, \quad \widetilde{\mathcal{L}}_\ell = \rho^{-\ell}\widetilde{\mathcal{L}}\rho^\ell, \quad P_\ell = \rho^{-\ell}P\rho^\ell,$$

where $\ell \in \mathbb{R}$. The notational advantages will be evident momentarily. Observe that, with respect to the volume form $\frac{d\rho}{\rho}x^2\,dx\,d\theta$, as long as $\ell \in \mathbb{R}$,

$$\mathcal{L}_\ell^* = \mathcal{L}_{1-\ell}, \quad \widetilde{\mathcal{L}}_\ell^* = \widetilde{\mathcal{L}}_{1-\ell}.$$

so that

$$P = \widetilde{\mathcal{L}}_1\mathcal{L} = \widetilde{\mathcal{L}}^*\mathcal{L},$$
$$\gamma^0 P_\ell \gamma^0 = \mathcal{L}_{\ell+1}\widetilde{\mathcal{L}}_\ell.$$

The latter relationship plays a role in the boundary section, as

$$P^* = \gamma^0 P \gamma^0. \tag{10}$$

In the sequel, we will require explicit expressions for our operators near C_+. For convenience, we record this below, with $x = r/t$ and $\rho = 1/t$:

$$\begin{aligned}\mathcal{L} &= i\rho^{-2-i\mathbf{Z}}\left(\partial_t + \frac{i\mathbf{Z}}{r} + \alpha_r\left(\partial_r + \frac{1}{r} - \frac{1}{r}\beta K\right)\right)\rho^{1+i\mathbf{Z}} \\ &= -i\rho\partial_\rho - ix\partial_x - i + \mathbf{Z} - \frac{\mathbf{Z}}{x} + \alpha_r\left(i\partial_x + \frac{i}{x} - \frac{i}{x}\beta K\right), \\ P &= \rho^{-3-i\mathbf{Z}}\left[-\left(\partial_t + \frac{i\mathbf{Z}}{r}\right)^2 + \sum \partial_j^2 - \frac{i\mathbf{Z}}{r^2}\alpha_r\right]\rho^{1+i\mathbf{Z}} \\ &= \left(\rho\partial_\rho + x\partial_x + 1 + i\mathbf{Z} - \frac{i\mathbf{Z}}{x}\right)^*\left(\rho\partial_\rho + x\partial_x + 1 + i\mathbf{Z} - \frac{i\mathbf{Z}}{x}\right) - D_x^* D_x - \frac{1}{x^2}\Delta_\theta - \frac{i\mathbf{Z}}{x^2}\alpha_r.\end{aligned}$$

The corresponding Mellin transformed and semiclassical $\mathcal{L}$ operator takes the forms

$$\begin{aligned}\widehat{\mathcal{L}}_\sigma &= xD_x + \sigma - i + \mathbf{Z} - \frac{\mathbf{Z}}{x} - \alpha_r\left(D_x - \frac{i}{x} + \frac{i}{x}\beta K\right), \\ \widehat{\mathcal{L}}_h &= h\widehat{\mathcal{L}}_\sigma = hxD_x + z - ih + h\mathbf{Z} - \frac{h\mathbf{Z}}{x} - \alpha_r\left(hD_x - \frac{ih}{x} + \frac{ih}{x}\beta K\right),\end{aligned}$$

where $h = |\sigma|^{-1}$ and $z = \frac{\sigma}{|\sigma|}$.

Similarly, the second-order operators are given by

$$\begin{aligned}\widehat{P}_\sigma = {}& \left(xD_x - \frac{\mathbf{Z}}{x} + \sigma + \mathbf{Z} - i\right)^*\left(xD_x - \frac{\mathbf{Z}}{x} + \sigma + \mathbf{Z} - i\right) + 2i(\operatorname{Im}\sigma)\left(xD_x - \frac{\mathbf{Z}}{x} + \sigma + \mathbf{Z} - i\right) \\ & - D_x^* D_x - \frac{1}{x^2}\Delta_{\mathbb{S}^2} - \alpha_r\frac{i\mathbf{Z}}{x^2}, \\ \widehat{P}_h = {}& h^2\widehat{P}_\sigma = \left(hxD_x - \frac{h\mathbf{Z}}{x} + z + h\mathbf{Z} - ih\right)^*\left(hxD_x - \frac{h\mathbf{Z}}{x} + z + h\mathbf{Z} - ih\right) \\ & + 2i(\operatorname{Im} z)\left(hxD_x - \frac{h\mathbf{Z}}{x} + z + h\mathbf{Z} - ih\right) - (hD_x)^*(hD_x) - \frac{h^2}{x^2}\Delta_{\mathbb{S}^2} - \alpha_r\frac{ih^2\mathbf{Z}}{x^2}.\end{aligned}$$

We do not need the precise forms of the operators near C_- until Section 8; as we use a different set of coordinates in that section, we do not record the forms here.

We observe now that as the indicial operator of $\widehat{\mathcal{L}}_\sigma$ agrees with that of $\mathcal{B}$ near $x = 0$, the domain of $\widehat{\mathcal{L}}_\sigma$ also consists of functions that are H^1 near the singularity. We provide a sketch of the proof in Section 6.1 below.

4.2. **The radial sets.** Classical propagation of singularities arguments show that wavefront set is propagated along integral curves of the Hamilton vector field within the characteristic set; the arguments become more complicated when the vector field is singular (as near the singularity of the potential) or proportional to the radial vector field. In the latter case, we call the subset the radial set.

Our treatment of the estimates near the radial set is in terms of the second-order operators P and $\widehat{P}_\sigma$. In both settings it is convenient to replace the coordinate x with the coordinate $v = 1 - x$.

We consider first the homogeneous (bulk) version (i.e., for P). As the b-principal symbol of P on mf agrees with that of the wave operator on Minkowski space, the set of radial points must agree as well. In particular, as in previous work [BVW15, Section 3.6], the radial set $\mathcal{R}$ is exactly

$$\mathcal{R} = \{(\rho, v, \theta, \sigma, \gamma, \eta) : \rho = v = 0, \eta = 0, \sigma = 0\}.$$

As it projects to $S_+ \cup S_-$, the set $\mathcal{R}$ naturally splits into two components $\mathcal{R}_\pm$ according to whether the component lies over S_+ or S_-. The radial set propagation arguments are structured along *thresholds*: near the past radial set S_-, we can propagate regularity *out* of the radial set provided we have enough a priori regularity of our solution. (This will be easily achieved as we consider the forward solution, which necessarily vanishes near S_-.) At the future radial set S_+, we can propagate regularity *in*, but only up to the threshold.

We now consider the operator on the boundary $\widehat{P}_\sigma$. The analysis of $\widehat{P}_\sigma$ near the radial sets follows from previous work on the wave operator on Minkowski space; specifically analysis of the conjugated, rescaled, Mellin-transformed normal operator $\widehat{N}(\rho^{-3}\Box\rho)(\sigma)$. Indeed, the operator $\widehat{P}_\sigma$ has the same semiclassical principal and subprincipal symbol as $\widehat{N}(\rho^{-3}\Box\rho)Id$. We can therefore use relevant results in [Vas13, BVW15]. In particular, the characteristic set of the semiclassical differential operator $\widehat{N}(\rho^{-3}\Box\rho)(\sigma)$ is not homogeneous, and is understood as a submanifold of the fiber-compactified cotangent bundle $\overline{T}^*(\mathrm{mf})$. (Note that away from the poles this is canonically identified with $T^*(\mathbb{S}^3)$.) It is shown in [BVW15] that the characteristic set of $\widehat{N}(\rho^{-3}\Box\rho)$ (and therefore of $\widehat{P}_\sigma$), admits smooth families of radial sets given by

$$\Lambda^\pm := \partial\overline{N}^* S_\pm, \tag{11}$$

Concretely, near the fiber boundary of $\overline{N}^* S_\pm$, one has coordinates $(v, \theta, \nu, \tilde{\eta})$ where (v, θ) are spatial coordinates on $\mathbb{S}^3$ near $S_\pm$ with θ coordinates on $S_\pm$ and v defining $S_\pm$, $\nu = 1/|\gamma|$ and $\tilde{\eta} = \eta/|\gamma|$ where γ is dual to v and η dual to θ. Then

$$\Lambda^\pm = \{(v, \theta, \nu, \eta) \mid \nu = 0, v = 0, \tilde{\eta} = 0\}.$$

The boundaries of $N^*S_\pm$, denoted $\partial N^*S_\pm$, in the radial compactification of the fibers of $T^*\,\mathrm{mf}$ act as sources or sinks for a rescaling of the Hamilton vector field.[5]

The global structure of the bicharacteristic flow of $\widehat{P}_\sigma$, which is explained in detail in [BVW15], has the following general structure. The (classical) characteristic set $\tilde{\Sigma}$ of $\widehat{P}_\sigma$ lies in $\overline{T}^*(\mathrm{mf})$ and lies entirely above C_0. It consists of two components $\tilde{\Sigma}^\pm$; along $\tilde{\Sigma}^\pm$ the Hamilton flow limits to $\Lambda^\pm$ as the flow parameter goes to ∞, and to $\Lambda^\mp$ as it goes to $-\infty$.

4.3. **Variable-order Sobolev spaces.** As in prior work [BVW15, BVW18, BM19], we aim to show that the operator $\widehat{\mathcal{L}}_\sigma$ on mf is Fredholm on appropriate spaces. In order to do this, we aim to propagate regularity from S_- to S_+ and so the spaces on which it is Fredholm should include functions that are more regular than some threshold at the past radial set and less regular than the threshold at the future radial set. As the two thresholds agree, we employ variable-order Sobolev spaces. Complicating the definition is our desire to guarantee enough regularity near the singularity so that we can carry out our propagation estimates there.

[5]Whether they are sources or sinks depends on the sign of γ nearby and so proving the estimates requires treating the two components of $N^*S_\bullet \setminus 0$ separately. As this argument is identical to the one in Minkowski space, we will omit it and so do not provide the components with unique names.

We therefore define a smooth regularity function $s_{\mathrm{ftr}} : \mathrm{mf} \to \mathbb{R}$.

(1) The function s_{ftr} is constant near $S_\pm$ and $s_{\mathrm{ftr}} \equiv 0$ in $\{x < 1/4\}$;
(2) Within C_0, s_{ftr} is monotonically decreasing as a function of t/r, and constant in neighborhoods of S_+ and S_-
(3) $s_{\mathrm{ftr}}|_{S_+} < 1/2 + \operatorname{Im}\sigma$ and $s_{\mathrm{ftr}}|_{S_-} > 1/2 + \operatorname{Im}\sigma$, the threshold exponents at $\Lambda^\pm$.

The monotonicity in (2) ensures that the lift of s_{ftr} to $T^*(\mathrm{mf})$ is monotonically decreasing along the flow from Λ^- to Λ^+ within the characteristic set of $\widehat{\mathcal{L}}_\sigma$. (This means that it is *increasing* as a function of the flow parameter on the component of $\tilde{\Sigma}$ for which Λ^- is a source.) As the (classical) characteristic set of $\widehat{\mathcal{L}}_\sigma$ lives above the closure of C_0, the condition at $x = 0$ is easy to satisfy.

We further define

$$s_{\mathrm{past}} = -s_{\mathrm{ftr}},$$

which satisfies related properties; it is monotonically *increasing* (hence decreasing along the flow in the opposite direction) and satisfies $\pm s_{\mathrm{past}}|_{S_\pm} > \mp(1/2 + \operatorname{Im}\sigma)$.

Motivated by previous work [BVW15, Appendix A], we define the variable order Sobolev spaces $H^{s_{\mathrm{ftr}}}(\mathbb{S}^3)$ and $H^{s_{\mathrm{past}}}(\mathbb{S}^3)$ on the sphere. As $L^2(\mathrm{mf}) = L^2(\mathbb{S}^3)$ and $s_{\mathrm{ftr}} = s_{\mathrm{past}} = 0$ near the poles, there is a canonical identification of the spaces $H^{s_{\mathrm{ftr}}}$ (or $H^{s_{\mathrm{past}}}$) and $L^2(\mathrm{mf})$ locally near the poles. We may therefore identify the variable order Sobolev spaces on the sphere as consisting of distributions on mf.

We now define the $\mathcal{X}$ and $\mathcal{Y}$ spaces based on the variable order spaces. In an abuse of notation, we use $\mathcal{X}^s$ and $\mathcal{Y}^s$ to denote spaces based on $H^{s_{\mathrm{ftr}}}$ and $\mathcal{X}^{s^*}$ and $\mathcal{Y}^{s^*}$ to denote those based on $H^{s_{\mathrm{past}}}$. We first define the $\mathcal{Y}$ spaces.

$$\mathcal{Y}^s = H^{s_{\mathrm{ftr}}}, \quad \mathcal{Y}^{s^*} = H^{s_{\mathrm{past}}},$$

equipped with the inherited norms. The $\mathcal{X}$ spaces are then defined by

$$\mathcal{X}^s = \{u \in \mathcal{Y}^s \mid \widehat{\mathcal{L}}_\sigma u \in \mathcal{Y}^s\},$$
$$\mathcal{X}^{s^*} = \{u \in \mathcal{Y}^{s^*} \mid \widehat{\mathcal{L}}_\sigma^* u \in \mathcal{Y}^{s^*}\}.$$

The norms on the $\mathcal{X}$ spaces are the graph norms, i.e.,

$$\|u\|^2_{\mathcal{X}^s} = \|u\|^2_{\mathcal{Y}^s} + \left\|\widehat{\mathcal{L}}_\sigma u\right\|^2_{\mathcal{Y}^s}.$$

Similarly, we define semiclassical spaces

$$\mathcal{Y}^s_h = H^{s_{\mathrm{ftr}}}_h, \qquad \mathcal{X}^s_h = \{u \in \mathcal{Y}^s_h \mid \widehat{\mathcal{L}}_h u \in \mathcal{Y}^s_h\},$$

and similarly for $s^* = s_{\mathrm{past}}$.

4.4. **Compressed characteristic set.** As in previous work [BW20], our treatment of the propagation of b-regularity is strongly influenced by the work of Vasy on manifolds with corners [Vas08].

In the bulk, the main propagation results near the singularity take place inside the *compressed characteristic set*, which is the appropriate extension of the ordinary characteristic set to the boundary setting. Near cf but away from mf, we refer the reader to prior work [BW20] for a discussion of this set. We limit our discussion here to a neighborhood of $C_+ \cap \mathrm{cf}$.

In coordinates associated to the canonical one-form

$$\underline{\sigma}\, d\rho + \underline{\xi}\, dx + \underline{\eta} \cdot d\theta$$

on T^*M, the characteristic set Σ is given by

$$\Sigma = \left\{ (\rho, x, \theta, \underline{\sigma}, \underline{\xi}, \underline{\eta}) \mid \left(\rho\underline{\sigma} + x\underline{\xi}\right)^2 - \underline{\xi}^2 - \frac{1}{x^2}\left|\underline{\eta}\right|^2 = 0 \right\}.$$

The *compressed characteristic set* $\dot{\Sigma}$, originally due to Melrose–Sjöstrand [MS78, MS82], is the closure of the image of the characteristic set under the natural map $T^*M \to {}^{\mathrm{b}}T^*M$. In the coordinates associated to the canonical one-form

$$\sigma\frac{d\rho}{\rho} + \xi\frac{dx}{x} + \eta \cdot d\theta$$

on ${}^{\mathrm{b}}T^*M$, $\dot{\Sigma}$ has the following form over $x = 0$:

$$\dot{\Sigma}|_{x=0} = \left\{ (\rho, x = 0, \theta, \sigma, \xi = 0, \eta = 0) \mid \theta \in \sigma^2, \sigma \neq 0 \right\}.$$

On the boundary mf, the characteristic sets of $\widehat{\mathcal{L}}_\sigma$ and $\widehat{P}_\sigma$ are bounded away from cf (as the operators are elliptic there). On the other hand, the operators $\widehat{\mathcal{L}}_h$ and $\widehat{P}_h$ are not *semiclassically* elliptic there. We therefore consider the analogous construction in the semiclassical setting on mf. In terms of the coordinates associated to the canonical one-form

$$\underline{\xi}\,dx + \underline{\eta} \cdot d\theta$$

on $T^*\,\mathrm{mf}$, the semiclassical characteristic set near $x = 0$ is given by

$$\Sigma_h = \left\{ (x, \theta, \underline{\xi}, \underline{\eta}) \mid (x\underline{\xi} + z)^2 - \underline{\xi}^2 - \frac{1}{x^2}\left|\underline{\eta}\right|^2 = 0 \right\}.$$

The semiclassical compressed characteristic set is again the closure of the image of Σ_h under the map $T^*\,\mathrm{mf} \to {}^{\mathrm{b}}T^*\,\mathrm{mf}$. In terms of the coordinates given by

$$\xi\frac{dx}{x} + \eta \cdot d\theta,$$

on ${}^{\mathrm{b}}T^*\,\mathrm{mf}$, $\dot{\Sigma}_h = \{x^2(\xi + z)^2 - (\xi^2 + |\eta|^2) = 0\}$, so over $x = 0$ we get the simple expression

$$\dot{\Sigma}_h|_{x=0} = \left\{ (x = 0, \theta, \xi = 0, \eta = 0) \mid \theta \in \mathbb{S}^2 \right\}. \tag{12}$$

5. Propagation in the bulk

The aim of this section is to prove that the forward solutions lie in an appropriate weighted Sobolev space and possess additional regularity; this regularity is expressed in terms of iterated application of "module derivatives", which we define now.

Definition 13. Let $\mathcal{M} \subset \Psi^1_{\mathrm{b}}(M)$ denote the $\Psi^0_{\mathrm{b}}(M)$-module of pseudodifferential operators with principal symbol vanishing on the radial set $\mathcal{R}$.

The module $\mathcal{M}$ is closed under commutators and is generated over Ψ^0_{b} by $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, ∂_θ, and Id.

Theorem 14. *If ψ is the forward solution of $i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g$, where $g \in C^\infty_c(M)$, then there are $m, \gamma \in \mathbb{R}$ with $1 + m + \gamma < 1/2$ so that $\psi \in H^{1,m,\gamma}_{\mathrm{b}}$ and for each $N \in \mathbb{N}$ and $A \in \mathcal{M}^N$, we have $A\psi \in H^{1,m,\gamma}_{\mathrm{b}}$.*

Before we proceed to the proof we recall some facts about the domain of the Hamiltonian $\mathcal{B}$ which are discussed in detail in [BW20]. The operator $\mathcal{B}$ is essentially self-adjoint (with core domain $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$) and its unique self-adjoint domain is

$$\mathcal{D} = \mathcal{D}(\mathcal{B}) = r^{-1/2} H_{\mathrm{b}}^1(\mathbb{R}^3),$$

where H_{b}^1 is the Sobolev space based on $L^2(\frac{dr}{r} d\theta)$ which is "b" at $r = 0$ and "scattering" at infinity, meaning $u \in H_{\mathrm{b}}^1$ if, for $\chi = \chi(r) \in C_0^\infty(\mathbb{R}_+)$ is identically one near $r = 0$ then $\{1, r\partial_r, \partial_{\theta_i}\}\chi u \in L^2$ while $\{1, \partial_r, \frac{1}{r}\partial_{\theta_i}\}(1-\chi)u \in L^2$. The powers $\mathcal{D}^m = \mathrm{Dom}(\mathrm{Id} + \mathcal{B}^2)^{m/2}$ are preserved by the forward propagator. Following [BW20], we call a solution $i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g$ **admissible** provided $\psi \in C(\mathbb{R}; \mathcal{D}^m)$ for some m. (This statement is local in time. For our analysis near the poles we assume that ψ is the forward solution.)

The first part of the statement follows from the following proposition:

Proposition 15. *If ψ is the forward solution of $i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g$, where $g \in C_c^\infty(M)$, then ψ is admissible and $\psi \in H_{\mathrm{b}}^{1,m,\gamma}$ for some m, γ.*

Proof. As the Hamiltonian is self-adjoint and the inhomogeneous term is compactly supported, the spatial L^2 norm of ψ is bounded for all time, i.e., there is some constant (depending on g) so that

$$\int_{\mathbb{R}^3} |\psi|^2 r^2 \, dr \, d\theta < C(g).$$

In particular, we have that

$$\int_{\mathbb{R}\times\mathbb{R}^3} t^{-1-2\epsilon} |\psi|^2 r^2 \, dr \, d\theta \, dt < \widetilde{C}(g, \epsilon).$$

In terms of $\rho = 1/t$ and $x = r/t$ (i.e., near the northern cap), we therefore have that

$$\int \rho^{-3+2\epsilon} |\psi|^2 \frac{d\rho}{\rho} \, dx \, d\theta < \widetilde{C}(g, \epsilon),$$

i.e., $\psi \in \rho^{\frac{3}{2}-\epsilon} L^2$ (recall that we use $\frac{d\rho}{\rho} \, dx \, d\theta$ as the volume form for this space).

Note also that the characterization of the domain (described in Section 4.1) of the operator shows that ψ has bounded energy for all time, i.e., there is some bound (depending on g) so that

$$\int_{\mathbb{R}^3} \left(|\partial_t \psi|^2 + |\nabla \psi|^2 \right) \, dx < C(g)$$

for all time. Integrating this bound (with a weight in t) shows that in fact $\psi \in t^{1/2+\epsilon} H^1(\mathbb{R} \times \mathbb{R}^3)$ and therefore lies in $H_{\mathrm{b}}^{1,0,1/2-\epsilon}$ near the pole $(x = 0, \rho = 0)$.

As ψ is tempered and lies in $H_{\mathrm{b}}^{1,0,1/2-\epsilon}$ near the pole, we can conclude that $\psi \in H_{\mathrm{b}}^{1,m,\gamma}$ for some m and γ. □

As a corollary to the proof of the above proposition, we record a result useful in the estimation of error terms below.

Corollary 16. *Suppose ψ the forward solution of $i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g$ and $g \in C_c^\infty$. If $\chi \in C^\infty(M)$ is supported in $\{x < 1\}$, then $\chi\psi \in H_{\mathrm{b}}^{1,0,1/2-\epsilon}$ for all $\epsilon > 0$.*

Once we know that ψ lies in some $H_{\mathrm{b}}^{1,m,\gamma}$, we can decrease the weight to guarantee that $m+\gamma<1/2$. The remainder of this section is devoted to the proof of module regularity, which proceeds by propagation of singularities arguments. In regions where the operator $\mathcal{L}$ is microlocally elliptic, classical microlocal estimates suffice. Similarly, where it is microlocally hyperbolic we use the b-version of Hörmander's theorem to propagate the regularity from one region to another. We therefore focus our attention on the radial points $N^*S_\pm$ (where the operator is neither elliptic nor hyperbolic) and on the regions near the singularity of the Coulomb potential where the operator is singular.

5.1. **The radial set.** At N^*S_+, the Hamilton vector field of the operator P is radial and so we appeal to radial points estimates of Vasy [Vas13]. The following proposition has the same proof as in previous work of the first author [BVW18]. To match prior work, we phrase the result first in terms of $\mathcal{L}$ rather than $\not{\partial}_{\mathbf{Z}/r}$.

Proposition 17 (c.f. [BVW18, Proposition 5.4])**.** *Suppose $u\in H_{\mathrm{b}}^{1,-\infty,\ell}$ for some ℓ and $\mathcal{L}u\in H_{\mathrm{b}}^{0,m,\ell}$. If we further assume that $u\in H_{\mathrm{b}}^{1,m-1,\ell}$ on a punctured neighborhood $U\setminus N^*S_+$ of N^*S_+ in S^*M, then for $m'\leq m$ with $m'+\ell<1/2$, $u\in H_{\mathrm{b}}^{1,m'-1,\ell}$ at N^*S_+. Further, for $N\in\mathbb{N}$ with $m'+N\leq m$ and for $A\in\mathcal{M}^N$, $Au\in H_{\mathrm{b}}^{1,m'-1,\ell}$ at N^*S_+ as well.*

In particular if $\mathcal{L}u\in H_{\mathrm{b}}^{0,\infty,\ell}$ and $u\in H_{\mathrm{b}}^{1,\infty,\ell}$ on the punctured neighborhood, then we conclude that u has infinite order module regularity, i.e. that N in $A\in\mathcal{M}^N$ is arbitrary.

Proof. From the assumptions, we have that $Pu\in H_{\mathrm{b}}^{0,m-1,\ell}$, and the proposition follows from the cited source. □

Remark 18. Since the conclusion of the theorem is drawn away from the singularity of the potential, the regularity at the poles is irrelevant, and one could equivalently assume that $u\in H_{\mathrm{b}}^{0,m,\ell}$ on the punctured neighborhood and conclude that $u\in H_{\mathrm{b}}^{0,m,\ell}$ at N^*S_+.

5.2. **Near the singularity of the potential.** Previous work of the first author [BW20] establishes the following propagation of singularities estimate. Note that this is a statement at finite times.

Theorem 19 ([BW20, Theorem 22])**.** *If ψ is an admissible solution of $i\not{\partial}_{\mathbf{Z}/r}\psi=g\in C_c^\infty(\mathbb{R}\times(\mathbb{R}^3\setminus 0))$ and $|\mathbf{Z}|<1/2$. For each m, $\mathrm{WF}_{\mathrm{b}}^{1,m}\psi\subset\dot{\Sigma}$. Away from $r=0$, $\mathrm{WF}_{\mathrm{b}}^{1,m}\psi$ is invariant under bicharacteristic flow.*

For $q_0=\{(t_0,r=0,\theta\in\mathbb{S}^2,\tau_0,\xi_r=0,\eta=0)\}\subset\dot{\Sigma}$ and let U denote a neighborhood of $q_0\in\dot{\Sigma}$. If

$$U\cap\{\xi_r/\tau>0\}\cap\mathrm{WF}_{\mathrm{b}}^{1,m}\psi=\emptyset,$$

then

$$q_0\cap\mathrm{WF}_{\mathrm{b}}^{1,m}\psi=\emptyset.$$

Note that, as $\mathrm{WF}_{\mathrm{b}}^{1,m}\psi$ is closed, the second part of the theorem yields regularity at the outgoing points ($\xi_r/\tau<0$) sufficiently near ρ_0.

The analogous statement near the poles ($\rho=0$, $x=0$) has the form:

Theorem 20. *If ψ is the forward solution of $i\not{\partial}_{\mathbf{Z}/r}\psi=g$, $g\in C_c^\infty(\mathbb{R}\times(\mathbb{R}^3\setminus 0))$, and $|\mathbf{Z}|<1/2$, then $\mathrm{WF}_{\mathrm{b}}^{1,m,\ell}\psi\subset\dot{\Sigma}$. For*

$$q_0=\{(\rho=0,x=0,\theta\in\mathbb{S}^2,\sigma_0,\xi=0,\eta=0)\}\subset\dot{\Sigma}\cap\{\rho=0\}$$

and let U denote a neighborhood of $q_0 \in \dot{\Sigma}$. If

$$U \cap \{\xi/\sigma > 0\} \cap \mathrm{WF}_{\mathrm{b}}^{1,m,\ell} \psi = \emptyset,$$

then

$$q_0 \cap \mathrm{WF}_{\mathrm{b}}^{1,m} \psi = \emptyset.$$

As g is smooth and compactly supported, $\mathrm{WF}_{\mathrm{b}}^{0,m} g = \emptyset$; accounting for the presence of non-compactly supported inhomogeneous terms would require adding statements analogous to those in Theorem 19.

Recall that, as in previous work [BW20], Theorem 20 suffices to show that (in the notation of the theorem)

$$U \cap \{\xi/\sigma < 0\} \cap \mathrm{WF}_{\mathrm{b}}^{1,m} \psi = \emptyset.$$

As the proof of Theorem 20 follows [BW20] closely, we defer this proof to the appendix.

6. The boundary operator

The aim of this section is to show that the normal operator $\widehat{\mathcal{L}}_\sigma$ is Fredholm, a key step in the iterative argument below. **For the remainder of this section, we let** $s = s_{\mathrm{ftr}}, s^* = s_{\mathrm{past}} \in C^\infty(\mathrm{mf})$ **be as in Section 4.3.** We will prove the following theorem:

Theorem 21. *The family $\widehat{\mathcal{L}}_\sigma$ enjoys the following mapping properties:*

1. *$\widehat{\mathcal{L}}_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$ and $\widehat{\mathcal{L}}_\sigma^* : \mathcal{X}^{s^*} \to \mathcal{Y}^{s^*}$ are Fredholm.*
2. *The operators $\widehat{\mathcal{L}}_\sigma$ form a holomorphic Fredholm family on these spaces in the strips*

$$\mathbb{C}_{s_+,s_-} = \left\{ \sigma \in \mathbb{C} \mid s_+ < \frac{1}{2} + \operatorname{Im}\sigma < s_- \right\},$$

with $s_{\mathrm{ftr}}|_{S_\pm} = s_\pm$. The formal adjoint $\widehat{\mathcal{L}}_\sigma^*$ is antiholomorphic in the same region.

3. *$\widehat{\mathcal{L}}_\sigma^{-1}$ has only finitely many poles in each strip $a < \operatorname{Im}\sigma < b$.*
4. *For all a and b, there is a constant C so that*

$$\left\| \widehat{\mathcal{L}}_\sigma^{-1} \right\|_{\mathcal{Y}^s_{|\sigma|^{-1}} \to \mathcal{X}^s_{|\sigma|^{-1}}} \leq C \langle \operatorname{Re}\sigma \rangle^{-1}$$

for $a < \operatorname{Im}\sigma < b$, $\langle \operatorname{Re}\sigma \rangle > C$, with a similar estimate holding for $(\widehat{\mathcal{L}}_\sigma^*)^{-1}$.

The proof of the first two parts of Theorem 21 follows from estimates of the form

$$\|u\|_{\mathcal{X}^s} \leq C \left\| \widehat{\mathcal{L}}_\sigma u \right\|_{\mathcal{Y}^s} + C \|u\|_{H^{-N}},$$
$$\|u\|_{\mathcal{X}^{s^*}} \leq C \left\| \widehat{\mathcal{L}}_\sigma^* u \right\|_{\mathcal{Y}^{s^*}} + C \|u\|_{H^{-N}},$$

where N is sufficiently large. At elliptic and hyperbolic points away from the singularity of the potential, microlocalizations of these estimates follow from standard techniques. Section 6.1 is therefore devoted to the proof of these estimates near the singularity and near the radial points $\Lambda^\pm$.

The last two parts of Theorem 21 follow from the semiclassical versions of these estimates, namely

$$\|u\|_{\mathcal{X}_h^s} \leq C\left(\frac{1}{h}\left\|\widehat{\mathcal{L}}_h u\right\|_{\mathcal{Y}_h^s} + h\,\|u\|_{H_h^{-N}}\right),$$
$$\|u\|_{\mathcal{X}_h^{s*}} \leq C\left(\frac{1}{h}\left\|\widehat{\mathcal{L}}_h^* u\right\|_{\mathcal{Y}_h^{s*}} + h\,\|u\|_{H_h^{-N}}\right).$$

These estimates again follow from standard techniques at semiclassical elliptic and hyperbolic points away from the singularity of the potential. Section 6.2 is devoted to these estimates near the radial set and the singularity of the potential. Though the analysis near the radial points is essentially identical to the non-semiclassical version, the estimates near the singularity are a bit more complicated due to the failure of semiclassical ellipticity near the singularity.

6.1. **The Fredholm statement.** We first show that $\widehat{\mathcal{L}}_\sigma$ is Fredholm on the desired spaces. Throughout this section we use the notation H^1 to denote the standard Sobolev space H^1 on the sphere obtained by blowing down the lift of cf within mf.

Away from the radial sets and the singularity at the poles, standard elliptic parametrix arguments and hyperbolic propagation arguments apply. Near the singularity at the poles, the following lemma yields the desired estimate.

Lemma 22. *Fix $\chi \in C^\infty$ supported in $\{x < 1/4\}$. For any N, there is a C so that*

$$\|\chi u\|_{H^1} \leq C\left(\left\|\widehat{\mathcal{L}}_\sigma(\chi u)\right\|_{L^2} + \|\chi u\|_{H_g^{-N}}\right).$$

In particular, as distributions in $\mathcal{X}^s$ lie in L^2 near the pole, for any N, we may estimate

$$\|\chi u\|_{\mathcal{X}^s} \leq C\left(\left\|\widehat{\mathcal{L}}_\sigma(\chi u)\right\|_{\mathcal{Y}^s} + \|\chi u\|_{H^{-N}}\right).$$

In the above lemma it is convenient to use the blow-down map to identify classical Sobolev spaces on the boundary as we can then use the compactness of the inclusion of standard Sobolev spaces.

Proof. In this proof we use $L^2_{\mathrm{b}}(x^{-1}dxd\theta)$ to denote the L^2 space associated to the b-density $\frac{dx}{x}d\theta$, while L^2 continues to denote $L^2(x^2dxd\theta)$.

The operator $\widehat{\mathcal{L}}_\sigma$ is elliptic on $\{x < 1/4\}$ and so we may use the b-calculus to construct a good (large calculus) parametrix G for $\widehat{\mathcal{L}}_\sigma$. (Indeed, following [BW20, Sect. 4.1], the reduced normal operator of $\widehat{\mathcal{L}}_\sigma$ agrees with that of the stationary elliptic operator $\mathcal{B}$ from (2) (acting in x). One can check that $[1/2, 3/2]$ does not intersect the boundary spectrum of $x\mathcal{B}$, and thus there is a large calculus parametrix Gx^{-1} for $x\widehat{\mathcal{L}}_\sigma$.) From [Mel93], we conclude that there is an operator G such that

$$I = G\widehat{\mathcal{L}}_\sigma + R,$$

where $G, R : r^{-3/2}L^2_{\mathrm{b}} \to r^{-1/2}H^1_{\mathrm{b}}(x^{-1}dxd\theta)$. As $r^{-3/2}L^2_{\mathrm{b}} = L^2$ and $r^{-1/2}H^1_{\mathrm{b}}(x^{-1}dxd\theta) \cap L^2 = H^1$, we may estimate

$$\|\chi u\|_{H^1} \leq C\left(\left\|\widehat{\mathcal{L}}_\sigma(\chi u)\right\|_{L^2} + \|u\|_{L^2}\right).$$

Interpolation inequalities on the sphere then allow us to bound

$$\|\chi u\|_{L^2} \leq \epsilon\,\|\chi u\|_{H^1} + C\epsilon^{-N}\,\|\chi u\|_{H^{-N}},$$

finishing the proof. □

Near the radial points $\Lambda^\pm$ (see Section 4.2), we use the following estimates.

Lemma 23 ([Vas13, Proposition 2.3 and 2.4])**.** *For all N and for $s_0 \geq m > \frac{1}{2} + \operatorname{Im}\sigma$, and for all $A, B, G \in \Psi_{\mathrm{b}}^0(\mathrm{mf})$ supported near Λ^- with A, G elliptic at Λ^- and bicharacteristics from the microsupport of B tend to Λ^- in one direction with closure in the elliptic set of G, we have*

$$\text{If } Au \in H^m \text{ then } \|Bu\|_{H^{s_0}} \leq C\left(\left\|G\widehat{P}_\sigma u\right\|_{H^{s_0-1}} + \|u\|_{H^{-N}}\right).$$

For $s_0 < \frac{1}{2} + \operatorname{Im}\sigma$ and all $A, B, G \in \Psi_{\mathrm{b}}^0(\mathrm{mf})$ supported near Λ^+ with B, G elliptic at Λ^+ and bicharacteristics from $\mathrm{WF}'(B) \setminus \Lambda^+$ reach the microsupport of A in one direction while remaining in the elliptic set of G, we have

$$\|Bu\|_{H^{s_0}} \leq C\left(\left\|G\widehat{P}_\sigma u\right\|_{H^{s_0-1}} + \|Au\|_{H^{s_0}} + \|u\|_{H^{-N}}\right).$$

Because $\widehat{P}_\sigma = \widehat{N}\left(\rho^{-2}\widetilde{\mathcal{L}}\rho\right)\widehat{\mathcal{L}}_\sigma$, estimates for $\widehat{P}_\sigma$ in H^{s^0-1} immediately yield estimates for $\widehat{\mathcal{L}}_\sigma$ in H^{s_0}, i.e. we conclude that, assumptions as in the lemma, that

$$\text{If } Au \in H^m \text{ then } \|Bu\|_{H^{s_0}} \leq C\left(\left\|G\widehat{\mathcal{L}}_\sigma u\right\|_{H^{s_0}} + \|u\|_{H^{-N}}\right). \tag{13}$$

and

$$\|Bu\|_{H^{s_0}} \leq C\left(\left\|G\widehat{\mathcal{L}}_\sigma u\right\|_{H^{s_0}} + \|Au\|_{H^{s_0}} + \|u\|_{H^{-N}}\right). \tag{14}$$

Given A, B, G as in the lemma, this follows directly from the lemma by choosing a $G' \in \Psi_{\mathrm{b}}^0(\mathrm{mf})$ for which B is still microsupported in the elliptic set of G' and with $\mathrm{WF}'\,G' \subset \ell G$ and applying the lemma to A, B, G'. Then the fact that $G'\widehat{\mathcal{L}}_\sigma \in \Psi_{\mathrm{b}}^1$ and elliptic regularity give (13) and (14).

Although similar estimates hold for $\widehat{P}_\sigma^*$, these do not immediately give the desired estimates for $\widehat{\mathcal{L}}_\sigma^*$. To obtain these, we recall the formulas for the "ℓ-based" operators (i.e., those obtained via conjugation by ρ^ℓ) in Section 4.1. Specifically, recall that

$$(P_1)^* = \gamma^0 P_{-1}\gamma^0 = \mathcal{L}\widetilde{\mathcal{L}}_{-1}.$$

From this we can obtain estimates for $\widehat{\mathcal{L}}_\sigma^*$ by taking the adjoint of the Mellin-transformed normal operator:

$$\widehat{N}_\sigma(P_1^*) = \widehat{\mathcal{L}}_\sigma \widehat{N}_\sigma(\mathcal{L}_{-1}),$$

so using that in general for b-operators $\widehat{N}_\sigma(B^*) = \widehat{N}_{\overline{\sigma}}(B)$, we conclude that

$$\widehat{N}_{\overline{\sigma}-i}(P) = \widehat{N}_{\sigma+i}(P^*)^* = \widehat{N}_\sigma(P_1^*)^* = \left(\widehat{\widetilde{\mathcal{L}}}_{-1,\sigma}\right)^* \widehat{\mathcal{L}}_\sigma^*.$$

The radial estimates for $\widehat{P}_{\overline{\sigma}-i}$ are read off in the obvious way (i.e. by exchanging $\overline{\sigma} - i$ for σ) in the two Lemmas above. This immediately yields estimates for $\widehat{\mathcal{L}}_\sigma^*$. We may therefore conclude that the adjoint $\widehat{\mathcal{L}}_\sigma^*$ satisfies analogous radial point estimates with thresholds $\frac{1}{2} - 1 - \operatorname{Im}\sigma = -\left(\frac{1}{2} + \operatorname{Im}\sigma\right)$, i.e., in the dual spaces. Concretely, for A, B, G as in the theorem, if $s_0^* > -(1/2 + \operatorname{Im}\sigma)$ then

$$\text{If } Au \in H^m \text{ then } \|Bu\|_{H^{s_0^*}} \leq C\left(\left\|G\widehat{\mathcal{L}}_\sigma^* u\right\|_{H^{s_0^*}} + \|u\|_{H^{-N}}\right). \tag{15}$$

and is $s_0^* < -(1/2 + \operatorname{Im}\sigma)$ then

$$\|Bu\|_{H^{s_0^*}} \leq C\left(\left\|G\widehat{\mathcal{L}}_\sigma^* u\right\|_{H^{s_0^*}} + \|Au\|_{H^{s_0^*}} + \|u\|_{H^{-N}}\right). \tag{16}$$

Taking microlocal partitions of unity as appropriate, we thus have the two estimates

$$\|u\|_{\mathcal{X}^s} \leq C\left(\left\|\widehat{\mathcal{L}}_\sigma u\right\|_{\mathcal{Y}^s} + \|u\|_{H^{-N}}\right),$$
$$\|u\|_{\mathcal{X}^{s^*}} \leq C\left(\left\|\widehat{\mathcal{L}}_\sigma^* u\right\|_{\mathcal{Y}^{s^*}} + \|u\|_{H^{-N}}\right).$$

As the inclusions $\mathcal{X}^s, \mathcal{X}^{s^*} \hookrightarrow H^{-N}$ are compact for sufficiently large N, the operators $\widehat{\mathcal{L}}_\sigma$ and $\widehat{\mathcal{L}}_\sigma^*$ are Fredholm on the stated spaces, proving the first part of the theorem. The second part of the theorem follows by inspection of the coefficients together with the observation that we may choose N uniformly on these strips.

6.2. **The semiclassical statements.** The other two statements of the theorem follow from a semiclassical estimate of the form

$$\|u\|_{\mathcal{X}_h^s} \leq \frac{C}{h}\left\|\widehat{\mathcal{L}}_h u\right\|_{\mathcal{Y}_h^s} + \mathcal{O}(h)\,\|u\|_{H_h^{-N}}, \tag{17}$$

with a corresponding estimate for $\widehat{\mathcal{L}}_h^*$, where $h = |\sigma|^{-1}$. This estimate immediately implies that $\widehat{\mathcal{L}}_h$ is invertible for small h and provides a bound on the norm, proving the last two statements of the theorem.

For the rest of this section, recall that $\Psi_{\mathrm{b},h}$ (without a superscript) denotes the space of semiclassical b-pseudodifferential operators with *compactly supported* symbols.

6.2.1. *The radial set.* The estimates near the radial sets $\Lambda^\pm$ follow as in earlier work [BM19, BVW15, BVW18, Vas13]:

Proposition 24 (c.f. [Vas13, Propositions 2.8 and 2.9])**.** *For $s|_{S_-} > m > \frac{1}{2} + \operatorname{Im}\sigma$ and $A, B, G \in \Psi_{\mathrm{b},h}^0$ supported near Λ^- with A, G elliptic at Λ^- and so that semiclassical bicharacteristics from the microsupport of B tend to Λ^- in one direction with closure in the elliptic set of G, we have*

$$\text{If } Au \in H^m \text{ then } \|Bu\|_{\mathcal{X}_h^s} \leq \frac{C}{h}\left\|G\widehat{\mathcal{L}}_h u\right\|_{\mathcal{Y}_h^s} + Ch\,\|u\|_{H_h^{-N}}.$$

For $s|_{S_+} < \frac{1}{2} + \operatorname{Im}\sigma$, and for all $A, B, G \in \Psi_{\mathrm{b},h}^0$ supported near Λ^+ with B, G elliptic at Λ^+ and so that semiclassical bicharacteristics from $\mathrm{WF}_{\mathrm{b},h}'(B) \setminus \Lambda^+$ reach the microsupport of A in one direction while remaining in the elliptic set of G, we have

$$\|Bu\|_{\mathcal{X}_h^s} \leq \frac{C}{h}\left\|G\widehat{\mathcal{L}}_h u\right\|_{\mathcal{Y}_h^s} + C\,\|Au\|_{\mathcal{X}_h^s} + Ch\,\|u\|_{H_h^{-N}}.$$

Analogous estimates hold for $\widehat{\mathcal{L}}_\sigma^*$ on the dual spaces as well; these follow as in Section 6.1 above.

6.2.2. *Commutators with semiclassical* b*-pseudodifferential operators.*

Lemma 25. *If $C \in \Psi^0_{\mathrm{b},h}$ is invariant with real-valued scalar principal symbol, then*

$$\frac{1}{h}\left[\widehat{P}_h, C\right] \in \left\{\frac{h^2}{x^2}\Delta_\theta, h^2 D_x^2, \frac{h^2}{x}D_x, \frac{h^2}{x^2}\right\}\Psi^{-1}_{\mathrm{b},h} + \left\{hD_x, \frac{h}{x}\right\}\Psi^0_{\mathrm{b},h} + \Psi^1_{\mathrm{b},h},$$

The possibility that the commutator of C and K is non-vanishing leads to a different result for the first-order operator:

$$\frac{1}{h}\left[\widehat{\mathcal{L}}_h, C\right] \in \left\{\frac{h}{x}\beta K, \frac{h}{x}, hD_x\right\}\Psi^0_{\mathrm{b},h}.$$

Lemma 26. *If $C \in \Psi^0_{\mathrm{b},h}(X)$ is invariant with real-valued scalar principal symbol c, then*

$$\frac{1}{ih}\left[\widehat{\mathcal{L}}_h, C\right] = A_0\left(\alpha_r\left(ih\partial_x + \frac{ih}{x} - \frac{ih}{x}\beta K\right) - \frac{h\mathbf{Z}}{x}\right) + B_0 + \alpha_r B_1 + \mathbf{B}_2\frac{1}{x} + \mathbf{B}_3 hD_x + h\mathbf{B}_4$$

where

- $A_0 \in \Psi^{-1}_{\mathrm{b},h}$, *with* $\sigma_{\mathrm{b},h}(A_0) = -\partial_\xi(c)$,
- $B_0 \in \Psi^0_{\mathrm{b},h}$, *with* $\sigma_{\mathrm{b},h}(B_0) = -x\partial_x(c)$,
- $B_1 \in \Psi^0_{\mathrm{b},h}$, *with* $\sigma_{\mathrm{b},h}(B_0) = -\partial_x(c)$
- $\mathbf{B}_2 \in \Psi^0_{\mathrm{b},h}$, *with* $\operatorname{supp}\sigma_{\mathrm{b},h}(\mathbf{B}_2) \subset \operatorname{supp}\partial_\eta(c)$,
- $\mathbf{B}_3 \in \Psi^{-1}_{\mathrm{b},h}$, *with* $\operatorname{supp}\sigma_{\mathrm{b},h}(\mathbf{B}_3) \subset \operatorname{supp}\partial_\eta(c)$,
- $\mathbf{B}_4 \in \left\{\frac{h}{x}, hD_x\right\}\Psi^{-2}_{\mathrm{b},h}$.

Proof. Recall that we may write

$$\widehat{\mathcal{L}}_h = hxD_x + z + h\mathbf{Z} - ih - \frac{h\mathbf{Z}}{x} - \alpha_r\left(hD_x - \frac{ih}{x} + \frac{ih}{x}\beta K\right).$$

We first consider the angular term and write

$$\frac{1}{ih}\left[-\frac{ih}{x}\alpha_r\beta K, C\right] = -\frac{i}{x}\alpha_r\frac{1}{ih}\left[h\beta K, C\right] - \frac{i}{x}\frac{1}{ih}\left[\alpha_r, C\right] - \frac{1}{ih}\left[\frac{i}{x}, C\right]\alpha_r h\beta K.$$

As C is invariant and α_r and K have only angular dependence, the first and second terms are microsupported in the support of $\partial_\eta(c)$ and contribute to $\mathbf{B}_2$, while the third term yields the angular part of the term involving A_0. (Indeed, we take $A_0 = \frac{1}{ih}[1/x, C]$ as the *definition* of A_0.) Lemma 11 then shows that $\sigma_{\mathrm{b,h}}(A_0) = -\partial_\xi(c)$.

We now consider the term involving $i\alpha_r\left(h\partial_x + \frac{h}{x}\right)$. Because α_r depends only on the angular variables, its commutator with C is again microsupported in $\operatorname{supp}\partial_\eta(c)$ and therefore contributes to the $\mathbf{B}_2$ and $\mathbf{B}_3$ terms. Now, by Lemma 11 the commutator of $-hD_x$ with C yields a term of the form $A_1\alpha_r\partial_x$ as well as a term of the form $\alpha_r B_1$, where $\sigma_{\mathrm{b},h}(A_1) = \sigma_{\mathrm{b},h}(A_0)$ and $\sigma_{\mathrm{b},h}(B_1) = \partial_x(c)$. At the cost of a contribution to the $\mathbf{B}_4$ term, we may replace A_1 by A_0. The commutator of ih/x with C yields the corresponding part of the A_0 term.

Finally, the commutator of $-h\mathbf{Z}/x$ with C yields the corresponding piece of the A_0 term, while the commutator of hxD_x with C yields the B_0 term. The $h\mathbf{Z}$ part of the operator commutes with A and so does not contribute. □

6.2.3. *Elliptic estimates near the singularity.* We now establish the semiclassical elliptic estimates near the conic singularity. *Throughout this section, we assume that all pseudodifferential operators and distributions are supported in* $\{x < 1/4\}$. We further present arguments only near the "north pole" (i.e., in C_+) as the proofs are nearly identical near the "south pole" (i.e., in C_-).

Recall that s continues to satisfy the assumptions of Section 4.3. The core of the semiclassical elliptic estimate near the singularity is the following proposition:

Proposition 27. *Suppose* $|\mathbf{Z}| < 1/2$, $A \in \Psi^0_{\mathrm{b},h}$ *is invariant and satisfies* $\mathrm{WF}'_{\mathrm{b},h}(A) \cap \dot{\Sigma}_h = \emptyset$. *For any* $G \in \Psi^0_{\mathrm{b},h}$ *with* $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b},h}(G)$, *there is a constant* C *so that*

$$\|Au\|_{\mathcal{X}^s_h} \leq C \left\| G\widehat{\mathcal{L}}_h u \right\|_{\mathcal{Y}^s_h} + Ch^{1/2} \|Gu\|_{\mathcal{X}^s_h} + \mathcal{O}(h^\infty) \|u\|_{\mathcal{X}^s_h}.$$

In fact, by enlarging the microsupport of G, we can improve the factor of h in the estimate:

Corollary 28. *If* $|\mathbf{Z}| < 1/2$, $A \in \Psi^0_{\mathrm{b},h}$ *is invariant and satisfies* $\mathrm{WF}'_{\mathrm{b},h}(A) \cap \dot{\Sigma}_h = \emptyset$, *then for any* N *and* $G \in \Psi^0_{\mathrm{b},h}$ *with* $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b},h}(G)$, *there is a constant* C *so that*

$$\|Au\|_{\mathcal{X}^s_h} \leq C \left\| G\widehat{\mathcal{L}}_h u \right\|_{\mathcal{Y}^s_h} + Ch^{N} \|Gu\|_{\mathcal{X}^s_h} + \mathcal{O}(h^\infty) \|u\|_{\mathcal{X}^s_h}.$$

The following proposition provides a convenient way to estimate error terms of the form $\|u\|_{H^1_h}$:

Proposition 29. *If* χ *is a smooth radial cut-off function supported in* $\{x < \delta\}$ *for* $\delta > 0$ *sufficiently small, then there is a* C *so that for all* $u \in \mathcal{X}^s_h$, $\chi u \in H^1_h$ *and*

$$\|\chi u\|_{H^1_h} \leq C \|\chi u\|_{\mathcal{X}^s_h}.$$

Proof. The proposition is a reflection of the observation that $\widehat{\mathcal{L}}_\sigma$ is elliptic near $x = 0$. Recall that the $\mathcal{X}^s_h$ norm is given by

$$\|v\|^2_{\mathcal{X}^s_h} = \|v\|^2_{H^s_h} + \left\|\widehat{\mathcal{L}}_h v\right\|^2_{H^s_h}.$$

and thus by our assumptions on s, if v is supported in the region $\{x < \delta\}$ with $\delta \leq 1/4$, it has the form

$$\|v\|^2_{\mathcal{X}^s_h} = \|v\|^2 + \left\|\widehat{\mathcal{L}}_h v\right\|^2,$$

where both norms are the L^2 norm (taken with respect to the volume form $x^2\, dx\, d\theta$).

Consider now the real part of the pairing $\langle \widehat{P}_h v, v\rangle$. Using the form of $\widehat{P}_h$ in Section 4.1, integrating by parts shows that this is equal to

$$\|hD_x v\|^2 + \left\|\frac{h}{x}\nabla_\theta v\right\|^2 - \left\|\left(hxD_x - \frac{h\mathbf{Z}}{x} + z + h\mathbf{Z} - ih\right) v\right\|^2 + 2(\operatorname{Im} z)(h + 2\operatorname{Im} z)\|v\|^2.$$

In particular, as

$$\left|\operatorname{Re}\left\langle \widehat{P}_h v, v\right\rangle\right| \leq \epsilon' \|v\|^2_{H^1_h} + \frac{1}{\epsilon'}\left\|\widehat{P}_h\right\|^2_{H^{-1}_h},$$

we may bound

$$\|v\|^2_{H^1_h} \leq \frac{C}{\epsilon}\|hxD_x v\|^2 + \frac{C}{\epsilon}\|v\|^2 + (1+\epsilon)\left\|\frac{h\mathbf{Z}}{x}v\right\|^2 + \frac{C}{\epsilon'}\left\|\widehat{P}_h v\right\|^2_{H^{-1}_h} + \epsilon'\|v\|^2_{H^1_h}.$$

For $|\mathbf{Z}| < 1/2$, the Hardy inequality yields

$$\left\| \frac{h\mathbf{Z}}{x} v \right\|^2 \leq 4\mathbf{Z}^2 \left\| hD_x v \right\|^2 ,$$

so, for $\epsilon > 0$ sufficiently small and fixed, the third term on the right can be absorbed into the left hand side. Similarly, if v is supported in $\{x < \delta\}$, the first term on the right is bounded by $(C/\epsilon)\delta^2 \left\| hD_x v\right\|^2$, so for δ sufficiently small this can also be absorbed on the left, as can the last term for ϵ' small, leaving

$$\|v\|^2_{H^1_h} \leq C \left(\|v\|^2 + \left\| \widehat{P}_h v \right\|^2_{H^{-1}_h} \right).$$

Recall that

$$\widehat{P}_h = \left(\frac{1}{|\sigma|} \widehat{N}(\widetilde{\mathcal{L}}_1) \right) \widehat{\mathcal{L}}_h,$$

and so we obtain the bound

$$\|v\|^2_{H^1_h} \leq C \left(\|v\|^2 + \left\| \widehat{\mathcal{L}}_h v \right\|^2 \right).$$

As v is supported in $\{x < \delta\}$, the right side is a multiple of $\|v\|^2_{\mathcal{X}^s_h}$. □

A more careful look at the real part of the pairing yields the following:

Lemma 30. *Suppose $A, G \in \Psi^0_{\mathrm{b},h}$ with A invariant, $\sigma_{\mathrm{b}}(A)$ scalar and real-valued, $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b},h}(G)$ and δ as in Proposition 29. There is a constant C so that*

$$\left| \|hD_x Au\|^2 + \left\| \frac{h}{x} \nabla_\theta Au \right\|^2 - \left\| \left(hxD_x + z - \frac{h\mathbf{Z}}{x} + h\mathbf{Z} - ih \right) Au \right\|^2 \right|$$
$$\leq \epsilon \|Au\|^2_{H^1_h} + \frac{C}{\epsilon} \left\| G\widehat{\mathcal{L}}_h u \right\|^2_{L^2_g} + Ch \|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty) \|\chi u\|^2_{H^1_h} + \mathcal{O}(h^\infty) \|(1-\chi)u\|_{H^{-N}_h},$$

where χ is a cut-off function supported in $x < \delta$.

Proof. Considering the real part of the pairing $\left\langle Au, \widehat{P}_h Au \right\rangle$ shows that we may bound

$$\left| \|hD_x Au\|^2 + \left\| \frac{h}{x} \nabla_\theta Au \right\|^2 - \left\| \left(hxD_x - \frac{h\mathbf{Z}}{x} + z + h\mathbf{Z} - ih \right) Au \right\|^2 + 2(\operatorname{Im} z)(h + 2(\operatorname{Im} z)) \|Au\|^2 \right|$$
$$\leq \left| \left\langle Au, \widehat{P}_h Au \right\rangle \right|.$$

Using the factorization $\widehat{P}_h = \widetilde{L}\widehat{\mathcal{L}}_h$ with $\widetilde{L} \in \mathrm{Diff}^1_h$, the term on the right is bounded by

$$\epsilon \|Au\|^2_{H^1_h} + \frac{1}{\epsilon} \left\| A\widetilde{L}\widehat{\mathcal{L}}_h u \right\|_{H^{-1}_h} + \frac{1}{\epsilon} \left\| [\widehat{P}_h, A]u \right\|^2_{H^{-1}_h},$$

and so, by Lemma 25 and Lemma 10, can be estimated by

$$\epsilon \|Au\|^2_{H^1_h} + \frac{C}{\epsilon} \left\| A\widehat{\mathcal{L}}_h u \right\| + \frac{C}{\epsilon} h \|Gu\|^2_{H^1_h} + \mathcal{O}(h^\infty) \|\chi u\|_{H^1_h} + \mathcal{O}(h^\infty) \|(1-\chi)u\|_{H^{-N}_h},$$

where χ is a radial cut-off function supported near $x = 0$. □

Proof of Proposition 27. For $x > 0$, the proposition follows from standard proofs of semiclassical elliptic regularity. We may therefore assume that the support and microsupport of A are contained in $\{x < \delta\}$.

We begin by establishing a version of the estimate with $\mathcal{X}_h^s$ replaced by H_h^1 and $\mathcal{Y}_h^s$ replaced by L^2. Because $\mathrm{WF}'_{\mathrm{b},h}(A) \cap \dot{\Sigma}_h = \emptyset$, we may assume there is a constant $C > 0$ so that

$$2\left((\xi+z)^2+1\right) < C\left(\xi^2+|\eta|^2\right)$$

on the microsupport of A (see (12)). We may therefore bound

$$\begin{aligned}
\|(hxD_x+z)\,Au\|^2+\|Au\|^2 &\leq \left\langle C\,\mathrm{Op}_h(\xi^2+|\eta|^2)Au, Au\right\rangle + Ch\,\|Gu\|_{H_h^1}^2 \\
&= C\left(\|hxD_xAu\|^2+\left\|x(\frac{h}{x}\nabla_\theta Au)\right\|^2\right)+Ch\,\|Gu\|_{H_h^1}^2 \\
&\leq C\delta^2\,\|Au\|_{H_h^1}^2+Ch\,\|Gu\|_{H_h^1}^2\,,
\end{aligned} \tag{18}$$

where in the last line we have used that A is supported in $\{x < \delta\}$.

Adding $\left\|\left(hxD_x+z-\frac{h\mathbf{Z}}{x}-ih\right)Au\right\|^2+\|Au\|^2$ to both sides of the estimate in Lemma 30 then yields

$$\begin{aligned}
\|Au\|_{H_h^1}^2 \leq \epsilon\,\|Au\|_{H_h^1}^2+\frac{C}{\epsilon}\left\|G\widehat{\mathcal{L}}_hu\right\|^2+Ch\,\|Gu\|_{H_h^1}+\|Au\|^2+\left\|(hxD_x+z-\frac{h\mathbf{Z}}{x}+h\mathbf{Z}-ih)Au\right\|^2 \\
+\mathcal{O}(h^\infty)\,\|\chi u\|_{H_h^1}^2+\mathcal{O}(h^\infty)\,\|u\|_{H_h^{-N}}^2
\end{aligned}$$

Using the triangle and Hardy inequalities together with the estimate (18) yields

$$\|Au\|_{H_h^1} \leq \left(4\mathbf{Z}^2+\epsilon+C\delta^2\right)\|Au\|_{H_h^1}+C\left\|G\widehat{\mathcal{L}}_hu\right\|^2+Ch\,\|Gu\|_{H_h^1}^2+\mathcal{O}(h^\infty)\left(\|\chi u\|_{H_h^1}^2+\|u\|_{H_h^{-N}}^2\right).$$

Provided that $|\mathbf{Z}| < 1/2$ and ϵ, δ are sufficiently small, the first term can be absorbed into the left, finishing the proof of the estimate.

To finish the proof of the proposition, we note that on the support of A and G, the $\mathcal{X}_h^s$ norm is equivalent to the H_h^1 norm while the $\mathcal{Y}_h^s$ norm is equivalent to the L_g^2 norm. Finally, the error terms are both controlled by the $\mathcal{X}_h^s$ norm provided that N is sufficiently large. □

6.2.4. *Hyperbolic estimates near the singularity.* In this section we establish the semiclassical propagation result for $\widehat{\mathcal{L}}_h$ in the hyperbolic regime near the singularity. The result for the adjoint problem is essentially identical. In particular, we prove the following:

Proposition 31. *If $G \in \Psi_{\mathrm{b},h}$ is elliptic at $\dot{\Sigma}_h \cap \{x=0\}$ then there are $Q, Q_1 \in \Psi_{\mathrm{b},h}$ with Q also elliptic at $\dot{\Sigma}_h \cap \{x=0\}$ and*

$$\begin{aligned}
\mathrm{WF}'_{\mathrm{b},h}(Q) &\subset \mathrm{ell}_{\mathrm{b},h}(G), \\
\mathrm{WF}'_{\mathrm{b},h}(Q_1) &\subset \mathrm{ell}_{\mathrm{b},h}(G) \cap \{-\xi > 0\},
\end{aligned}$$

so that for all $u \in \mathcal{X}_h^s$,

$$\|Qu\|_{\mathcal{X}_h^s} \leq \frac{C}{h}\left\|G\widehat{\mathcal{L}}_hu\right\|_{\mathcal{Y}_h^s}+C_1\,\|Q_1u\|_{\mathcal{X}_h^s}+Ch^{1/2}\,\|Gu\|_{\mathcal{X}_h^s}+\mathcal{O}(h^\infty)\,\|u\|_{\mathcal{X}_h^s}\,.$$

As the characteristic set near the singularity is in a bounded region of phase space, we need only pseudodifferential operators with *compactly supported* symbols in this section. Recall that we use the notation $\Psi_{\mathrm{b},h}$ (without a subscript) to denote this space.

In an approach similar to the one taken in Section A.3, we introduce an operator $A \in \Psi_{b,h}$ with compactly supported symbol given by

$$a = \chi_0(2 - \phi/\delta)\chi_1(2 - \xi/\delta)\chi_2(\xi^2 + |\eta|^2),$$

where $\chi_\bullet$ are the same functions as in that section and here $\phi = -\xi + \frac{1}{\beta^2\delta}x^2$. For convenience we recall the relevant properties of the $\chi_\bullet$:

- $\chi_\bullet$ and $\chi_\bullet'$ have smooth square roots,
- χ_0 is supported in $[0, \infty)$ with $\chi_0(s) = \exp(-1/s)$ for $s > 0$,
- χ_1 is supported in $[0, \infty)$ with $\chi_1(s) = 1$ for $s \geq 1$ and $\chi_1' \geq 0$, and
- χ_2 is supported in $[-2c_1, 2c_1]$ and is equal to 1 on $[-c_1, c_1]$.

We think of the symbol a as being determined by the three localizing parameters c_1, β, and δ.

We further choose an invariant operator $Q \in \Psi_{b,h}$ with compactly supported symbol given by

$$q = \frac{\sqrt{2}}{\sqrt{\delta}} \left(\chi_0\chi_0'\right)^{1/2} \chi_1\chi_2, \tag{19}$$

where the arguments of $\chi_\bullet$ are the same as those in the definition of a. The symbol q will appear when derivatives land on the χ_0 term in a.

Lemma 32. *For A and Q defined as above,*

$$\frac{1}{ih}\left[\widehat{\mathcal{L}}_h, A^*A\right] = \widetilde{R}\widehat{\mathcal{L}}_h + Q\left(hxD_x + z + h\mathbf{Z} - ih\right)Q + B_0 + \alpha_r B_1 + E' + E'' + h\mathbf{R},$$

where

- *All listed pseudodifferential operators have compact support,*
- *$Q \in \Psi_{b,h}$ is invariant, self-adjoint, and has principal symbol q defined in equation* (19)*,*
- *$\widetilde{R} \in \Psi_{b,h}$, with $\sigma_{b,h}(\widetilde{R}) = -\partial_\xi(a^2)$,*
- *$B_0, B_1 \in \Psi_{b,h}$ have principal symbols bounded by $C\beta^{-1}q^2$,*
- *$E' \in \Psi_{b,h}$, with $\mathrm{WF}'_{b,h}(E') \subset \{\delta \leq \xi \leq 2\delta, x^2 < 4\delta^2\beta^2\}$,*
- *$E'' \in \frac{1}{x}\Psi_{b,h}$, with $\mathrm{WF}'_{b,h}(E'') \cap \dot{\Sigma}_h = \emptyset$, and*
- *$\mathbf{R} \in \frac{1}{x}\Psi_{b,h}$.*

Proof. We carefully apply Lemma 26 with $C = A^*A$. The main term in Lemma 26 (the one involving A_0) is due to the near-homogeneity of the operator $\widehat{\mathcal{L}}_h$ in x. The principal symbol of A_0 there is $-\partial_\xi(a^2)$; we use that its coefficient includes all of the homogeneous terms of degree -1 in $\widehat{\mathcal{L}}_h$ and trade it for one of the form

$$\widetilde{R}\widehat{\mathcal{L}}_h + A_1\left(hxD_x + z + h\mathbf{Z} - ih\right)A_1,$$

where $\sigma_{b,h}(A_1^2) = \partial_\xi(a^2)$. We now split A_1 into three terms according to where the ξ-derivative lands. Those terms where the derivative falls on χ_0 we write as Q^2 modulo a lower order error (which we absorb into $h\mathbf{R}$). Those where the derivative falls on χ_1 are absorbed into E' and those for which it falls on χ_2 are absorbed into E''.

The B_1 term arising in Lemma 26 has principal symbol

$$-\partial_x(a^2) = -2\chi_0\chi_0'\chi_1^2\chi_2^2 \cdot \left(\frac{-1}{\delta}\right)\left(\frac{2}{\beta^2\delta}x\right).$$

As $0 \leq x \leq 2\beta\delta$ on the support of a, this term can be estimated by a multiple of $\beta^{-1}\delta^{-1}\chi_0'\chi_0\chi_1^2\chi_2^2$ and hence by a multiple of $\beta^{-1}q^2$. The B_0 term there is estimated similarly (and in fact is even smaller owing to the additional factor of x.)

The $\mathbf{B}_2$ and $\mathbf{B}_3$ terms in Lemma 26 have symbols proportional to $\partial_\eta(a^2)$ and so are absorbed into E''. The remaining terms constitute the $\mathbf{R}$ term. □

We now record a few consequences of the symbol calculus:

Lemma 33. *With A, Q, B_0, and B_1 as above, there are positive constants C and c so that the following estimates hold:*

$$\|Au\| \leq C\sqrt{\delta}\,\|Qu\| + \mathcal{O}(h^\infty)\,\|u\|_{\mathcal{X}_h^s},$$
$$|\langle B_0u, u\rangle| \leq C\beta^{-1}\,\|Qu\|^2 + \mathcal{O}(h^\infty)\,\|u\|^2_{\mathcal{X}_h^s},$$
$$|\langle B_1u, u\rangle| \leq C\beta^{-1}\,\|Qu\|^2 + \mathcal{O}(h^\infty)\,\|u\|^2_{\mathcal{X}_h^s},$$
$$|\langle (hxD_x + z + h\mathbf{Z} - ih)Qu, Qu\rangle| \geq |z|\,\|Qu\|^2 - c\delta\,\|Qu\|^2 - \mathcal{O}(h)\,\|Qu\|^2 - \mathcal{O}(h^\infty)\,\|u\|_{\mathcal{X}_h^s}.$$

We now finish the proof of Proposition 31.

Proof. We first observe that $\widehat{\mathcal{L}}_h^* = \widehat{\mathcal{L}}_h - 2\operatorname{Im} z + 2ih = \widehat{\mathcal{L}}_h + \mathcal{O}(h)$.

Given $u \in \mathcal{X}^s$, we apply Lemma 32 and write

$$\begin{aligned}
-\frac{2}{h}\operatorname{Im}\left\langle A\widehat{\mathcal{L}}_h u, Au\right\rangle &= \frac{1}{ih}\left\langle \left[\widehat{\mathcal{L}}_h, A^*A\right]u, u\right\rangle + \mathcal{O}(1)\,\|Au\|^2 \\
&= \left\langle \widetilde{R}\widehat{\mathcal{L}}_h u, u\right\rangle + \langle (hxD_x + z - ih)Qu, Qu\rangle + \langle (B_0 + \alpha_r B_1)u, u\rangle \\
&\quad + \langle E'u, u\rangle + \langle E''u, u\rangle + h\,\langle \mathbf{R}u, u\rangle + \mathcal{O}(1)\,\|Au\|^2.
\end{aligned}$$

Because $z = \pm 1 + \mathcal{O}(h)$, applying Lemma 33 shows that

$$\begin{aligned}
\|Qu\|^2 \leq{}& \frac{C}{h}\left|\left\langle A\widehat{\mathcal{L}}_h u, Au\right\rangle\right| + \left(C\delta + C\beta^{-1} + \mathcal{O}(h)\right)\|Qu\|^2 \\
&+ \left|\left\langle \widetilde{R}\widehat{\mathcal{L}}_h u, u\right\rangle\right| + |\langle E'u, u\rangle| + |\langle E''u, u\rangle| + h\,|\langle \mathbf{R}u, u\rangle| + \mathcal{O}(h^\infty)\,\|u\|_{\mathcal{X}_h^s}.
\end{aligned}$$

We turn our attention to the second line. The first term can be estimated by

$$\frac{C}{h^2}\left\|G\widehat{\mathcal{L}}_h u\right\|^2 + Ch\,\|Gu\|^2 + \mathcal{O}(h^\infty)\,\|u\|^2_{\mathcal{X}_h^s},$$

while the second term is bounded by

$$\|Q_1\|^2 + Ch\,\|Gu\|^2 + \mathcal{O}(h^\infty)\,\|u\|^2_{\mathcal{X}_h^s}.$$

The last term can be similarly estimated by $Ch\,\|Gu\|^2 + \mathcal{O}(h^\infty)\,\|u\|^2_{\mathcal{X}_h^s}$. We finally consider the term involving E''. Writing

$$E'' = \frac{1}{x}E_0,$$

with $E_0 \in \Psi_{\mathrm{b},h}$ microsupported away from $\dot{\Sigma}_h$, the Hardy inequality yields

$$\begin{aligned}\left|\left\langle \frac{1}{x} E_0 u, u\right\rangle\right| &\leq C \left\|\widetilde{G} u\right\|^2 + C\left\|\frac{1}{x}\widetilde{G}u\right\|^2 + \mathcal{O}(h^\infty)\left\|u\right\|^2_{\mathcal{X}^s_h} \\ &= \frac{C}{h^2}\left\|\frac{h}{x}\widetilde{G}u\right\|^2 + C\left\|\widetilde{G}u\right\|^2 + \mathcal{O}(h^\infty)\left\|u\right\|^2_{\mathcal{X}^s_h} \\ &\leq \frac{C}{h^2}\left\|\widetilde{G}u\right\|^2_{\mathcal{X}^s_h} + \mathcal{O}(h^\infty\left\|u\right\|^2_{\mathcal{X}^s_h},\end{aligned}$$

where $\widetilde{G}$ is elliptic on the microsupport of E_0. As E_0 is microsupported in the elliptic set, we can guarantee that $\widetilde{G}$ is also microsupported in the elliptic set. Corollary 28 then applies to yield

$$\left|\langle E'' u, u\rangle\right| \leq \frac{C}{h^2}\left\|G\widehat{\mathcal{L}}_h u\right\|^2_{\mathcal{Y}^s_h} + Ch\left\|Gu\right\|^2_{\mathcal{X}^s_h} + \mathcal{O}(h^\infty)\left\|u\right\|^2_{\mathcal{X}^s_h}.$$

Taking δ small and β large then provides the desired bound for $\|Qu\|$. ☐

7. Polyhomogeneity

In this section we show that the forward solution ψ of $i\not{\partial}_{\mathbf{Z}/r}\psi = g$ with $g \in C_c^\infty$ is polyhomogeneous at $\mathcal{I}^+$ and mf.

In particular we prove the following theorem:

Theorem 34. *Let $\mathcal{E}$ denote the index set arising from the poles (with multiplicity) of $\widehat{\mathcal{L}}_\sigma^{-1}$ from Theorem 21, i.e.,*

$$\mathcal{E} = \{(\sigma, m) : \sigma \text{ is a pole of } \widehat{\mathcal{L}}_\sigma^{-1} \text{ of order } m\},$$

set $\mathcal{E}(\varsigma)$ to consist of those elements $(\sigma, m) \in \mathcal{E}$ with $\operatorname{Im}\sigma < -\varsigma$, and let $\mathcal{E}_0 = \{(k, 0) \in \mathbb{C}\times\mathbb{N} : k \in \mathbb{N}\}$ denote the index set describing smooth functions. If ψ is the forward solution of $i\not{\partial}_{\mathbf{Z}/r}\psi = g$ with $g \in C_c^\infty$, then there is a ς so that ψ is polyhomogeneous on $[M; S_+]$ with index sets

$$\begin{cases}\emptyset & \text{at } C_- \cup C_0 \\ -i(1+i\mathbf{Z}) + \mathcal{E}_0 & \text{at } \mathcal{I}^+ \\ -i(1+i\mathbf{Z}) + \mathcal{E}(\varsigma) & \text{at } C_+\end{cases}.$$

Conjugating and rescaling, we work instead with $\mathcal{L}u = f$, where $u = \rho^{-1-i\mathbf{Z}}\psi$ and $f = \rho^{-2-i\mathbf{Z}}\gamma^0 g \in C_c^\infty(M)$. The argument in this section closely mirrors the "short-range" case for the radiation field in previous work [BVW18] and so we provide only an abbreviated version. In fact, the argument here is simplified in comparison with [BVW15, BVW18], owing to the scale-invariance of the operator $\mathcal{L}$ in ρ. Indeed, because $N(\mathcal{L}) = \mathcal{L}$, we are able to avoid the complications of the remainder terms arising in the contour deformation argument.

We apply the Mellin transform (in ρ) to $\mathcal{L}u = f$ to obtain $\widehat{\mathcal{L}}_\sigma\widetilde{u_\sigma} = \widetilde{f}_\sigma$, where $\widehat{\mathcal{L}}_\sigma = N(\mathcal{L})$ is the indicial operator of $\mathcal{L}$. As we are considering the forward problem, ψ is supported in the forward light cone from g, and we may therefore assume that $\widetilde{u_\sigma}$ is supported in $\overline{C_+}$.

The conormal spaces we consider are adapted to our variable-order spaces $\mathcal{Y}^s$:

Definition 35. For a distribution u on mf, we say that $u \in I^{(s)}(S_+)$ if $u \in \mathcal{Y}^s$ and $A_1 \dots A_k u \in \mathcal{Y}^s$ for all $A_j \in \mathcal{M}$ with $\mathcal{M}$ as in Definition 13 and $k \in \mathbb{N}$.

In particular, elements of $I^{(s)}(S_+)$ lie in $H_{\mathrm{b}}^{1,\infty}$ away from S_+ and are therefore smooth away from S_+ and $x = 0$. The order s influences the regularity microlocally at N^*S_+, in the sense that, if $u \in I^{(s)}(S_+)$ and $s' > s$ then $\mathrm{WF}_{\mathrm{b}}^{1,s'}(u) \subset N^*S_+$.

To assist in bookkeeping, we recall a bit of notation from previous work [BVW18]:

Definition 36. For $\varsigma, s \in \mathbb{R}$, we let $\mathbb{C}_\varsigma$ denote the upper half-plane $\operatorname{Im}\sigma > -\varsigma$ and define

$$\mathcal{B}(\varsigma, s) = \mathcal{H}(\mathbb{C}_\varsigma) \cap \langle\sigma\rangle^{-\infty} L^\infty_{\operatorname{Im}\sigma} L^2(\mathbb{R}, I^{(s)}(S_+)).$$

As in [BVW15, Def. 2.1], for any Fréchet space $\mathcal{F}$ and $k \in \mathbb{R}$, $\mathcal{H}(\mathbb{C}_\varsigma) \cap \langle\sigma\rangle^k L^\infty_{\operatorname{Im}\sigma} L^2(\mathbb{R}, \mathcal{F})$ is the space of holomorphic functions g of σ with values in $\mathcal{F}$ such that for each fixed line $\operatorname{Im}\sigma = C > -\varsigma$, with $\mu = \operatorname{Re}\sigma$, for each seminorm $|\cdot|_\bullet$ of $\mathcal{F}$, we have $\langle\mu\rangle^k |g(\mu + iC)|_\bullet \in L^2(d\mu)$ with L^2-norm bounded uniformly in C.

Given $f \in C_c^\infty(M)$, $\widetilde{f_\sigma} \in \mathcal{B}(\varsigma, s)$ for all ς and s. Our main use of Section 5 is summarized in the following lemma:

Lemma 37 (c.f. [BVW15, BVW18, BM19])**.** *There are ς_0 and s_0 so that*

$$\widetilde{u_\sigma} \in \mathcal{B}(\varsigma_0, s_0 - 0).$$

The proof is essentially identical to one provided in previous work [BVW15, Section 9] and follows from the interpolation of the propagation and module regularity estimates in Section 5 together with the mapping properties of the Mellin transform.

We know already that $\widehat{\mathcal{L}}_\sigma \widetilde{u_\sigma} = \widetilde{f_\sigma}$ and $\widetilde{f_\sigma}$ is entire. As $\widetilde{f_\sigma} \in \mathcal{Y}^s$ and $\widetilde{u_\sigma} \in \mathcal{X}^s$, we may apply Theorem 21 to conclude that $\widetilde{u_\sigma}$ is a meromorphic function on the strip $s_+ < \operatorname{Im}\sigma < s_-$ taking values in $\mathcal{X}^s$ provided that $s_{\mathrm{ftr}}|_{S_+} \leq s_+$ and $s_{\mathrm{ftr}}|_{S_-} > s_-$. Shifting the contour, applying the residue theorem, and interpolating then yields

$$\widetilde{u_\sigma} \in \mathcal{B}\left(\varsigma_0 + N, \min(s_0 - 0, \frac{1}{2} + \varsigma_0 - N - 0)\right) + \sum_{(\sigma_j, m_j) \in \mathcal{E}, \operatorname{Im}\sigma_j > -\varsigma_0 - N} (\sigma - \sigma_j)^{-m_j} a_j,$$

for any N, where

$$a_j \in \mathcal{B}(\varsigma_0 + N, \frac{1}{2} + \operatorname{Im}\sigma_j - 0).$$

Here the regularity of the a_j follows from Cauchy's integral formula. Here $\mathcal{E}$ denotes the set of poles of $\widehat{\mathcal{L}}_\sigma^{-1}$ lying below our initial contour $\operatorname{Im}\sigma = -\varsigma_0$. Inverting the Mellin transform then yields an asymptotic expansion:

Proposition 38. *With $\mathcal{E}$ the resonance index set and ς_0 chosen to ignore those resonances where $\widetilde{u_\sigma}$ is known to be holomorphic, we have*

$$u = \sum_{(\sigma_j, m) \in \mathcal{E}, \operatorname{Im}\sigma_j > -\ell} \rho^{i\sigma_j} (\log\rho)^m a_{jm} + u',$$

where for $C = s_0 + \varsigma_0$,

$$u' \in \rho^\ell H_{\mathrm{b}}^{1, \min(C - \ell - 1 - 0, -1/2 - \varsigma_0 - \ell - 0), \gamma}.$$

The coefficients a_{jm} are smooth functions of ρ taking values in $I^{(1/2 - \operatorname{Re}(i\sigma_j) - 0)}$.

Remark 39. Because $\widetilde{u_\sigma}$ is supported in $\overline{C_+}$, the a_{jm} are also supported in $\overline{C_+}$ for ρ near 0. We see in Section 8 below that in fact all $m = 0$.

One consequence of Proposition 38 is that

$$u' = \left(\prod_{(\sigma_j, m)\in\mathcal{E}(\varsigma_0), \operatorname{Im}\sigma_j > -\ell} (\rho D_\rho - \sigma_j) \right) u \in \rho^\ell H_{\mathrm{b}}^{1,\min(C-\ell-1-0,-1/2-\varsigma_0-\ell-0),\gamma}.$$

By Theorem 14, u enjoys module regularity with respect to $\rho^\ell H_{\mathrm{b}}^{1,s',\gamma}$ for some s' and so we can interpolate to show that in fact u' enjoys module regularity with respect to the space $\rho^\ell H_{\mathrm{b}}^{1,\min(C-\ell-1-0,-1/2-\varsigma_0-\ell-0),\gamma}$.

The module $\mathcal{M}$ includes a basis for $\mathcal{V}_{\mathrm{b}}$ with the exception of ∂_v, though $v\partial_v$ lies in $\mathcal{M}$, leading to

Lemma 40. *For all N and ℓ, if $\mathcal{M}^{N+\ell} w \in H_{\mathrm{b}}^{1,p,\gamma}$, then $\mathcal{M}^N v^\ell w \in H_{\mathrm{b}}^{1,p+\ell,\gamma}$.*

We may then commute $\rho^{-\ell}$ through the module factors to see that

$$\mathcal{M}^N \rho^{-\ell} v^\ell \left(\prod_{(\sigma_j, m)\in\mathcal{E}(\varsigma_0), \operatorname{Im}\sigma_j > -\ell} (\rho D_\rho - \sigma_j) \right) u \in H_{\mathrm{b}}^{1,\min(C-1-0,-1/2-\varsigma_0-0),\gamma}.$$

In other words, if we set $\varpi = |\rho/v|$ to be the defining function of the faces C_+ and C_0 in the blow-up $[M; S_+]$, then we have the following result:

Proposition 41. *On C_+, uniformly up to the corner $C_+ \cap \mathcal{I}^+$ in $[M; S_+]$, u enjoys an asymptotic expansion with powers given by the resonance index set:*

$$\left(\prod_{(\sigma_j, m)\in\mathcal{E}(\varsigma_0), \operatorname{Im}\sigma_j > -\ell} (\varpi D_\varpi - \sigma_j) \right) u \in \varpi^\ell H_{\mathrm{b}}^{1,\infty,*,*}([M; S]), \tag{20}$$

where the $*$'s represent fixed (independent of ℓ) growth orders at C_+ and $\mathcal{I}^+$.

We now turn our attention to the expansion at $\mathcal{I}^+$. The following lemma is instrumental in that discussion:

Lemma 42. *With $\mathcal{M}$ denoting the module above,*

$$P + 2D_v\left(\rho D_\rho + v D_v\right) \in \mathcal{M}^2.$$

At this stage, the argument proceeds exactly as in previous work [BVW18, Section 9.2], commuting powers of the radial vector field through the operator and then integrating from $v = -\rho/2$ to show that

$$\mathcal{M}^N \left(\prod_{j=0}^{k} (\rho D_\rho + v D_v + ji) \right) u \in (v + C\rho)^{k+1} H_{\mathrm{b}}^{1,s-1,\gamma}(M)$$

for all k and N. Lifting to the blown up space $[M; S_+]$ then yields

$$\left(\prod_{j=0}^{k} (\rho D_\rho + v D_v + ji) \right) u \in (v + C\rho)^{k+1} H_{\mathrm{b}}^{1,\infty,*,*}([M, S_+]), \tag{21}$$

for some fixed (k-independent) weights $*$. Applying Lemma 12 with the results of equations (20) and (21) yields the result.

8. Characterization of the exponents

Having established that the solution is polyhomogeneous at C_+ and $\mathcal{I}_+$, we now aim to characterize the exponents σ_j in the expansion at C_+. As polyhomogeneity is preserved under coordinate changes, we use a more convenient system of coordinates.

In particular, in a neighborhood of $\overline{C_+}$, we want to use coordinates $(\bar\rho, \bar x, \theta)$, where

$$\bar\rho = \frac{1}{t+r}, \quad \bar x = \frac{2r}{t+r}, \quad \theta \in \mathbb{S}^2. \tag{22}$$

Similarly, near $\overline{C_-}$, it is convenient to use the coordinate system with

$$\underline{\rho} = \frac{1}{r-t}, \quad \underline{x} = \frac{2r}{r-t}, \quad \theta \in \mathbb{S}^2.$$

As $\widehat{\mathcal{L}}_\sigma$ depends on the choice of defining function $\bar\rho$, let us now, in an abuse of notation, fix a function $\widetilde\rho$ near mf so that

$$\widetilde\rho = \begin{cases} \bar\rho = \frac{1}{r+t} & t > \frac{1}{2}r \\ \underline{\rho} = \frac{1}{r-t} & t < -\frac{1}{2}r, \end{cases}$$

and so that $\widetilde\rho$ remains a defining function for the boundary for $|t| \leq \frac{1}{2}r$. We may even guarantee that $\widetilde\rho$ is homogeneous of degree -1 in (t,r) near $\widetilde\rho = 0$. Note that this is necessarily a different boundary defining function than the one discussed in the previous sections.

In another abuse of notation, we now redefine the operator $\widehat{\mathcal{L}}_\sigma$ of the previous sections to be the normal operator

$$\widehat{\mathcal{L}}_\sigma = \widehat{N}(\widetilde\rho^{-2-i\mathbf{Z}} i\gamma^0 \partial\!\!\!/_V \widetilde\rho^{1+i\mathbf{Z}}),$$

which is related to our earlier operator by changes of coordinates, conjugation by a non-vanishing smooth function, and a change in the definition of the Mellin transform (from ρ to $\widetilde\rho$). In particular, the Fredholm properties of $\widehat{\mathcal{L}}_\sigma$ and the locations of the poles of its inverse are unchanged. The operator $\widehat{\mathcal{L}}_\sigma$ is conformally "Dirac-type" in the sense that its square has principal part which is conformal to the hyperbolic Laplacian, as can be seen from setting $V = 0$ and recalling from [Vas13, Section 5] that the Mellin-transformed operator $\widehat{\rho^{-2}\Box}(\sigma)$ is conformally equivalent to $\Delta_{\mathbb{H}^3} - \overline{\sigma}^2 - 1$ on the cap C_+. Whether $\widehat{\mathcal{L}}_\sigma$ is given (conformally) by a canonical Dirac operator on $\mathbb{H}^3$ plus a potential is irrelevant for our purposes here.

In a still further abuse of notation, we use the letter x to denote both the coordinates $\bar x$ and $\underline{x}$ above in the relevant regions of interest.

With our choice of $\widetilde\rho$ above, we now record the form of the operator $\widehat{\mathcal{L}}_\sigma$ in the caps $C_\pm$. In a neighborhood of $\overline{C_+}$, we have

$$\widehat{\mathcal{L}}_\sigma = xD_x + \sigma + \mathbf{Z} - i - \frac{2\mathbf{Z}}{x} - \alpha_r\left((2-x)D_x - \frac{2i}{x} + \frac{2i}{x}\beta K - \sigma - \mathbf{Z} + i\right),$$

while in a neighborhood of $\overline{C_-}$, the signs of the first three terms flip:

$$\widehat{\mathcal{L}}_\sigma = -xD_x - \sigma - \mathbf{Z} + i - \frac{2\mathbf{Z}}{x} - \alpha_r\left((2-x)D_x - \frac{2i}{x} + \frac{2i}{x}\beta K - \sigma - \mathbf{Z} + i\right).$$

Before continuing, we further observe (though we do not record it explicitly), that $\widehat{\mathcal{L}}_\sigma$ is a hyperbolic operator on the interior of C_0.

We now use separation of variables to show that the operators $\widehat{\mathcal{L}}_\sigma$ reduce to hypergeometric operators in $C_\pm$. Recall from Section 2.2 that for $\kappa \in \mathbb{Z} \setminus \{0\}$, the $-\kappa$ eigenspace of K is spanned by

$$a \begin{pmatrix} \Omega_{\kappa,\mu} \\ \Omega_{-\kappa,\mu} \end{pmatrix} + b \begin{pmatrix} \Omega_{\kappa,\mu} \\ -\Omega_{-\kappa,\mu} \end{pmatrix} \equiv a\mathbf{\Omega}_- + b\mathbf{\Omega}_+,$$

where $\Omega_{\kappa,\mu}$ are spinor spherical harmonics and $\mathbf{\Omega}_\pm$ lies in the ± 1-eigenspace of α_r. The operator $\widehat{\mathcal{L}}_\sigma$ respects these eigenspaces and we may therefore compute the action of $\widehat{\mathcal{L}}_\sigma$ on an element of the eigenspace. In a neighborhood of $\overline{C_+}$, this has the form

$$\begin{aligned} \widehat{\mathcal{L}}_\sigma a\mathbf{\Omega}_- &= 2\left(D_x - \frac{i}{x} - \frac{\mathbf{Z}}{x}\right) a\mathbf{\Omega}_- + \frac{2i\kappa}{x} a\mathbf{\Omega}_+ \\ \widehat{\mathcal{L}}_\sigma b\mathbf{\Omega}_+ &= -2\left((1-x)D_x - \frac{i}{x} + \frac{\mathbf{Z}}{x} - (\sigma + \mathbf{Z} - i)\right) b\mathbf{\Omega}_+ - \frac{2i\kappa}{x} b\mathbf{\Omega}_- \end{aligned} \tag{23}$$

Similarly, in a neighborhood of $\overline{C_-}$, it has the form

$$\begin{aligned} \widehat{\mathcal{L}}_\sigma a\mathbf{\Omega}_- &= 2\left((1-x)D_x - \frac{i}{x} - \frac{\mathbf{Z}}{x} - (\sigma + \mathbf{Z} - i)\right) a\mathbf{\Omega}_- + \frac{2i\kappa}{x} a\mathbf{\Omega}_+ \\ \widehat{\mathcal{L}}_\sigma b\mathbf{\Omega}_+ &= -2\left(D_x - \frac{i}{x} + \frac{\mathbf{Z}}{x}\right) b\mathbf{\Omega}_+ - \frac{2i\kappa}{x} b\mathbf{\Omega}_- \end{aligned} \tag{24}$$

In particular, separating into eigenspaces of K, if

$$\widehat{\mathcal{L}}_\sigma(a\mathbf{\Omega}_- + b\mathbf{\Omega}_+) = h_1\mathbf{\Omega}_- + h_2\mathbf{\Omega}_+,$$

then a and b must solve a pair of coupled ordinary differential equations.

Indeed, after setting $\nu = \sqrt{\kappa^2 - \mathbf{Z}^2}$ and

$$F = x^{1-\nu}a, \quad G = x^{1-\nu}b, \quad H_1 = \frac{i}{2}x^{1-\nu}h_1, \quad H_2 = -\frac{i}{2}x^{1-\nu}h_2,$$

these equations have a particularly nice form.

Near $\overline{C_+}$, these equations have the form

$$\begin{aligned} \left(\partial_x + \frac{\nu - i\mathbf{Z}}{x}\right) F + \frac{\kappa}{x} G &= H_1, \\ \left((1-x)\partial_x + \frac{\nu + i\mathbf{Z}}{x} - (\nu + i\sigma + i\mathbf{Z})\right) G + \frac{\kappa}{x} F &= H_2. \end{aligned}$$

Similarly, near $\overline{C_-}$, they have the form

$$\begin{aligned} \left((1-x)\partial_x + \frac{\nu - i\mathbf{Z}}{x} - (\nu + i\sigma + i\mathbf{Z})\right) F + \frac{\kappa}{x} G &= H_1, \\ \left(\partial_x + \frac{\nu + i\mathbf{Z}}{x}\right) G + \frac{\kappa}{x} F &= H_2 \end{aligned}$$

After substituting one equation into the other, we find that F and G satisfy decoupled second order equations on $C_\pm$.

In a neighborhood of $\overline{C_+}$, F and G satisfy the following equations:

$$x(1-x)\partial_x^2 F + (1+2\nu - x(1+2\nu+i\sigma))\,\partial_x F - (\nu + i\sigma + i\mathbf{Z})(\nu - i\mathbf{Z})F \tag{25}$$

$$= (1-x)x\partial_x H_1 + (1+\nu+i\mathbf{Z} - x(1+\nu+i\sigma+i\mathbf{Z}))H_1 - \kappa H_2, \tag{26}$$

$$x(1-x)\partial_x^2 G + (1+2\nu - x(2+2\nu+i\sigma))\,\partial_x G - (\nu + i\sigma + i\mathbf{Z})(1+\nu - i\mathbf{Z})G \tag{27}$$
$$= x\partial_x H_2 + (1+\nu - i\mathbf{Z})H_2 - \kappa H_1.$$

In other words, F and G satisfy inhomogeneous hypergeometric equations with parameters (a, b, c) and $(a+1, b, c)$ respectively, where

$$a = \nu - i\mathbf{Z}, \quad b = \nu + i\sigma + i\mathbf{Z}, \quad c = 1 + 2\nu.$$

Near the other cap $\overline{C_-}$, F and G satisfy a slightly different pair of hypergeometric equations:

$$x(1-x)\partial_x^2 F + (1+2\nu - x(2+2\nu+i\sigma+2i\mathbf{Z}))\,\partial_x F - (\nu+i\sigma+i\mathbf{Z})(1+\nu+i\mathbf{Z})F$$
$$= x\partial_x H_1 + (1+\nu+i\mathbf{Z})H_1 - \kappa H_2,$$
$$x(x-1)\partial_x^2 G + (1+2\nu - x(1+2\nu+i\sigma+2i\mathbf{Z}))\,\partial_x G - (\nu+i\sigma+i\mathbf{Z})(\nu+i\mathbf{Z})G$$
$$= (1-x)x\partial_x H_2 + (1+\nu+i\mathbf{Z} - x(1+\nu+i\sigma+i\mathbf{Z}))H_2 - \kappa H_1.$$

This is again a pair of inhomogeneous hypergeometric equations for F and G but now with parameters $(a+1, b, c)$ and (a, b, c) respectively, with

$$a = \nu + i\mathbf{Z}, \quad b = \nu + i\sigma + i\mathbf{Z}, \quad c = 1 + 2\nu.$$

We now exploit the hypergeometric structure of $\widehat{\mathcal{L}}_\sigma$ to characterize the support of $\widehat{\mathcal{L}}_\sigma^{-1} f$ when f is supported in $\overline{C_+}$.

Lemma 43. *Suppose σ is a regular point of $\widehat{\mathcal{L}}_\sigma^{-1}$. If f is supported in $\overline{C_+}$, then $\widehat{\mathcal{L}}_\sigma^{-1} f$ is also supported in $\overline{C_+}$.*

Proof. Suppose $u = \widehat{\mathcal{L}}_\sigma^{-1} f$. We proceed by separation of variables and decompose both u and f into spinor spherical harmonics; without loss of generality, we may assume both u and f lie in the span of

$$\mathbf{\Omega}_- = \begin{pmatrix} \Omega_{\kappa,\mu} \\ \Omega_{-\kappa,\mu} \end{pmatrix} \text{ and } \mathbf{\Omega}_+ = \begin{pmatrix} \Omega_{\kappa,\mu} \\ -\Omega_{-\kappa,\mu} \end{pmatrix}$$

for κ and μ fixed.

Consider first the solution in C_- and write u in terms of the above basis of spinor spherical harmonics. By the above computation, a rescaling of the components of u in C_- must solve homogeneous hypergeometric equations as f vanishes identically in C_-.

As u must lie in L^2 near $x = 0$, the $\mathbf{\Omega}_-$ component of $x^{1-\nu}u$ must be a multiple of the hypergeometric function $F(a+1, b, c; x)$, where $a = \nu + i\mathbf{Z}$, $b = \nu + i\sigma + i\mathbf{Z}$, and $c = 1 + 2\nu$. Similarly, the $\mathbf{\Omega}_+$ component of $x^{1-\nu}u$ must be a multiple of $F(a, b, c; x)$.

By contrast, as u must be regular at $x = 1$ (i.e., at S_-), the corresponding components of $x^{1-\nu}u$ must be multiples of $F(a+1, b, a+b+2-c; 1-x)$ and $F(a, b, a+b+1-c; 1-x)$ respectively. As these pairs of solutions of the corresponding equations are linearly independent for a dense open subset of σ with $\operatorname{Im}\sigma > 0$ and $\widehat{\mathcal{L}}_\sigma^{-1}$ is meromorphic, u must vanish identically in $\overline{C_-}$.

In fact, u must vanish identically in a neighborhood of S_- as well because the hypergeometric form of the equation continues to hold there in our explicit coordinate systems. We may thus conclude that u vanishes on C_0 as well by finite speed of propagation, as $\widehat{\mathcal{L}}_\sigma$ is a hyperbolic operator in C_0 and u vanishes on a Cauchy surface. □

Lemma 44. *For $0 < |\mathbf{Z}| < 1/2$, and f compactly supported in the interior of C_+, the poles of $\widehat{\mathcal{L}}_\sigma^{-1} f$ occur at*

$$-i(1+\nu+m), \quad \nu = \sqrt{\kappa^2 - \mathbf{Z}^2}, m = 0, 1, 2, \dots.$$

If $\mathbf{Z} = 0$, $\widehat{\mathcal{L}}_\sigma^{-1} f$ has no poles.

Proof. We proceed by explicitly characterizing the kernel of $\widehat{\mathcal{L}}_\sigma^{-1}$.

We first consider $\operatorname{Im}\sigma \gg 0$. As f is compactly supported in C_+, the computations above show that near S_+ but within C_+, the components of $x^{1-\nu}u$ must be linear combinations of hypergeometric functions. In particular, the $\mathbf{\Omega}_-$ component of $x^{1-\nu}u$ must be a linear combination of the following two hypergeometric functions:

$$\begin{aligned} &F(\nu - i\mathbf{Z}, \nu + i\sigma + i\mathbf{Z}, i\sigma; 1-x), \\ (1-x)^{1-i\sigma+i\mathbf{Z}} &F(1+\nu+i\mathbf{Z}, 1+\nu-i\sigma-i\mathbf{Z}, 2-i\sigma; 1-x). \end{aligned}$$

Observe that the function given on $(0,\infty)$ by

$$\begin{cases} F(\nu - i\mathbf{Z}, \nu + i\sigma + i\mathbf{Z}, i\sigma; 1-x) & 0 < x < 1 \\ 0 & x \geq 1 \end{cases},$$

is not regular enough to lie in the image of $x^{1-\nu}\widehat{\mathcal{L}}_\sigma^{-1}$ for $\operatorname{Im}\sigma \gg 0$. In particular, this component of u must in fact be a multiple of

$$(1-x)^{1-i\sigma} F(1+\nu+i\mathbf{Z}, 1+\nu-i\sigma-i\mathbf{Z}, 2-i\sigma; 1-x).$$

Near $x = 0$, u must lie in L^2 and so the components of $x^{1-\nu}u$ must be multiples of the regular solution there, e.g., the $\mathbf{\Omega}_-$ component of $x^{1-\nu}u$ must be a multiple of

$$F(\nu - i\mathbf{Z}, \nu + i\sigma + i\mathbf{Z}, 1+2\nu; x)$$

near $x = 0$.

Using the notation of the calculation above, we may therefore write down explicitly each component of $u = \widehat{\mathcal{L}}_\sigma^{-1} f$; for f in a single (κ,μ)-eigenspace of K,

$$\widehat{\mathcal{L}}_\sigma^{-1} f = \widehat{\mathcal{L}}_\sigma^{-1}\left(f_1\mathbf{\Omega}_- + f_2\mathbf{\Omega}_+\right) = u_-\mathbf{\Omega}_- + u_+\mathbf{\Omega}_+$$

where u_+ and u_- are given explicitly.

The coefficient u_- is given in terms of the following functions (whose names are consistent with the DLMF [DLMF, 15.10(ii)]):

$$\begin{aligned} w_1 &= F(\nu - i\mathbf{Z}, \nu + i\sigma + i\mathbf{Z}, 1+2\nu; x) \\ w_4 &= (1-x)^{1-i\sigma} F(1+\nu+i\mathbf{Z}, 1+\nu-i\sigma-i\mathbf{Z}, 2-i\sigma; 1-x), \end{aligned}$$

where the F functions denote hypergeometric functions of the given parameters and arguments. One consequence of Kummer's connection formulas [DLMF, 15.10(ii)] is a formula for the Wronskian of w_1 and w_4:

$$w_1(x)w_4'(x) - w_1'(x)w_4(x) = -\frac{\Gamma(1+2\nu)\Gamma(2-i\sigma)}{\Gamma(1+\nu+i\mathbf{Z})\Gamma(1+\nu-i\sigma-i\mathbf{Z})} x^{-1-2\nu}(1-x)^{-i\sigma}.$$

We use that $x^{1-\nu}u_-$ satisfies the hypergeometric equation (25) (F in the above calculation) and must have the w_1 behavior near $x = 0$ and the w_4 behavior at $x = 1$. A standard variation of parameters argument then shows that

$$x^{1-\nu}u_- = -\frac{\Gamma(1+\nu+i\mathbf{Z})\Gamma(1+\nu-i\sigma-i\mathbf{Z})}{\Gamma(1+2\nu)\Gamma(2-i\sigma)}\left(w_4(x)\int_0^x y^{1+2\nu}(1-y)^{i\sigma}w_1(y)f_+(y)\,dy\right.$$
$$\left.-w_1(x)\int_1^x y^{1+2\nu}(1-y)^{i\sigma}w_4(y)f_+(y)\,dy\right),$$

where, for ease of display, we have introduced notation for the inhomogeneous term in equation (25):

$$f_+ = \frac{i}{2}\partial_x(x^{1-\nu}f_1) + \frac{i}{2}\frac{1+\nu+i\mathbf{Z}}{x}(x^{1-\nu}f_1) + \frac{i}{2}\frac{-i\sigma}{1-x}(x^{1-\nu}f_1) + \frac{i}{2}\frac{\kappa f_2}{x(1-x)}.$$

A similar expression[6] is available for u_+, which satisfies a different hypergeometric equation:

$$x^{1-\nu}u_+ = -\frac{\Gamma(\nu+i\mathbf{Z})\Gamma(1+\nu-i\sigma-i\mathbf{Z})}{\Gamma(1+2\nu)\Gamma(1-i\sigma)}\left(\widetilde{w}_4\int_0^x y^{1+2\nu}(1-y)^{1+i\sigma}\widetilde{w}_1(y)f_-(y)\,dy\right.$$
$$\left(-\widetilde{w}_1(x)\int_1^x y^{1+2\nu}(1-y)^{1+i\sigma}\widetilde{w}_4(y)f_-(y)\right),$$

where f_- is the same rescaling of the right side of the second equation in (25) and $\widetilde{w}_1, \widetilde{w}_4$ are the analogous hypergeometric functions here:

$$\begin{aligned}
f_- &= -\frac{i}{2(1-x)}\partial_x(x^{1-\nu}f_2) - \frac{i}{2}\frac{1+\nu-i\mathbf{Z}}{x(1-x)}f_2 - \frac{i}{2}\kappa f_1,\\
\widetilde{w}_1 &= F(\nu-i\mathbf{Z}+1, \nu+i\sigma+i\mathbf{Z}, 1+2\nu; x),\\
\widetilde{w}_4 &= (1-x)^{-i\sigma}F(\nu+i\mathbf{Z}, 1+\nu-i\sigma-i\mathbf{Z}, 1-i\sigma; 1-x).
\end{aligned}$$

As these expressions are meromorphic in σ and hold on a dense open subset of $\operatorname{Im}\sigma \gg 0$, they also characterize $\widehat{\mathcal{L}}_\sigma^{-1}f$ at all regular σ.

In particular, the poles of $\widehat{\mathcal{L}}_\sigma^{-1}$ occur precisely when one of the leading coefficients has a pole, which occurs only when $1+\nu-i\sigma-i\mathbf{Z}$ is a pole of the Gamma function,[7] i.e., only when

$$\sigma = -\mathbf{Z} - i(1+\nu+m), \quad m = 0, 1, 2, \ldots.$$

□

Appendix A. Estimates in the bulk near the singularity, and the proof of Theorem 20

We now generalize the results of [BW20] to our setting. The proof of Theorem 20 is summarized at the end of Section A.3.

[6]This expression could also be derived from formulas for contiguous hypergeometric functions.
[7]If $\mathbf{Z} = 0$, then the poles all cancel; this is a reflection of Huygens' principle.

A.1. **Commutators with b-pseudodifferential operators.** The following lemma allows us to essentially ignore the distinction between the operators $\mathcal{L}$ (and P) and their conjugated counterparts.

Lemma 45. *For any* $\ell \in \mathbb{R}$,

$$\rho^{-\ell} P \rho^{\ell} - P \in \frac{1}{x}\Psi_{\mathrm{b}}^{0} + \Psi_{\mathrm{b}}^{1}$$

and

$$\rho^{-\ell}\mathcal{L}\rho^{\ell} - \mathcal{L} \in \Psi_{\mathrm{b}}^{0}.$$

Lemma 46 (c.f. [BW20, Lemma 25])**.** *If* $C \in \Psi_{\mathrm{b}}^{m}$ *is invariant with scalar, real-valued principal symbol, then*

$$[P, C] \in \left\{\frac{1}{x^2}\Delta_\theta, D_x^2, \frac{1}{x}D_x, \frac{1}{x^2}\right\}\Psi_{\mathrm{b}}^{m-1} + \left\{D_x, \frac{1}{x}\right\}\Psi_{\mathrm{b}}^{m} + \Psi_{\mathrm{b}}^{m+1},$$

with the microsupport of the right side contained in $\mathrm{WF}_{\mathrm{b}}' C$.

Similarly,

$$[\mathcal{L}, C] \in \left\{\frac{1}{x}\beta K, D_x, \frac{1}{x}\right\}\Psi_{\mathrm{b}}^{m-1} + \Psi_{\mathrm{b}}^{m},$$

also with the microsupport of the right side contained in $\mathrm{WF}_{\mathrm{b}}' C$.

Lemma 47 (c.f. [BW20, Lemma 26])**.** *If* $C \in \Psi_{\mathrm{b}}^{m}$ *is invariant with scalar, real-valued principal symbol* $\sigma_{\mathrm{b},m}(C) = c$*, then near the north pole* $\{\rho = 0, x = 0, t > 0\}$,

$$\frac{1}{i}[\mathcal{L}, C] = A_0\left(\alpha_r\left(i\partial_x + \frac{i}{x} - \frac{i}{x}\beta K\right) - \frac{\mathbf{Z}}{x}\right) + B_0 + \alpha_r B_1 + \mathbf{B}_2\frac{1}{r} + \mathbf{B}_3 D_r + \mathbf{B}_4 \tag{28}$$

where

- $A_0 \in \Psi_{\mathrm{b}}^{m-1}$*, with* $\sigma_{\mathrm{b}}(A_0) = -\partial_\xi c$,
- $B_0 \in \Psi_{\mathrm{b}}^{m}$*, with* $\sigma_{\mathrm{b}}(B_0) = -\rho\partial_\rho c - x\partial_x c$,
- $B_1 \in \Psi_{\mathrm{b}}^{m}$*, with* $\sigma_{\mathrm{b}}(B_1) = \partial_x c$*, and*
- $\mathbf{B}_2 \in \Psi_{\mathrm{b}}^{m}$*, with* $\operatorname{supp}\sigma_{\mathrm{b}}(\mathbf{B}_2) \subset \operatorname{supp}\partial_\eta c$.
- $\mathbf{B}_3 \in \Psi_{\mathrm{b}}^{m-1}$*, with* $\operatorname{supp}\sigma_{\mathrm{b}}(\mathbf{B}_2) \subset \operatorname{supp}\partial_\eta c$.
- $\mathbf{B}_4 \in \left\{\frac{1}{x}, D_x\right\}\Psi_{\mathrm{b}}^{m-2} + \Psi_{\mathrm{b}}^{m-1}$.

As in [BW20], bold letters are used to denote nonscalar operators.

Proof. The proof follows [BW20, Lemma 26] closely. Briefly, using the expression for $\mathcal{L}$ from Section 4.1, one can take $B_0 = [\frac{1}{i}\rho\partial_\rho + \frac{1}{i}x\partial_x, C]$ and $[1/x, C] = A_0(i/x)$. The lemma follows from repeated application of Lemma 7. □

A.2. **Elliptic regularity.** The main result of this section is the following proposition establishing the first part of Theorem 20.

Proposition 48. *If* $\psi \in H_{\mathrm{b}}^{1,-\infty,\gamma}$ *is the forward solution of* $i\partial\!\!\!/_{\mathbf{Z}/r}\psi = g$ *with* $g \in C_c^\infty$*, then in a neighborhood of* $\{\rho = 0, x = 0\}$,

$$\mathrm{WF}_{\mathrm{b}}^{1,m,\gamma}\psi \subset \dot{\Sigma}$$

for all m.

As g is compactly supported ψ is a solution of $\partial\!\!\!/\psi = 0$ near $\{\rho = 0, x = 0\}$ and so $\mathrm{WF}_{\mathrm{b}}^{0,*,*} g$ plays no role there, though an analogous statement is true with $\mathrm{WF}_{\mathrm{b}}^{0,m+1,\gamma+1} g$ in the correct place.

Proposition 48 essentially follows by an integration by parts argument with several useful consequences. The argument is nearly identical to the one given in previous work [BW20] and so we omit many of the details.

As the result is local near the pole, we replace ψ by $\chi\psi$, where χ is a smooth function localizing to this region. By Corollary 16, we know that $\chi\psi \in H_{\mathrm{b}}^{1,0,1/2-\epsilon}$. As we may assume $\gamma < 1/2$, we then have $\rho^{-\gamma}\chi\psi \in H_{\mathrm{b}}^{1,0,0}$; it therefore suffices to prove the analogous theorem about $\mathrm{WF}_{\mathrm{b}}^{1,m,0}$ for $u \in H_{\mathrm{b}}^{1,0,0}$; the difference between the equation satisfied by ψ and that of u is of lower order and can be absorbed into the error terms in the estimate.

The proof of Proposition 48 is essentially identical to the proof away from the boundary [BW20, Lemma 28] and will be mostly omitted for brevity. We include only the main tool in the proof: the following lemma, which is proved by considering the real part of the pairing $\langle PA_\lambda u, A_\lambda u\rangle$.

Lemma 49. *Suppose that $K \subset U \subset {}^{\mathrm{b}}S^*M$ with K compact and U open, and suppose that A_λ are a bounded family of invariant elements in Ψ_{b}^m with $\mathrm{WF}'_{\mathrm{b}} A_\lambda \subset K$ in the sense of uniform wavefront set of families, and $A_\lambda \in \Psi_{\mathrm{b}}^{m-1}$ for all $\lambda \in (0,1)$. There exist $G \in \Psi_{\mathrm{b}}^{m-1/2}$, $\widetilde{G} \in \Psi_{\mathrm{b}}^m$, both microsupported in U, and C_0 so that for all $\epsilon > 0$, $\lambda \in (0,1)$, $u \in H_{\mathrm{b}}^{1,0,0}$ with $\mathrm{WF}_{\mathrm{b}}^{1,m-1/2,0} u \cap U = \emptyset$, and $\mathrm{WF}_{\mathrm{b}}^{0,m,0}(\mathcal{L}u) \cap U = \emptyset$,*

$$\left| \|D_x A_\lambda u\|^2 + \left\|\frac{1}{x}\nabla_\theta A_\lambda u\right\|^2 - \left\|\left(\rho\partial_\rho + x\partial_x + 1 + i\mathbf{Z} - \frac{i\mathbf{Z}}{x}\right)A_\lambda u\right\|^2 \right|$$
$$\leq C_0\left(\epsilon \|A_\lambda u\|^2_{H_{\mathrm{b}}^{1,0,0}} + \|u\|^2_{H_{\mathrm{b}}^{1,0,0}} + \|Gu\|^2_{H_{\mathrm{b}}^{1,0,0}} + \frac{1}{\epsilon}\left(\|\mathcal{L}u\|^2 + \left\|\widetilde{G}\mathcal{L}u\right\|^2\right)\right).$$

Proof. We start by fixing $G, \widetilde{G}$ of the appropriate order microsupported in U and so that the principal symbols of both operators are identically 1 on K.

The pairing

$$\mathrm{Re}\,\langle PA_\lambda u, A_\lambda u\rangle$$

is finite for all $\lambda > 0$ by our wavefront set hypothesis, which implies that $PA_\lambda u \in H_{\mathrm{b}}^{-1,0,0}$ and $A_\lambda u \in H_{\mathrm{b}}^{1,0,0}$. We now write

$$|\mathrm{Re}\,\langle PA_\lambda u, A_\lambda u\rangle| \leq |\langle [P, A_\lambda]u, A_\lambda u\rangle| + |\langle A_\lambda Pu, A_\lambda u\rangle|. \tag{29}$$

The second term is estimated by

$$|\langle A_\lambda Pu, A_\lambda u\rangle| \leq \|A_\lambda Pu\|_{H_{\mathrm{b}}^{-1,0,0}} \|A_\lambda u\|_{H_{\mathrm{b}}^{1,0,0}} \leq \epsilon \|A_\lambda u\|^2_{H_{\mathrm{b}}^{1,0,0}} + \epsilon^{-1}\|A_\lambda Pu\|^2_{H_{\mathrm{b}}^{-1,0,0}}.$$

We now write $P = \widetilde{\mathcal{L}}_1\mathcal{L}$. By Lemma 7, $[A_\lambda, \widetilde{\mathcal{L}}_1] \in \frac{1}{x}\Psi_{\mathrm{b}}^s$ with uniform estimates in this space and so, by the elliptic regularity in Lemma 8 we have the bound

$$\epsilon \|A_\lambda u\|^2_{H_{\mathrm{b}}^{1,0,0}} + C\epsilon^{-1}\left(\|\mathcal{L}u\|^2 + \left\|\widetilde{G}\mathcal{L}u\right\|^2\right).$$

We now turn our attention to the commutator term in equation (29). By Lemma 46, we know

$$[P, A_\lambda] \in \left\{\frac{1}{x^2}\Delta_\theta, D_x^2, \frac{1}{x}D_x, \frac{1}{x^2}\right\}\Psi_{\mathrm{b}}^{m-1} + \left\{\frac{1}{x}\beta K, D_x, \frac{1}{x}\right\}\Psi_{\mathrm{b}}^m + \Psi_{\mathrm{b}}^{m+1},$$

and so again by Lemma 8, we have

$$|\langle [P, A_\lambda]u, A_\lambda u\rangle| \lesssim \|Gu\|^2_{H_{\mathrm{b}}^{1,0,0}} + \|u\|^2_{H_{\mathrm{b}}^{1,0,0}}.$$

□

A.3. **Hyperbolic propagation.** As in the previous subsection, we work near the poles as the statement away from $\rho = 0$ is in previous work [BW20, Theorem 22].

Lemma 49 and the Hardy inequality show that if $u \in H_{\mathrm{b}}^{1,0,0}$ and $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{0,m+1} \mathcal{L}u$, then

$$q_0 \in \mathrm{WF}_{\mathrm{b}}^{1,m} u \text{ if and only if } q_0 \in \mathrm{WF}_{\mathrm{b}}^{0,m+1} u.$$

The proof of the hyperbolic part of the estimate near the pole is similar to setting for finite time and exploits the near-homogeneity (in x) of $\mathcal{L}$. We denote by U a neighborhood of q_0 in $\dot{\Sigma}$ with

$$U \cap \{\xi/\sigma_0 > 0\} \cap \mathrm{WF}_{\mathrm{b}}^{0,m+1/2} u, \ U \cap \mathrm{WF}_{\mathrm{b}}^{0,m+1/2}(\mathcal{L}u) = \emptyset.$$

For our inductive hypothesis, we assume that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{0,m}(u)$ and aim to show that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{0,m+1/2} u$. We assume for the text below that $\sigma_0 = 1$; only minor modifications are needed for $\sigma_0 = -1$.

Set $\omega = x^2 + \rho^2$ and let

$$\phi = -\widehat{\xi/\sigma_0} + \frac{1}{\beta^2\delta}\omega.$$

Fix cutoff functions χ_0, χ_1, and χ_2 so that

- χ_0 is supported in $[0, \infty)$ with $\chi_0(s) = \exp(-1/s)$ for $s > 0$,
- χ_1 is supported in $[0, \infty)$ with $\chi_1(s) = 1$ for $s \geq 1$ and $\chi_1' \geq 0$, and
- χ_2 is supported in $[-2c_1, 2c_1]$ and is equal to 1 on $[-c_1, c_1]$.

Here c_1 is chosen so that $\hat{\xi}^2 + |\hat{\eta}|^2 < c_1 < 2$ in $\dot{\Sigma} \cap U$. We now set

$$a = |\sigma|^{m+1/2} \chi_0(2 - \phi/\delta)\chi_1(2 - \hat{\xi}/\delta)\chi_2(\hat{\xi}^2 + |\hat{\eta}|^2)\mathbf{1}_{\operatorname{sgn}\sigma = \operatorname{sgn}\sigma_0}, \tag{30}$$

and let A be its quantization to an invariant element of $\Psi_{\mathrm{b}}^{m+1/2}$. Note that

$$\operatorname{supp} a \subset \left\{ \left|\hat{\xi}\right| < 2\delta, \omega < 4\beta^2\delta^2 \right\},$$

and so the support of a in ${}^{\mathrm{b}}T^*M$ can be made arbitrarily close to q_0.

The following lemma is proved by a careful application of Lemma 47.

Lemma 50. *For A defined as above,*

$$\frac{1}{i}[\mathcal{L}, A^*A] = \widetilde{R}\mathcal{L} + \operatorname{sgn}(\sigma_0)Q^*Q + \mathbf{R} + B_0 + \alpha_r B_1 + E' + E''$$

where

- $Q \in \Psi_{\mathrm{b}}^{m+1/2}$ *is invariant and self-adjoint with*
$$\sigma_{\mathrm{b}}(Q) = \sqrt{2}\,|\sigma|^m\,|\sigma + \xi|^{1/2}\,\delta^{-1/2}(\chi_0'\chi_0)^{1/2}\chi_1\chi_2\mathbf{1}_{\operatorname{sgn}\sigma = \operatorname{sgn}\sigma_0},$$
- $\widetilde{R} \in \Psi_{\mathrm{b}}^{2m}$,
- $\mathbf{R} \in \left\{\frac{1}{x}\beta K, D_x, \frac{1}{x}\right\}\Psi_{\mathrm{b}}^{2m-1} + \Psi_{\mathrm{b}}^{2m}$,
- $B_0, B_1 \in \Psi_{\mathrm{b}}^{2m+1}$ *with* $|\sigma_{\mathrm{b}}(B_\bullet)|$ *equal to an order* 0 *symbol times* $C\beta^{-1}\sigma_{\mathrm{b}}(Q)^2$,
- $E' \in \Psi_{\mathrm{b}}^{2m+1}$ *with* $\mathrm{WF}_{\mathrm{b}}' E' \subset \{\delta \leq \hat{\xi} \leq 2\delta, \omega \leq 4\beta^2\delta^2\}$, *and*

- $E'' \in \frac{1}{x}\Psi_{\mathrm{b}}^{2m+1} + D_x\Psi_{\mathrm{b}}^{2m} + \Psi_{\mathrm{b}}^{2m+1}$ *with* $\mathrm{WF}'_{\mathrm{b}}\, E'' \cap \dot{\Sigma} = \emptyset$.

Proof. We carefully apply Lemma 47 and use its notation. The term A_0 arising there has principal symbol $-\partial_\xi(a^2)$ and arises from $\mathcal{L}$ being nearly homogeneous in r of degree -1. We rewrite the A_0 term in (28) as $A_0(\mathcal{L} + i\rho\partial_\rho + ix\partial_x)$, modulo A_0 times smooth lower-order terms (which are absorbed into $\mathbf{R}$). We then split the symbol of A_0 into three terms: those terms where the ξ derivative falls on χ_0 can be written in the form $\widetilde{Q}^2(i\rho\partial_\rho + ix\partial_x)$, which we write as the product of $\operatorname{sgn}(\tau_0)$ times squares Q^2 modulo a lower-order term to be absorbed into $\mathbf{R}$. Those terms where the derivative falls on χ_1 are absorbed into E' and those where it falls on χ_2 form part of E''. Thus, modulo further commutators (to be absorbed into $\mathbf{R}$), the first term on the right side of (28) is given by $\widetilde{R}\mathcal{L} + \operatorname{sgn}(\sigma_0)Q^*Q$.

The B_1 term satisfies the stated symbol bound because x derivatives on a^2 may fall only on the χ_0 term, giving

$$2\,|\sigma|^{2m+1}\,(\chi_0'\chi_0)\chi_1^2\chi_2^2(-2x)(\beta^{-2}\delta^{-2}).$$

As $0 \le x \le 2\beta\delta$ on the support of a, this term can be estimated by a multiple of

$$\beta^{-1}\delta^{-1}\,|\sigma|^{2m+1}\,\chi_0'\chi_0\chi_1^2\chi_2^2,$$

which is a constant multiple of $\frac{|\tau|}{|\tau+\xi|}\sigma_{\mathrm{b}}(Q)^2$. This prefactor is a symbol of order zero (provided the neighborhood U is small enough). Likewise, the B_0 term in the Lemma 47 becomes the B_0 term here and is estimated similarly, as the ρ derivative may also only hit the χ_0 term.

Finally, the remaining $\mathbf{B}_2$ term in Lemma 47 is proportional to $\partial_\eta(a^2)$ and therefore the derivative must fall on χ_2 and so these terms are absorbed into E''. □

The rest of the proof of Theorem 20 is a positive commutator estimate. We pair $\frac{1}{i}[\mathcal{L}, A^*A]u$ with u and regularize as in the elliptic setting. On the one hand, as $\mathcal{L}^* = \mathcal{L} - i$, we may bound

$$|\langle[\mathcal{L}, A^*A]u, u\rangle| \le 2\,\|Au\|\,\|A\mathcal{L}u\| + \|Au\|^2 \le 2\,\|Au\|^2 + \|A\mathcal{L}u\|^2,$$

while on the other hand we apply Lemma 50.

The main term is $\operatorname{sgn}(\sigma_0)\,\langle Q^*Qu, u\rangle = \operatorname{sgn}(\sigma_0)\,\|Qu\|^2$, which has a definite sign. We then bound

$$\|Qu\|^2 \le 2\,\|Au\|^2 + \|A\mathcal{L}u\|^2 + \left|\left\langle \widetilde{R}\mathcal{L}u, u\right\rangle\right| + |\langle\mathbf{R}u, u\rangle| + |\langle B_0u, u\rangle| + |\langle\alpha_r B_1 u, u\rangle| + |\langle E'u, u\rangle| + |\langle E''u, u\rangle|.$$

The $\widetilde{R}$ term is bounded by $\|G_m\mathcal{L}u\|\,\|G_m u\|$ for some $G_m \in \Psi_{\mathrm{b}}^m$, while the $\mathbf{R}$ term can be estimated by $\|G_{m-1}u\|_{H_{\mathrm{b}}^{1,0,0}}\,\|G_m u\|$ for $G_{m-1} \in \Psi_{\mathrm{b}}^{m-1}$ and $G_m \in \Psi_{\mathrm{b}}^m$. This leaves the terms involving B_0, B_1, E', and E''.

The terms involving B_0 and B_1 are estimated by the symbol calculus, i.e.,

$$|\langle B_j u, u\rangle| \le C\beta^{-1}\,\|Qu\|^2 + C\,\|Gu\|^2 + C\,\|u\|^2_{H_{\mathrm{b}}^{1,0,0}}$$

for some $G \in \Psi_{\mathrm{b}}^m$ elliptic on the support of B_j.

The term involving E' is bounded by $\left\|G_{m+1/2}u\right\|^2$, where $G_{m+1/2} \in \Psi_{\mathrm{b}}^{2+1/2}$ has $\mathrm{WF}'_{\mathrm{b}}\, G_{m+1/2} \subset \{\delta \le \hat{\xi} \le 2\delta, \omega \le 4\beta^2\delta^2\}$. The hypothesis that $U \cap \{\xi > 0\} \cap \mathrm{WF}_{\mathrm{b}}^{1,m-1/2}\, u = \emptyset$ implies that this term is finite.

Finally, we turn to the term involving E''. The microsupport of E'' is contained in the elliptic set of $\mathcal{L}$, so we may use elliptic regularity to bound this term by

$$C\left(\|G_{m-1}u\|^2_{H_{\mathrm{b}}^{1,0,0}} + \|G_m\mathcal{L}u\|^2 + \|u\|^2_{H_{\mathrm{b}}^{1,0,0}}\right),$$

where $G_r \in \Psi_{\mathrm{b}}^r$ are microsupported in the elliptic region within U.

As $\sigma_{\mathrm{b}}(A)$ is bounded by a small multiple of $\sigma_{\mathrm{b}}(Q)$, we then know that $\|Qu\|^2$ is finite; since Q is elliptic at q_0, we know that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{0,m+1/2} u$ (and hence not in $\mathrm{WF}_{\mathrm{b}}^{1,m-1/2} u$), finishing the proof.

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