RADIATION FIELDS ON SCHWARZSCHILD SPACETIME

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ABSTRACT. In this paper we define the radiation field for the wave equation on the Schwarzschild black hole spacetime. In this context it has two components: the rescaled restriction of the time derivative of a solution to null infinity and to the event horizon. In the process, we establish some regularity properties of solutions of the wave equation on the spacetime. In particular, we prove that the regularity of the solution across the event horizon and across null infinity is determined by the regularity and decay rate of the initial data at the event horizon and at infinity. We also show that the radiation field is unitary with respect to the conserved energy and prove support theorems for each piece of the radiation field.

1. INTRODUCTION

In this paper we define the radiation field for the wave equation on the Schwarzschild black hole spacetime. The radiation field is a rescaled restriction of the time derivative of a solution and in this case has two components: one corresponding to null infinity and one corresponding to the event horizon. In the process, we establish some regularity properties of solutions of the wave equation on the spacetime. In particular, we prove that the regularity of the solution across the event horizon and across null infinity is determined by the regularity and decay rate of the initial data at the event horizon and at infinity. We further show that the radiation field is unitary with respect to the conserved energy and prove support theorems for each component of the radiation field.

The radiation field for a solution of the wave equation describes the radiation pattern seen by distant observers. On Minkowski space $\mathbb{R} \times \mathbb{R}^n$, it is the rescaled restriction of a solution to null infinity. More precisely, one introduces polar coordinates (r, ω) in the spatial variables as well as the "lapse" parameter $s = t - r$. The forward radiation field of a solution u of $(\partial_t^2 - \Delta)u = 0$ with smooth, compactly supported initial data is given by

$$\lim_{r \to \infty} \partial_s r^{\frac{n-1}{2}} u(s + r, r\omega).$$

The map taking the initial data to the radiation field of the corresponding solution provides a unitary isomorphism from the space of finite energy initial data to $L^2(\mathbb{R}_s \times \mathbb{S}^{n-1}_\omega)$. The radiation field is a translation representation of the wave group and was initially defined by Friedlander [Fri80], though it is implicit in the work of Lax–Phillips (e.g., [LP89]) and Helgason (e.g., [Hel99]). Its definition, structure, and properties have been studied in a variety of geometric contexts [SB03, SB05, SBW05, SB08, BSB12], including settings of interest in general relativity [Wan13, BVW12].

We now recall the structure of the Schwarzschild black hole spacetime. (For a more thorough discussion, including many different coordinate systems, we direct the reader to the the book of Hawking and Ellis [HE73] or to the lecture notes of Dafermos and Rodnianski [DR13].) The Schwarzschild spacetime is diffeomorphic to $\mathbb{R}_t \times (2M, \infty)_r \times \mathbb{S}^2_\omega$ with Lorentzian metric given by

$$g_S = -\left(\frac{r - 2M}{r}\right) dt^2 + \left(\frac{r}{r - 2M}\right) dr^2 + r^2 \, d\omega^2.$$

Here $d\omega^2$ is the round metric on the unit sphere $\mathbb{S}^2$.

We consider the Cauchy problem:

(1.1) $$\Box_S u = 0, \quad (u, \partial_t u)|_{t=0} = (\phi, \psi)$$

where $\Box_S$ is the Laplace–Beltrami (D'Alembertian) operator for g_S:

$$\Box_S = -\left(\frac{r}{r - 2M}\right) \partial_t^2 + \left(\frac{r - 2M}{r}\right) \partial_r^2 + \frac{1}{r^2}\Delta_\omega + \frac{2(r - M)}{r^2}\partial_r.$$

Solutions u of equation (1.1) possesses a conserved energy $E(t)$:

$$E(t) = \int_{2M}^\infty \int_{\mathbb{S}^2} e(t) r^2 \, d\omega \, dr$$

where

$$e(t) = \left(1 - \frac{2M}{r}\right)^{-1} (\partial_t u)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r u)^2 + \frac{1}{r^2}\left|\nabla_\omega u\right|^2.$$

Observe that $e(t)$ is positive definite but is ill-behaved at $r = 2M$.

As the Schwarzschild black hole has two spatial ends, there are two ends through which null geodesics (i.e., light rays) can "escape": the event horizon (at the $r = 2M$ end) and null infinity (at the $r = \infty$ end). (There are also "trapped" null geodesics tangent to the photon sphere $r = 3M$.) In terms of incoming Eddington–Finkelstein coordinates,

$$(\tau = t + r + 2M \log(r - 2M), r, \omega),$$

the event horizon corresponds to $r = 2M$. Similarly, in terms of outgoing Eddington–Finkelstein coordinates,

$$(\bar{\tau} = t - r - 2M \log(r - 2M), r, \omega),$$

null infinity corresponds to $r = \infty$. Figure 1 depicts the Penrose diagram of the Schwarzschild black hole exterior; $E_1^{\pm}$ corresponds to the event horizons and $S_1^{\pm}$ to null infinity.
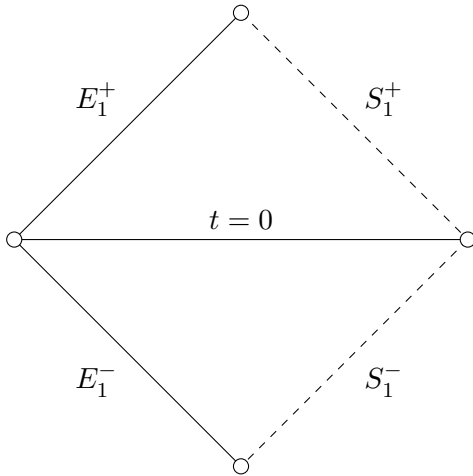

FIGURE 1. The Penrose diagram of the Schwarzschild exterior. $E_1^{\pm}$ are the event horizons for the black and white holes, while $S_1^{\pm}$ are the future and past null infinities.

In this paper we study the behavior of solutions to the Cauchy problem in terms of their radiation fields. We first partially compactify the Schwarzschild spacetime to a manifold with corners and find expansions for solutions of equation (1.1) at each boundary hypersurface. Two of the boundary hypersurfaces correspond to null infinity and the event horizon. Currently we do not include a full compactification, as we omit temporal infinity. A full compactification on which solutions are well-behaved is expected to be somewhat complicated, as it must take into account the different expected behaviors of solutions at the event horizon, null infinity, and the photon sphere ($r = 3M$).

The main regularity result of this paper is the following:

Theorem 1.1. *If the initial data (ϕ, ψ) has an asymptotic expansion at $r = 2M$, i.e., if*

$$\phi = (r - 2M)^{\lambda/2}\tilde{\phi}(\sqrt{r - 2M}, \omega), \quad \psi = (r - 2M)^{\lambda/2}\tilde{\psi}(\sqrt{r - 2M}, \omega),$$

where $\tilde{\phi}, \tilde{\psi} \in C^{\infty}([0, \infty) \times \mathbb{S}^2)$, then the solution u has an asymptotic expansion in terms of $e^{\tau/4M}$ at $\tau = -\infty$ near the event horizon and the k-th term (defined by equation (3.3)) in the expansion is $C^{l,\delta}$ up to the event horizon, where $0 < \lambda + k = l + \alpha$ and $\delta = \min\{\alpha - \epsilon, 1/2\}$ for any $\epsilon > 0$.

Similarly, if the initial data has a classical asymptotic expansion at infinity, i.e., if

$$\phi = r^{-\lambda-1}\tilde{\phi}(1/r, \omega), \quad \psi = r^{-\lambda-2}\tilde{\psi}(1/r, \omega),$$

where $\tilde{\phi}, \tilde{\psi} \in C^\infty([0, 1/2M) \times \mathbb{S}^2)$, *then the solution u has a (polyhomogeneous) asymptotic expansion in terms of $-1/\bar{\tau}$ at $\bar{\tau} = -\infty$ near null infinity and the k-th term (defined by equation (3.5)) in the expansion is $C^{l,\delta}$ up to null infinity, where l and δ are as above.*

Remark 1.2. *The above theorem shows that if the initial data are smooth and have asymptotic expansions at $r = 2M$ and $r = \infty$, then the solution has an asymptotic expansion at E_0 and S_0 (as pictured in Figure 2). The regularity at E_1^+ (respectively, S_1^+) of each term in this expansion is determined by its rate of decay at E_0 (respectively, S_0) and hence by the expansion of the initial data. The structure of the partial compactification is discussed in Section 2.*
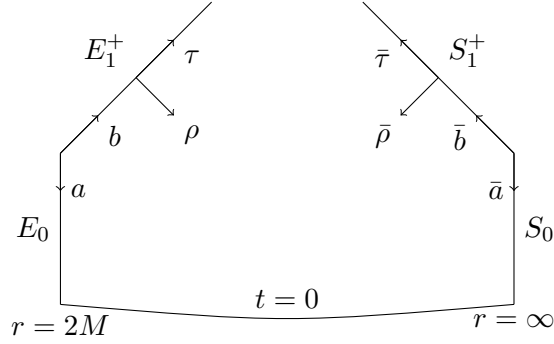

FIGURE 2. The compactification of Schwarzschild spacetime for $t \geq 0$: E_1^+ is the event horizon; S_1^+ is null infinity; E_0 and S_0 are from the blow-up of the spatial ends in the Penrose diagram of the spacetime.

One consequence of Theorem 1.1 is the existence of the radiation field, i.e., that solutions of equation (1.1) may be restricted to the event horizon (E_1^+) and null infinity (S_1^+). As there are two spatial ends of the Schwarzschild black hole exterior, our definition of the radiation field has two components. For smooth initial data (ϕ, ψ) compactly supported in $(2M, \infty) \times \mathbb{S}^2$, we define the two components of the forward radiation field as follows:

$$\mathcal{R}_{E_1^+}(\phi, \psi)(\tau, \omega) = \lim_{r \to 2M} \partial_t u(\tau - r - 2M \log(r - 2M), r, \omega),$$
$$\mathcal{R}_{S_1^+}(\phi, \psi)(\bar{\tau}, \omega) = \lim_{r \to \infty} r \partial_t u(\bar{\tau} + r + 2M \log(r - 2M), r, \omega).$$

The backward radiation fields $\mathcal{R}_{E_1^-}$ and $\mathcal{R}_{S_1^-}$ are defined analogously in equation (5.1).

We show that, under our definition, the radiation field is unitary (i.e., norm-preserving):

Theorem 1.3. *Given (ϕ, ψ) with finite energy, the radiation field of the solution u of equation (1.1) is unitary, i.e.,*

$$4M^2 \left\| \mathcal{R}_{E_1^+}(\phi, \psi) \right\|^2_{L^2(\mathbb{R} \times \mathbb{S}^2)} + \left\| \mathcal{R}_{S_1^+}(\phi, \psi) \right\|^2_{L^2(\mathbb{R} \times \mathbb{S}^2)} = E(0).$$

The unitarity of this map strongly depends on the fact that the Schwarzschild background is static. As such, it is unlikely that that the radiation field remains unitary under non-static metric perturbations. Within some classes of nonlinear problems, however, it is reasonable to expect that the norm of the radiation field agrees with the energy norm of the solution (see, e.g., recent work of the first author and Sá Barreto [BSB12]).

Bachelot [Bac94] (for the Klein-Gordon equation) and Dimock [Dim85] (for the wave equation) carried out related work on the unitarity of wave operators in a more abstract scattering-theoretic setting.

We also prove the following support theorem for the radiation field (stated more precisely in Section 5):

Theorem 1.4. *Suppose that $\phi, \psi \in C_c^\infty((2M, \infty) \times \mathbb{S}^2)$. If $\mathcal{R}_{E_1^+}(\phi, \psi)$ vanishes for $\tau \leq \tau_0$ and $\mathcal{R}_{E_1^-}(\phi, \psi)$ vanishes for $\tau \geq -\tau_0$, then both ϕ and ψ are supported in $[r_0, \infty) \times \mathbb{S}^2$, where r_0 is given implicitly by*

$$r_0 + 2M \log(r_0 - 2M) = \tau_0.$$

An analogous statement holds for the component of the radiation field corresponding to null infinity.

Remark 1.5. *The smoothness hypothesis above is not essential; we can relax the assumption to $C^{2,\alpha}$ for $\alpha > 0$. In particular, we require only enough smoothness to ensure that the rescaled solution is $C^{2,\alpha}$ after being extended by zero across the event horizon or null infinity. In fact, the support hypothesis near the event horizon can be relaxed as well; if the initial data has enough decay there (taking $\lambda > 2$ in Theorem 1.1 should suffice), then the rescaled solution will still have enough smoothness for the uniqueness theorems to apply.*

Near infinity, however, it is important that we take compactly supported data, as the past and future null infinities do not meet smoothly. Even if this difficulty were overcome, a strong decay condition must be assumed to rule out the counter-examples that exist already in Minkowski space. In particular, for any $m \in \mathbb{N}$, there are smooth functions $f(z)$ in $\mathbb{R}^n$ which are not compactly supported, decay like $|z|^{-m}$, and whose Radon transform (and hence radiation field) is compactly supported (see, e.g., [Hel99]).

In the setting of Minkowski space, the Fourier transform of the forward radiation field in the s variable is given in terms of the Fourier transform of the initial data (and is related to the Radon transform). In other settings, one may think of the Fourier transform of the radiation field as a distorted Fourier transform. The support theorem can then be seen as a Paley–Wiener theorem for a distorted Fourier-type transform on the Schwarzschild

spacetime. Similar support theorems in other contexts have been established by Helgason [Hel99], Sá Barreto [SB08, SB05], and the first author and Sá Barreto [BSB12].

When the wave evolution can be compared to a fixed background evolution (as is the case for perturbations of Minkowski space), one may define the Møller wave operator. Given a solution of the perturbed problem, one may often find a solution to the free problem to which the solution of the perturbed problem scatters. The Møller wave operators associate the initial data for the solution of the perturbed problem to the initial data for the solution of the "free" problem. (We refer the reader to the book of Lax–Phillips for more details [LP89].) In many such settings, the radiation field and the Møller wave operators carry the same information and thus the existence, unitary, and smoothness of the radiation field are equivalent to the corresponding statements for the Møller wave operator. In fact, in those cases, it may even be possible to obtain the support theorem from the Møller wave operator. We refer the reader to the recent paper of Donninger–Krieger [DK13] for results in this direction. In the setting of the Schwarzschild background, however, there is not an obvious natural evolution upon which to base the Møller wave operator, and so the radiation field should be thought of as a stand-in for this scattering-theoretic object.

Although we are unable to characterize the range of the radiation field, we do show that our definition of the radiation field captures "too much" information. In particular, the support theorem above implies the following:

Corollary 1.6. *If $\psi \in C_c^\infty \left((2M, \infty) \times \mathbb{S}^2 \right)$ and $\mathcal{R}_{E_1^+}(0, \psi) \equiv 0$, then $\psi = 0$. Similarly, for such a ψ, if $\mathcal{R}_{S_1^+}(0, \psi) \equiv 0$, then $\psi = 0$.*

In other words, for odd, smooth, compactly supported data, knowing that one component of the radiation field vanishes implies that it must vanish on the other component as well.

The study of the asymptotic behavior of solutions of the wave equation on the Schwarzschild background has been an active field of research (see, e.g., the work of Dafermos–Rodnianski [DR09], as well as the works of Blue–Sterbenz [BS06], Blue–Soffer [BS09], Marzuola et al. [MMTT10], and Luk [Luk10]) and we refer the reader to the lecture notes of Dafermos and Rodnianski [DR13] for many references. In Section 4 we view the unitarity of the radiation field as a consequence of the proofs of Price's law (see the works of Tataru [Tat13], Metcalfe–Tataru–Tohaneanu [MTT12], and Donninger–Schlag–Soffer [DSS12]), though we remark that a weaker energy decay result would suffice.

Section 2 describes the partial compactification on which we work, while in Section 3, we prove Theorem 1.1 via energy estimates. Section 4 is devoted to the proof of Theorem 1.3 and relies on the pointwise decay of solutions established by the proof of Price's law. Finally, in Section 5, we prove the support theorems.

1.1. **Notation.** In this section we lay out some notation common to the entire paper.

The regularity and asymptotic behavior of solutions near E_1^+ and S_1^+ may be obtained via energy estimates. As a preparation, for any function v and time-like function T, we give a name to the vector field obtained by contracting a modified stress-energy tensor[1] of v (with respect to a Lorentzian metric g) with the gradient of T:

$$\mathcal{F}_g(T, v) = \langle \nabla T, \nabla v \rangle_g \nabla v - \frac{1}{2} \left(\langle \nabla v, \nabla v \rangle_g + v^2 \right) \nabla T.$$

We also record

$$\mathrm{div}_g(\mathcal{F}_g(T, v)) = \langle \nabla T, \nabla v \rangle_g \left(\Box_g - 1 \right) v + \mathcal{Q}_g(T, v),$$

where

$$\mathcal{Q}_g(T, v) = \mathrm{Hess}_g(T)(dv, dv) - \frac{1}{2} \Box_g T \left(\langle \nabla v, \nabla v \rangle_g + v^2 \right).$$

For a manifold with corners M, we further require the spaces of uniformly degenerate (0-) and tangential (b-) vector fields:

$$\mathcal{V}_0 = \{ X \in C^\infty(M, TM) : X \text{ vanishes at } \partial M \}$$
$$\mathcal{V}_b = \{ X \in C^\infty(M, TM) : X \text{ is tangent to } \partial M \}$$

For a given measure $d\mu$, the 0-Sobolev space $H_0^1(M, d\mu)$ is the space of functions $u \in L^2(d\mu)$ so that $Xu \in L^2(d\mu)$ for all $X \in \mathcal{V}_0$. For an integer N and a given measure $d\mu$, the b-Sobolev space $H_b^N(M, d\mu)$ consists of functions $u \in L^2(d\mu)$ so that $X_1 \ldots X_k u \in L^2(d\mu)$ for $X_j \in \mathcal{V}_b$ and $k \leq N$. The mixed $0, b$-Sobolev space $H_{0,b}^{1,N}(M, d\mu)$ consists of those $u \in H_0^1(M, d\mu)$ so that $X_1 \ldots X_k u \in H_0^1(M, d\mu)$ for $X_j \in \mathcal{V}_b$ and $k \leq N$. For an introduction to b-Sobolev spaces and b-geometry, we refer the reader to the book of Melrose [Mel93].

2. A PARTIAL COMPACTIFICATION

In this section we describe the partial compactification of Schwarzschild spacetime for $t \geq 0$ (see Figure 2) on which we work. It can be obtained by a suitable blow-up (with a logarithmic correction) of the Penrose diagram in Figure 1. We describe the smooth structure on this partial compactification by taking explicit local coordinates near the boundary. Each set of local coordinates is valid in the corresponding domain as in Figure 2.

We now describe coordinates giving the smooth structure on the partial compactification (and the domains in which they are valid).

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[1]We modify the stress-energy tensor by including a v^2 term in order to control inhomogeneous Sobolev norms.

- Near $r = 2M$ but away from temporal infinity, i.e., for $r - 2M < \infty$ and $t + r + 2M \log(r - 2M) < \infty$, we choose coordinates (a, b, ω) with

$$a = e^{-\frac{t+r}{2M}}, \ b = e^{\frac{t+r}{4M}} \sqrt{r - 2M}.$$

- Near the interior of the event horizon E_1^+, i.e., $r - 2M < \infty$ and $-\infty < t + r + 2M \log(r - 2M) < \infty$, we also can use coordinates (τ, ρ, ω) with

$$\tau = t + r + 2M \log(r - 2M), \ \rho = r - 2M.$$

Here the initial surface $\{t = 0\}$ for r close to $2M$ is equivalent to

$$\{a = e^{-\frac{2M + ab^2}{2M}}\}$$

for a close to e and b close to 0, which intersect with E_0 smoothly.

For r close to ∞, we choose corresponding coordinates as follows:

- For r large and close to the spatial end S_0, i.e., for $r - 2M > 0$ and $t - r - 2M \log(r - 2M) < 0$, we choose coordinates $(\bar{a}, \bar{b}, \omega)$ with

$$\bar{a} = \frac{-t + r + 2M \log(r - 2M)}{r}, \ \bar{b} = \frac{1}{-t + r + 2M \log(r - 2M)}.$$

- Near the interior of null infinity, i.e., $r - 2M > 0$ and $-\infty < t - r - 2M \log(r - 2M) < \infty$, we use coordinates $(\bar{\tau}, \bar{\rho}, \omega)$ with

$$\bar{\tau} = t - r - 2M \log(r - 2M), \ \bar{\rho} = 1/r.$$

Here the initial surface $\{t = 0\}$ for r large is equivalent to

$$\bar{a} = 1 + 2M\bar{a}\bar{b}\big(\log(1 - 2M\bar{a}\bar{b}) - \log(\bar{a}\bar{b})\big)$$

for $\bar{a}$ close to 1 and $\bar{b}$ close to 0. The initial surface does not intersect S_0 smoothly but instead has a logarithmic correction term. A direct consequence of this lack of smoothness is that for classical initial data which have pure Taylor expansions at $r = \infty$, the radiation field on S_1^+ will have an expansion including logarithmic terms at $\bar{\tau} = -\infty$. We refer the reader to Proposition 3.13 and the surrounding discussion for details.

3. Existence and regularity of the radiation field

In this section we establish regularity properties for solutions of equation (1.1) on a partial compactification of the Schwarzschild background. We also show that sufficiently regular solutions have asymptotic expansions at the boundary hypersurfaces of this compactification. Taken together, Propositions 3.5 and 3.13 prove Theorem 1.1.

3.1. **At the event horizon.** For $t > 0$ we change coordinates to

$$\tau = t + r + 2M \log(r - 2M), \ \rho = r - 2M.$$

The coordinates (τ, ρ) are essentially the incoming Eddington-Finkelstein coordinates. The function τ is a coordinate along the event horizon E_1^+, while ρ is a defining function for E_+^1 as long as τ is bounded away from $\pm\infty$. The metric and its D'Alembertian are then:

$$g_S = -\frac{\rho}{\rho + 2M} d\tau^2 + 2 \, d\tau \, d\rho + (\rho + 2M)^2 \, d\omega^2,$$

$$\Box_S = 2\partial_\tau\partial_\rho + \frac{\rho}{\rho + 2M}\partial_\rho^2 + \frac{1}{(\rho + 2M)^2}\Delta_\omega + \frac{2(\rho + M)}{(\rho + 2M)^2}\partial_\rho + \frac{2}{\rho + 2M}\partial_\tau.$$

We can thus extend g_S naturally as a Lorentzian metric to a slightly larger manifold (given in these coordinates by $\{\rho > -\epsilon, |\tau| \leq C\}$) with $\{\rho = 0\}$ a characteristic hypersurface. Moreover, if Z_{ij} are the rotations of $\mathbb{S}^2$, then

$$[\Box_S, \partial_\tau] = [\Box_S, Z_{ij}] = 0,$$

$$[\Box_S, \rho\partial_\rho] = \Box_S - \frac{3\rho + 2M}{(\rho + 2M)^3}\Delta_\omega + \frac{1}{(\rho + 2M)^2}(\rho\partial_\rho)^2 + V_1,$$

$$[\Box_S, \partial_\rho] = -\frac{2M}{(\rho + 2M)^2}\partial_\rho^2 + \frac{2}{(\rho + 2M)^3}\Delta_\omega + V_2,$$

where V_1, V_2 are vector fields that are tangent to the event horizon:

$$V_1 = -\frac{4M}{(\rho + 2M)^2}\partial_\tau + \frac{\rho - 2M}{(\rho + 2M)^3}(\rho\partial_\rho),$$

$$V_2 = \frac{2}{(\rho + 2M)^3}(\rho\partial_\rho) + \frac{2}{(\rho + 2M)^2}\partial_\tau.$$

In terms of the coordinates (τ, ρ, ω), we choose a time-like function by

$$T = \tau - \rho, \quad \langle\nabla T, \nabla T\rangle_{g_S} = -2 + \frac{\rho}{\rho + 2M} < -1.$$

We may then compute

$$\langle\mathcal{F}(T, v), \nabla T\rangle_{g_S} = \frac{1}{2}\left(|\partial_\tau v|^2 + \left|\partial_\tau v + \frac{\rho\partial_\rho v}{\rho + 2M}\right|^2\right) + \frac{2M |\partial_\rho v|^2}{\rho + 2M}$$

$$+ \frac{\rho + 4M}{2(\rho + 2M)}\left(\frac{|\nabla_\omega v|^2}{(\rho + 2M)^2} + v^2\right),$$

$$\mathcal{Q}(T, v) = -\frac{M}{(\rho + 2M)^2}\left(|\partial_\rho v|^2 + \frac{2\rho |\partial_\rho v|^2}{\rho + 2M} + 4\partial_\rho\partial_\tau v - \frac{|\nabla_\omega v|^2}{(\rho + 2M)^2} + v^2\right).$$

Using Friedlander's argument [Fri80], we now show that solutions of the wave equation with compactly supported smooth initial are smooth across the event horizon E_1^+.

Proposition 3.1. *If $(\phi, \psi) \in C^\infty \left((2M, \infty)_r \times \mathbb{S}^2 \right)$ are such that* $\operatorname{supp}(\phi) \cup$ $\operatorname{supp}(\psi) \subset (2M + \epsilon, \infty) \times \mathbb{S}^2$ *for some $\epsilon > 0$, then u is smooth down to* $\{\rho = 0\}$ *for all $\tau \in (-\infty, \infty)$.*
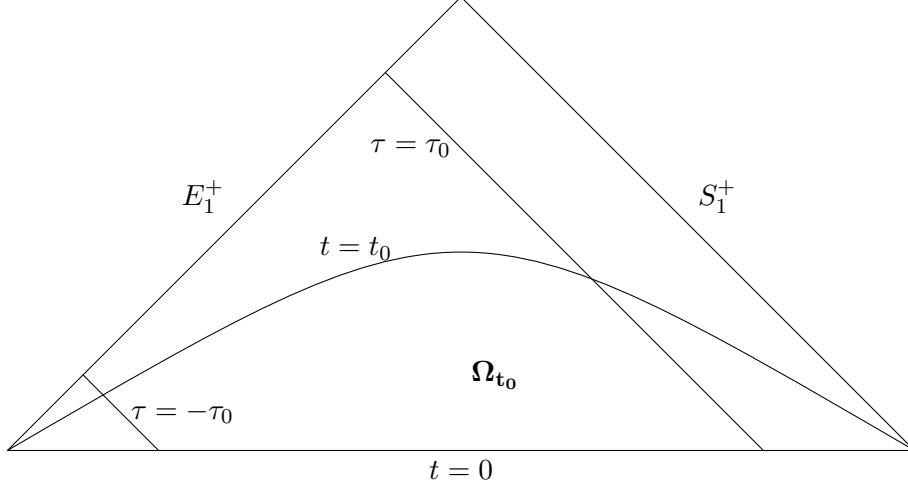

FIGURE 3. The domain Ω_{t_0}.

Proof. By finite speed of propagation, $u \equiv 0$ for $\tau \leq 2M + \epsilon + 2M \log \epsilon$. We now fix $\tau_0 > -(2M + \epsilon + 2M \log \epsilon)$ large. Let Ω_{t_0} be the domain bounded by $\{t = 0\}$, $\{t = t_0\}$, $\{\tau = \tau_0\}$, and $\{\tau = -\tau_0\}$ (pictured in Figure 3). Here $\{t = 0\}$, $\{t = t_0\}$ are space-like with defining function t and $\{\tau = \pm\tau_0\}$ are null with defining function τ. Moreover, $\langle \nabla T, \nabla t \rangle_{g_S} < 0$ and $\langle \nabla T, \nabla \tau \rangle_{g_S} < 0$ imply that

$$\langle \mathcal{F}(T, v), \nabla t \rangle_{g_S} \geq 0, \ \langle \mathcal{F}(T, v), \nabla \tau \rangle_{g_S} \geq 0.$$

Let $\Sigma_s = \{T = s\}$ with defining function T and $\Omega_{t_0}^s = \Omega_{t_0} \cap \{T \leq s\}$. Define now

$$M^N(u, s) = \left(\sum_{|I| \leq N} \int_{\Sigma_s \cap \Omega_{t_0}} e^{-cT} \langle \mathcal{F}\left(T, Z^I u \right), \nabla T \rangle_{g_S} \, d\mu_T \right)^{1/2}$$

where $Z \in \{\partial_\rho, \partial_\tau, Z_{ij}\}$ and $d\mu_T \wedge dT = dV_{g_S}$. Note that each term in sum above is positive, so M^N controls the first $N + 1$ derivatives of u. Choose c large enough so that

$$\begin{aligned} \operatorname{div} &\left(e^{-cT} \mathcal{F}(T, Z^I u) \right) \\ &= e^{-cT} \left(-c \langle \mathcal{F}(T, Z^I u) + \langle \nabla T, \nabla Z^I u \rangle (\square_{g_S} - 1) Z^I u + Q_{g_S}(T, Z^I u) \right) \\ &\leq 0 \end{aligned}$$

for all $|I| \leq N$. Here c depends on τ_0 and N. By Stokes' theorem, we then have that

$$\left(M^N(u,s)\right)^2 \leq \sum_{|I| \leq N} \int_{\{t=0\} \cap \Omega_{t_0}} e^{-cT} \langle \mathcal{F}(T, Z^I u), \nabla T \rangle \, d\mu_t.$$

Because T is bounded on Ω_∞ and $\{t = 0\} \cap \Omega_{t_0}$ is independent of t_0, we have that

$$\sum_{|I| \leq N} \int_{\Omega_{t_0}} e^{-cT} \langle \mathcal{F}(T, Z^I u), \nabla T \rangle_{g_S} \, dV_{g_S} = \int_{-\infty}^{\infty} \left(M^N(u,s)\right)^2 ds \leq C < \infty,$$

where C depends only on N and τ_0. Letting $t_0 \to \infty$, we obtain

$$\sum_{|I| \leq N} \int_{\Omega_\infty} e^{-cT} \langle \mathcal{F}(T, Z^I u), \nabla T \rangle_{g_S} \, dV_{g_S} \leq C_{N,\tau_0}.$$

Because N and τ_0 are arbitrary, u is smooth up to $\{\rho = 0\}$ for all $\tau \in (-\infty, \infty)$. $\qquad\square$

We now consider non-compactly supported data. On the partial compactification depicted in Figure 2, we use coordinates a and b, defined as follows:

$$a = \rho e^{-\frac{\tau}{2M}} = e^{-\frac{t+r}{2M}} \in [0, e^{-1}], \;\; b = e^{\frac{\tau}{4M}} = e^{\frac{t+r}{4M}} \sqrt{r - 2M}.$$

These are valid in a neighborhood of E_0, especially near the intersection of E_0 and E_1^+, where a is a defining function for E_1^+ and b is a defining function for E_0. (The function $r - 2M = ab^2$ vanishes on both E_0 and E_1^+.) Near the interior of E_1^+, (a, b) are equivalent to the coordinates (τ, ρ) as above. Near the initial surface $t = 0$ and r finite, (a, b) are equivalent to the coordinates

$$(\beta, t) = (\sqrt{r - 2M}, t).$$

In coordinates (a, b), the conformal metric $\tilde{g}_e = (2b\sqrt{M})^{-2} g_S$ and its wave operator are:

$$\tilde{g}_e = 2 \, da \frac{db}{b} + 2a \left(1 + \frac{ab^2}{2M + ab^2}\right) \left(\frac{db}{b}\right)^2 + \frac{(2M + ab^2)^2}{4Mb^2} \, d\omega^2,$$

$$\tilde{\Box}_e = 2\partial_a \left(b\partial_b - a\partial_a - 1\right) + \frac{4Mb^2}{(2M + ab^2)^2} \Delta_\omega - \frac{2b^2}{2M + ab^2} (a\partial_a)^2$$

$$- \frac{4b^2(M + ab^2)}{(2M + ab^2)^2} a\partial_a + \frac{2b^2}{2M + ab^2} b\partial_b.$$

We thus have that

$$\Box_{g_s} u = 0 \iff \left(\tilde{\Box}_e + \gamma_e\right) \tilde{u} = 0,$$

$$\tilde{u} = 2b\sqrt{M} u, \; \gamma_e = -b^{-1}\tilde{\Box}_e b = -\frac{2b^2}{2M + ab^2}.$$

Moreover,

$$(3.1) \quad \left[\tilde{\Box}_e, Z_{ij}\right] = 0, \ \left[\tilde{\Box}_e, b\partial_b\right] = b^2 \sum_{|I| \leq 2} c_I Z^I, \ \left[\tilde{\Box}_e, a\partial_a\right] = \tilde{\Box}_e + b^2 \sum_{|I| \leq 2} c'_I Z^I,$$

where $Z \in \{a\partial_a, b\partial_b, Z_{ij}\}$ and c_I, c'_I are smooth coefficients. In the above and what follows, we use I as a multi-index. In coordinates (a, b, ω), we choose time-like functions as follows:

$$T_1 = -a + \log b, \qquad \langle \nabla T_1, \nabla T_1 \rangle_{\tilde{g}_e} = -2 - 2a \left(1 + \frac{ab^2}{2M + ab^2}\right) < 0,$$

$$T'_1 = -a, \qquad\qquad \langle \nabla T'_1, \nabla T'_1 \rangle_{\tilde{g}_e} = -2a \left(1 + \frac{ab^2}{2M + ab^2}\right) < 0.$$

Here T'_1 is asymptotically null when approaching the event horizon. Moreover, we again compute

(3.2)

$$\langle \mathcal{F}_{\tilde{g}_e}(T_1, v), \nabla T'_1 \rangle_{\tilde{g}_e}$$

$$= \left(1 + \frac{ab^2}{2M + ab^2}\right) a |\partial_a v|^2 + \frac{1}{2} |b\partial_b v|^2$$

$$+ \frac{1}{2} \left| 2 \left(1 + \frac{ab^2}{2M + ab^2}\right) a\partial_a v - b\partial_b v \right|^2$$

$$+ \frac{1}{2} \left(1 + 2a \left(1 + \frac{ab^2}{2M + ab^2}\right)\right) \left(\frac{4M |b\nabla_\omega v|^2}{(2M + ab^2)^2} + v^2\right),$$

$$\langle \mathcal{F}_{\tilde{g}_e}(T_1, v), \nabla \log b \rangle_{\tilde{g}_e}$$

$$= |\partial_a v|^2 + \left(1 + \frac{ab^2}{2M + ab^2}\right) a |\partial_a v|^2 + \frac{1}{2} \left(\frac{4M |b\nabla_\omega v|^2}{(2M + ab^2)^2} + v^2\right),$$

$$\mathcal{Q}_{\tilde{g}_e}(T_1, v)$$

$$= |\partial_a v|^2 - \left(2 + \frac{2b^2}{2M + ab^2}\right) (\partial_a v)(b\partial_b v)$$

$$+ \left(4 + \frac{8Mb^2}{(2M + ab^2)^2}\right) a |\partial_a v|^2 + \Theta_2 (a\partial_a v, b\partial_b v, b\nabla_\omega v, v),$$

where $\Theta_2(v^1, \ldots, v^l)$ is a quadratic form of $(v^1, \ldots, v^l)$ with smooth coefficients.

Proposition 3.2. *Suppose that (ϕ, ψ) are in the following weighted Sobolev spaces*

$$\beta^{\lambda-1} H^{1,N}_{0,b} \left([0, \beta_0)_\beta \times \mathbb{S}^2, \frac{d\beta \, d\omega}{\beta^3}\right) \times \beta^{\lambda-1} H^{1,N-1}_{0,b} \left([0, \beta_0)_\beta \times \mathbb{S}^2, \frac{d\beta \, d\omega}{\beta^3}\right)$$

for some $\beta_0 > 0$, $N > 2$ and $\lambda > 0$. Then $\tilde{u} = 2b\sqrt{M}u$ is C^δ up to $\{a = 0\}$ for $b < \beta_0 e^{\frac{\beta_0^2 + 2M}{4M}}$, where $\delta = \min\{\lambda, \frac{1}{2}\}$.

Proof. The assumption on ϕ and ψ implies that on the Cauchy surface

$$(Z^I \tilde{u})|_{t=0} \in \beta^\lambda H_{0,b}^{1,N-|I|}\left([0,\beta_0) \times \mathbb{S}^2, \frac{d\beta \, d\omega}{\beta^3}\right)$$

for $Z \in \{a\partial_a, b\partial_b, Z^{ij}\}$.

Let $\Sigma_s = \{T_1' = s\}$, so that Σ_s is space-like with defining function T_1' for $s < 0$. As $s \to 0$, Σ_s approaches a characteristic surface. Let Ω be the domain bounded by Σ_0, $\{t = 0\}$, $\{b = 0\}$ and $S = \{T_1 = \log b_0\}$. We then define

$$M_1^N(\tilde{u}, s; \lambda) = \left(\sum_{|I| \le N} \int_{\Sigma_s \cap \Omega} b^{-2\lambda} e^{-cT_1'} \langle \mathcal{F}_{\tilde{g}_e}(T_1, Z^I \tilde{u}), \nabla T_1'\rangle_{\tilde{g}_e} \, d\mu_{T_1'}\right)^{\frac{1}{2}},$$

$$L_1^N(\tilde{u}, s; \lambda) = \left(\sum_{|I| \le N} \int_{S \cap \{T_1' < s\} \cap \Omega} b^{-2\lambda} e^{-cT_1'} \langle \mathcal{F}_{\tilde{g}_e}(T_1, Z^I \tilde{u}), \nabla T_1\rangle_{\tilde{g}_e} \, d\mu_{T_1}\right)^{\frac{1}{2}},$$

where $Z \in \{a\partial_a, b\partial_b, Z_{ij}\}$ and $d\mu_{T_1'} \wedge dT_1' = dV_{\tilde{g}_e}$, $d\mu_{T_1} \wedge dT_1 = dV_{\tilde{g}_e}$.

We first choose s_0 close to 0 so that $\Sigma_{s_0} \cap \{t = 0\} \cap \Omega = \emptyset$. Let $\Omega_{s_0} = \Omega \cap \{s \le s_0\}$ (illustrated in Figure 4) so that Ω_{s_0} is bounded by Σ_{s_0}, $\{t = 0\}$, $\{b = 0\}$, and S. Choose c large enough so that in Ω_{s_0} we have

$$\sum_{|I| \le N} \operatorname{div}_{\tilde{g}_e}\left(b^{-2\lambda} e^{-cT'} \mathcal{F}_{\tilde{g}_e}(T_1, Z^I \tilde{u})\right)$$

$$= b^{-2\lambda} e^{-cT_1'} \sum_{|I| \le N} \left(-2\lambda \langle \mathcal{F}_{\tilde{g}_e}(T_1, Z^I \tilde{u}), \nabla \log b\rangle_{\tilde{g}_e} + \mathcal{Q}_{\tilde{g}_e}(T_1, Z^I \tilde{u})\right.$$

$$\left. - c\langle \mathcal{F}_{\tilde{g}_e}(T_1, Z^I \tilde{u}), \nabla T_1'\rangle_{\tilde{g}_e} + \langle \nabla T_1, \nabla Z^I \tilde{u}\rangle_{\tilde{g}_e}\left(\tilde{\Box}_e - 1\right) Z^I \tilde{u}\right) \le 0.$$

Here c depends only on b_0, s_0, and N, and the dependence on s_0 is required to bound $|\partial_a Z^I \tilde{u}|^2$ in terms of $a|\partial_a Z^I \tilde{u}|^2$. In order to bound the term of the form

$$\langle \nabla T_1, \nabla Z^I \tilde{u}\rangle_{\tilde{g}_e}\left(\tilde{\Box}_e - 1\right) Z^I \tilde{u},$$

we have used that $\tilde{u}$ solves $\left(\tilde{\Box}_e + \gamma_e\right)\tilde{u} = 0$ and the expressions (3.1) for the commutators of $\tilde{\Box}$ with Z^I. By Stokes' theorem, we then have

$$\left(M_1^N(\tilde{u}, s_0)\right)^2 + \left(L_1^N(\tilde{u}, s_0)\right)^2$$

$$\le \sum_{|I| \le N} \int_{\{t=0\} \cap \Omega} b^{-2\lambda} e^{-cT_1'} \langle \mathcal{F}_{\tilde{g}_e}(T_1 Z^I \tilde{u}), \nabla t\rangle_{\tilde{g}_e} \, d\mu_t,$$

where $d\mu_t \wedge dt = dV_{\tilde{g}_e}$. Observe that the right hand side is equivalent to the square of the initial data norm.
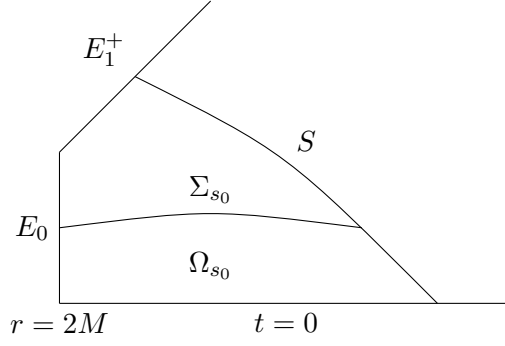
FIGURE 4. The domain Ω_{s_0}

For $s > s_0$, we now let $\Omega_{s_0}^s$ denote the domain bounded by Σ_{s_0}, Σ_s, $\{b = 0\}$, and S. Again Stokes' theorem implies that

$$\left(M_1^N(\tilde{u}, s)\right)^2 - \left(M_1^N(\tilde{u}, s_0)\right)^2 + \left(L_1^N(\tilde{u}, s)\right)^2 - \left(L_1^N(\tilde{u}, s_0)\right)^2$$

$$= \sum_{|I| \leq N} \int_{\Omega_{s_0}^s} \operatorname{div}\left(b^{-2\lambda} e^{-cT_1'} \mathcal{F}_{\tilde{g}_e}(T_1, Z^I \tilde{u})\right) \, dV_{\tilde{g}_e}.$$

Dividing by $s - s_0$, taking a limit, and then using the expressions (3.2) yields that

$$\partial_s \left(M_1^N(\tilde{u}, s)\right)^2 + \partial_s \left(L_1^N(\tilde{u}, s)\right)^2$$

$$\leq \begin{cases} \left((1 - 2\lambda)a^{-1} + Ca^{-\frac{1}{2}}\right)\left(M_1^N(\tilde{u}, s)\right)^2 & \lambda < \frac{1}{2} \\ Ca^{-\frac{1}{2}}\left(M_1^N(\tilde{u}, s)\right)^2 & \lambda \geq \frac{1}{2} \end{cases},$$

with C a constant depending only on b_0 and N. (Note that the power $a^{-1/2}$ arises from the need to estimate the terms of the form $(\partial_a Z^I \tilde{u})(b\partial_b Z^I \tilde{u})$.) Integrating in s, we find that for all $s > s_0$,

$$M_1^N(\tilde{u}, s) \leq \begin{cases} C'(-s)^{\lambda - \frac{1}{2}}(-s_0)^{\frac{1}{2} - \lambda} M_1^N(\tilde{u}, s_0) & \lambda < \frac{1}{2} \\ C' M_1^N(\tilde{u}, s_0) & \lambda \geq \frac{1}{2} \end{cases},$$

where $C' = \exp\left(\frac{1}{2}\int_0^{e^{-1}} Ca^{-1/2}\, da\right)$. Because $N > 2$, the Sobolev embedding theorem implies that for $a < -s_0$, we have

$$|\partial_a \tilde{u}| \leq \begin{cases} C'' M_1^N(\tilde{u}, s_0) a^{\lambda - 1} & \lambda < \frac{1}{2} \\ C'' M_1^N(\tilde{u}, s_0) a^{-\frac{1}{2}} & \lambda \geq \frac{1}{2} \end{cases},$$

where C'' only depends on b_0. Integrating in a then finishes the proof. $\square$

Corollary 3.3. *Suppose $\lambda = k + \alpha$ with $\alpha \in (0, 1]$ and $k \in \mathbb{N}_0$. If the pair (ϕ, ψ) lies in the following space*

$$\beta^{\lambda - 1} H_{0,b}^{1,N}\left([0, \beta_0) \times \mathbb{S}^2, \frac{d\beta\, d\omega}{\beta^3}\right) \times \beta^{\lambda - 1} H_{0,b}^{1,N-1}\left([0, \beta_0) \times \mathbb{S}^2, \frac{d\beta\, d\omega}{\beta^3}\right)$$

with $N > 2 + k$, then $\tilde{u}$ is $C^{k,\delta}$ up to $\{a = 0\}$ for $b < \beta_0 e^{\frac{\beta_0^2 + 2M}{4M}}$, where $\delta = \min\{\alpha, \frac{1}{2}\}$.

Proof. Notice that

$$\left[\tilde{\Box}_e, \partial_a^k\right] = 2k\partial_a^{k+1} + b^2 \sum_{|I|+i\leq k+1, i\leq k} c_{I,i} Z^I \partial_a^i,$$

$$\left[a\partial_a, \partial_a^k\right] = -k\partial_a^k, \quad \left[b\partial_b, \partial_a^k\right] = \left[Z_{ij}, \partial_a^k\right] = 0,$$

where $Z \in \{a\partial_a, b\partial_b, Z_{ij}\}$ and $c_{I,i}$ are smooth coefficients. We set

$$\widetilde{M}_1^N(\tilde{u}, s; \lambda) = \left(\sum_{i\leq k} \left(M_1^{N-i}(\partial_a^i \tilde{u}, s; \lambda)\right)^2\right)^{\frac{1}{2}},$$

$$\widetilde{L}_1^N(\tilde{u}, s; \lambda) = \left(\sum_{i\leq k} \left(L_1^{N-i}(\partial_a^i \tilde{u}, s; \lambda)\right)^2\right)^{\frac{1}{2}}.$$

According to the divergence formula in the proof of Proposition 3.2, we have that

$$\partial_s \left(\widetilde{M}_1^N(\tilde{u}, s)\right)^2 + \partial_s \left(\widetilde{L}_1^N(\tilde{u}, s)\right)^2$$

$$\leq \begin{cases} \left((1 - 2\alpha)a^{-1} + Ca^{-\frac{1}{2}}\right) \left(\widetilde{M}_1^N(\tilde{u}, s)\right)^2 & \alpha < \frac{1}{2} \\ Ca^{-\frac{1}{2}} \left(\widetilde{M}_1^N(\tilde{u}, s)\right)^2 & \alpha \geq \frac{1}{2} \end{cases}.$$

We thus obtain the following in a similar manner to the proof of Proposition 3.2:

$$\left|\partial_a^i \tilde{u}\right| \leq C' \widetilde{M}_1^N(\tilde{u}, s_0), \ a < -s_0, i \leq k;$$

$$\left|\partial_a^{k+1} \tilde{u}\right| \leq \begin{cases} C' \widetilde{M}_1^N(\tilde{u}, s_0) a^{\alpha-1} & \alpha < \frac{1}{2} \\ C' \widetilde{M}_1^N(\tilde{u}, s_0) a^{-\frac{1}{2}} & \alpha \geq \frac{1}{2} \end{cases},$$

which finishes the proof. $\qquad\qquad\square$

In particular, "Schwartz" initial data behaves in much the same way that compactly supported data does:

Corollary 3.4. *If (ϕ, ψ) lies in the space*

$$\beta^\infty H_{0,b}^{1,\infty}\left([0, \beta_0) \times \mathbb{S}^2, \frac{d\beta \, d\omega}{\beta^3}\right) \times \beta^\infty H_{0,b}^{1,\infty}\left([0, \beta_0) \times \mathbb{S}^2, \frac{d\beta \, d\omega}{\beta^3}\right),$$

then $\tilde{u}$ is smooth up to $\{a = 0\}$ for $b < \beta_0 e^{\frac{\beta_0^2 + 2M}{4M}}$.

We now claim that u has a classical asymptotic expansion if the initial data does. Indeed, suppose that

$$\phi, \psi \in \beta^\lambda C^\infty([0, \beta_0)_\beta \times \mathbb{S}^2),$$

and define the "k-th term" w^k by

$$(3.3) \qquad w^0 = \tilde{u} = 2b\sqrt{M}u, \quad w^k = \Pi_{i=1}^k (b\partial_b - \lambda - i)\,\tilde{u} \quad \text{for} \quad k \geq 1.$$

Notice that $b\partial b$ lifts to $\beta\partial_\beta - 2\beta^2\partial_t$ on initial surface $t = 0$. Hence

$$(Z^I w^k)|_{t=0} \in \beta^{\lambda+k} C^\infty([0, \beta_0)_\beta \times \mathbb{S}^2)$$

for $Z \in \{a\partial_a, b\partial_b, Z^{ij}\}$ and multi-index I.

Proposition 3.5. *If ϕ and ψ are as above with $0 < \lambda + k = l + \alpha$ for some integer l and $\alpha \in (0, 1]$, then w^k is $C^{l,\delta}$ up to $\{a = 0\}$ for $b < b_0$, where $\delta = \min\{\alpha - \epsilon, \frac{1}{2}\}$ with $\epsilon > 0$ arbitrarily small.*

Proof. Notice that because $\left(\tilde{\Box}_e + \gamma_e\right)\tilde{u} = 0$, we have

$$\tilde{\Box}_e w^i = b^2 \sum_{|I| \leq 2} c_I Z^I w^{i-1} + (b\partial_b + \lambda - i)\tilde{\Box}_e w^{i-1}$$

$$= b^2 \sum_{j=0}^{i} \sum_{|I| \leq 2} c_{I,j} Z^{I,j} w^j,$$

where $Z \in \{a\partial_a, b\partial_b, Z_{ij}\}$ and $c_{I,j}$ are smooth coefficients. We choose $0 < \epsilon_0 < \epsilon_1 < \ldots < \epsilon_k = \epsilon$ and let $\lambda_i = \lambda + i - \epsilon_i$. By Proposition 3.2, choosing N large and then $c = c(s_0, b_0, N)$, we may guarantee

$$M_1^N(w^i, s_0; \lambda_i)$$

$$\leq \sum_{|I| \leq N} \int_{\{t=0\} \cap \Omega} b^{-2\lambda_i} e^{-cT'} \langle \mathcal{F}_{\tilde{g}_e}(T_1, Z^I w^i), \nabla t \rangle_{\tilde{g}_e} \, d\mu_t \leq C_i < \infty$$

for all $i \leq k$. Assume now that

$$(3.4) \qquad M_1^{N-2i}\left(w^i, s; \lambda_i\right) \leq C_i'\left(1 + (-s)^{\lambda_i - \frac{1}{2}}\right),$$

for $s > s_0$ and $i \leq k - 1$. This then implies that

$$\left(\int_{\Sigma_s \cap \Omega} b^{-2\lambda_{i+1}} e^{-cT'}\left(b^2 \sum_{|I| \leq N-2i} c_I Z^I w^i\right)^2 d\mu_t\right)^{\frac{1}{2}} \leq C_i''\left(1 + (-s)^{\lambda_i}\right).$$

Then for $\lambda_{i+1} \leq \frac{1}{2}$, we have (as in the proof of Proposition 3.2)

$$\partial_s \left(M_1^{N-2i-2}\left(w^{i+1}, s; \lambda_{i+1}\right)\right)^2$$

$$\leq \left((1 - 2\lambda_{i+1})a^{-1} + Ca^{-\frac{1}{2}}\right)\left(M_1^{N-2i-2}(w^{i+1}, s; \lambda_{i+1})\right)^2$$

$$+ Ca^{-\frac{1}{2}} M_1^{N-2i-2}(w^{i+1}, s; \lambda_{i+1})(1 + (-s)^{\lambda_i}),$$

while if $\lambda_{i+1} \geq \frac{1}{2}$, we have

$$\partial_s \left(M_1^{N-2i-2}(w^{i+1}, s; \lambda_{i+1}) \right)^2$$
$$\leq Ca^{-\frac{1}{2}} \left(M_1^{N-2i-2}(w^{i+1}, s; \lambda_{i+1}) \right)^2$$
$$+ Ca^{-\frac{1}{2}} M_1^{N-2i-2}(w^{i+1}, s; \lambda_{i+1}) \left(1 + (-s)^{\lambda+k-1-\epsilon_{k-1}} \right).$$

Hence,

$$M_1^{N-2i-2}(w^k, s; \lambda_{i+1}) \leq C'_{i+1} \left(1 + (-s)^{\lambda_{i+1}-\frac{1}{2}} \right)$$

for all $s > s_0$. Now we know that equation (3.4) holds for $i = 0$. By induction and the arbitrariness of ϵ, we have thus shown that

$$M_1^{N-2k} \left(w^k, s; \lambda+k-\epsilon \right) \leq C'_k \left(1 + (-s)^{\lambda+k-\epsilon-\frac{1}{2}} \right)$$

for all $s > s_0$, where C'_k depends on the initial data and ϵ, N, and k. We thus have that if $\lambda + k = l + \alpha > 0$, then by inserting ∂_a^l as in the proof of Corollary 3.3, we see that for $\delta = \min\{\alpha-\epsilon, \frac{1}{2}\}$, w^k is $C^{l,\delta}$ up to $\{a = 0\}$. □

3.2. **At future null infinity.** We now adapt the argument from Section 3.1 to the region near null infinity. For $t > 0$ we introduce the coordinates

$$\bar\tau = t - r - 2M \log(r - 2M), \quad \bar\rho = \frac{1}{r}.$$

The conformal metric and its wave operator then become:

$$\tilde g_\infty = \bar\rho^2 g_S = -(1 - 2M\bar\rho)\bar\rho^2 \, d\tau^2 + 2d\bar\tau d\bar\rho + d\omega^2,$$
$$\tilde\Box_\infty = 2\partial_{\bar\tau}\partial_{\bar\rho} + (1 - 2M\bar\rho)(\bar\rho\partial_{\bar\rho})^2 + (1 - 4M\bar\rho)\bar\rho\partial_{\bar\rho} + \Delta_\omega.$$

We now calculate the commutators of $\tilde\Box_\infty$ with various vector fields:

$$\left[\tilde\Box_\infty, \partial_{\bar\tau}\right] = \left[\tilde\Box_\infty, Z_{ij}\right] = 0,$$
$$\left[\tilde\Box_\infty, \bar\rho\partial_{\bar\rho}\right] = \tilde\Box_\infty - (1 - 4M\bar\rho)(\bar\rho\partial_{\bar\rho})^2 - (1 - 8M\bar\rho)(\bar\rho\partial_{\bar\rho}) - \Delta_\omega,$$
$$\left[\tilde\Box_\infty, \partial_{\bar\rho}\right] = -2(1 - 3M\bar\rho)\bar\rho\partial_{\bar\rho}^2 - 2(1 - 6M\bar\rho)\partial_{\bar\rho}.$$

Moreover, we have that

$$\Box_S u = 0 \iff \left(\tilde\Box_\infty + \gamma_\infty\right)\tilde u = 0,$$
$$\tilde u = \bar\rho^{-1} u, \quad \gamma_\infty = -\bar\rho\tilde\Box_\infty\bar\rho^{-1} = -2M\bar\rho.$$

In the coordinates $(\bar\tau, \bar\rho, \omega)$, we choose the time-like function for $\bar\rho < \min\left\{1, \frac{1}{2M}\right\}$ as follows:

$$\bar T = \bar\tau - \bar\rho, \quad \langle\nabla\bar T, \nabla\bar T\rangle_{\tilde g_\infty} = -2 + \bar\rho^2(1 - 2M\bar\rho) < 0.$$

We then calculate the vector field $\mathcal{F}$ in this context:

$$\langle \mathcal{F}_{\tilde{g}_\infty}(\bar{T}, v), \nabla \bar{T} \rangle_{\tilde{g}_\infty}$$
$$= \left(1 - \bar{\rho}^2(1 - 2M\bar{\rho})\right) |\partial_{\bar{\rho}} v|^2 + \frac{1}{2} \left|\partial_{\bar{\tau}} v + \bar{\rho}^2(1 - 2M\bar{\rho})\partial_{\bar{\rho}} v\right|^2$$
$$+ \frac{1}{2} |\partial_{\bar{\tau}} v|^2 + \frac{1}{2} \left(2 - \bar{\rho}^2(1 - 2M\bar{\rho})\right) \left(|\nabla_\omega v|^2 + v^2\right),$$
$$\mathcal{Q}_{\tilde{g}_\infty}(\bar{T}, v) = -\bar{\rho}(1 - 3M\bar{\rho}) \left(|\partial_{\bar{\rho}} v|^2 + |\nabla_\omega|^2\right).$$

The above computation allows us to prove the following Proposition, whose proof is the same as that of Proposition 3.1 with the corresponding time-like function:

Proposition 3.6. *If* $(\phi, \psi) \in C^\infty\left((2M, \infty)_r \times \mathbb{S}^2\right)$ *satisfy* $\operatorname{supp}(\phi) \cup \operatorname{supp}(\psi) \subset (2M, \epsilon^{-1})$ *for some* $\epsilon > 0$, *then* $\tilde{u}$ *is smooth up to* $\{\bar{\rho} = 0\}$.

We now consider non-compactly supported data. Near S_0 in the partial compactification depicted in Figure 2, we may use coordinates $(\bar{a}, \bar{b})$ given by

$$\bar{a} = -\bar{\rho}\bar{\tau} = \frac{-t + r + 2M\log(r - 2M)}{r} \in [0, 1], \quad \bar{b} = -\frac{1}{\bar{\tau}}.$$

Together with the spherical coordinates ω, these are valid coordinates near the intersection of S_0 and S_1^+, where $\bar{a}$ is a defining function for E_1^+ and $\bar{b}$ is a defining function for E_0. Near the interior of E_1^+, we can take coordinates $(\bar{\tau}, \bar{\rho})$ as before. We extend $(\bar{a}, \bar{b})$ up to initial surface $t = 0$, which defines the smooth structure on this partial compactification near S_0. Notice that, at $t = 0$,

$$\bar{a} = \frac{\log(r - 2M)}{r} + 1.$$

In the coordinate system $(\bar{a}, \bar{b}, \omega)$, we have that

$$\tilde{g}_\infty = 2d\bar{a} \left(\frac{d\bar{b}}{\bar{b}}\right) + \bar{a} \left[2 - \bar{a}(1 - 2M\bar{a}\bar{b})\right] \left(\frac{d\bar{b}}{\bar{b}}\right)^2 + d\omega^2,$$
$$\tilde{\Box}_\infty = 2\partial_{\bar{a}} \left(\bar{b}\partial_{\bar{b}} - \bar{a}\partial_{\bar{a}}\right) + \left(1 - 2M\bar{a}\bar{b}\right) (\bar{a}\partial_{\bar{a}})^2 + \left(1 - 4M\bar{a}\bar{b}\right) \bar{a}\partial_{\bar{a}} + \Delta_\omega.$$

We also calculate the commutators of $\tilde{\Box}_\infty$ with relevant vector fields:

$$\left[\tilde{\Box}_\infty, Z_{ij}\right] = 0,$$
$$\left[\tilde{\Box}_\infty, \bar{b}\partial_{\bar{b}}\right] = 2M\bar{a}\bar{b}(\bar{a}\partial_{\bar{a}})^2 + 4M\bar{a}\bar{b}(\bar{a}\partial_{\bar{a}}),$$
$$\left[\tilde{\Box}_\infty, \bar{a}\partial_{\bar{a}}\right] = \tilde{\Box}_\infty - \left(1 - 4M\bar{a}\bar{b}\right) (\bar{a}\partial_{\bar{a}})^2 - (1 - 8M\bar{a}\bar{b})(\bar{a}\partial_{\bar{a}}) - \Delta_\omega.$$

In this coordinate system we now choose the time like functions $\bar{T}_1$ and $\bar{T}_1'$ as follows:

$$\bar{T}_1 = -\bar{a} + \log\bar{b}, \quad \langle\nabla\bar{T}_1, \nabla\bar{T}_1\rangle_{\tilde{g}_\infty} = -2 - \bar{a}\left(2 - \bar{a}(1 - 2M\bar{a}\bar{b})\right) < 0;$$
$$\bar{T}_1' = -\bar{a}, \quad \langle\nabla\bar{T}_1', \nabla\bar{T}_1'\rangle_{\tilde{g}_\infty} = -\bar{a}\left(2 - \bar{a}(1 - 2M\bar{a}\bar{b})\right) < 0.$$

The vector field $\mathcal{F}$ and the function $\mathcal{Q}$ are then given by

$$\langle \mathcal{F}_{\tilde{g}_\infty}(\bar{T}_1, v), \nabla \bar{T}_1' \rangle_{\tilde{g}_\infty}$$
$$= \frac{1}{2} \left(2 - \bar{a}(1 - 2M\bar{a}\bar{b}) \right) \bar{a} \, |\partial_{\bar{a}} v|^2 + \frac{1}{2} \left| \bar{a} \left(2 - \bar{a}(1 - 2M\bar{a}\bar{b}) \right) \partial_{\bar{a}} v - \bar{b} \partial_{\bar{b}} v \right|^2$$
$$+ \frac{1}{2} \left| \bar{b} \partial_{\bar{b}} v \right|^2 + \frac{1}{2} \left(1 + \bar{a} \left(2 - \bar{a}(1 - 2M\bar{a}\bar{b}) \right) \right) \left(|\nabla_\omega v|^2 + v^2 \right),$$
$$\mathcal{Q}_{\tilde{g}_\infty}(\bar{T}_1, v) = \left(1 - \bar{a}(1 - 3M\bar{a}\bar{b}) \right) \left(|\partial_{\bar{a}} v|^2 - |\nabla_\omega v|^2 - v^2 \right) + M\bar{a}\bar{b} \, |\bar{a} \partial_{\bar{a}} v|^2 \,.$$

We are now able to prove the following proposition:

Proposition 3.7. *If the initial data (ϕ, ψ) lie in the weighted Sobolev space*

$$\bar{\rho}^{\lambda+1} H_b^{N+1} \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right) \times \bar{\rho}^{\lambda+2} H_b^{N} \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right),$$

with $r_0 + 2M \log(r_0 - 2M) > 0$, $N > 2$ and $\lambda > 0$, then $\tilde{u}$ is C^δ up to $\{\bar{a} = 0\}$ for $\bar{b} < \bar{b}_0$, where $\delta = \min\{\lambda, \frac{1}{2}\}$ and $\bar{b}_0 = 1/(r_0 + 2M \log(r_0 - 2M))$.

Proof. The proof is the same as the proof of Proposition 3.2 with the corresponding time-like function. Because ∂_t lifts to $\bar{b} \left(\bar{b} \partial_{\bar{b}} - \bar{a} \partial_{\bar{a}} \right)$, the assumption on the initial data implies that at the Cauchy surface.

$$Z^I \tilde{u}|_{t=0} \in \bar{\rho}^\lambda H_b^{N+1-|I|} \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right)$$

for $Z \in \{\bar{a} \partial_{\bar{a}}, \bar{b} \partial_{\bar{b}}, Z^{ij}\}$ and multi-index I. $\qquad\square$

As in the previous section, we have the following corollary:

Corollary 3.8. *Suppose $\lambda = k + \alpha$ for some integer k and $\alpha \in (0, 1]$. If the initial data (ϕ, ψ) lie in the space*

$$\bar{\rho}^{\lambda+1} H_b^{N+1} \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right) \times \bar{\rho}^{\lambda+2} H_b^{N} \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right),$$

with $r_0 + 2M \log(r_0 - 2M) > 0$ and $N > 2 + k$, then $\tilde{u}$ is $C^{k,\delta}$ up to $\{\bar{a} = 0\}$ for $\bar{b} < \bar{b}_0$, where $\delta = \min\{\alpha, \frac{1}{2}\}$ and $\bar{b}_0 = 1/(r_0 + 2M \log(r_0 - 2M))$.

Proof. The proof is the same as the proof of Corollary 3.3 with the corresponding time-like function. This is because $\tilde{\Box}_\infty$ satisfies:

$$\left[\tilde{\Box}_\infty, \partial_{\bar{a}}^k \right] = 2k \partial_{\bar{a}}^{k+1} + \sum_{|I|+i \leq k+1, i \leq k} c_{I,i} Z^I \partial_{\bar{a}}^i,$$

where $Z \in \{\bar{a} \partial_{\bar{a}}, \bar{b} \partial_{\bar{b}}, Z_{ij}\}$ and $c_{I,i}$ are smooth coefficients. $\qquad\square$

Again as before, "Schwartz" data behaves in much the same way as compactly supported data:

Corollary 3.9. *For initial data (ϕ, ψ) in*

$$\bar{\rho}^\infty H_b^\infty \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right) \times \bar{\rho}^\infty H_b^\infty \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right),$$

with $r_0 + 2M \log(r_0 - 2M) > 0$, the rescaled solution $\tilde{u}$ is smooth up to $\{\bar{a} = 0\}$ for $\bar{b} < \bar{b}_0 = 1/(r_0 + 2M \log(r_0 - 2M))$.

The coordinate system given by $(\bar{a}, \bar{b}, \omega)$ valid only on half of null infinity, i.e. $\bar{\tau} < 0$. To extend Proposition 3.7 to the other half, we may use similar energy estimates in the $(\bar{\tau}, \bar{\rho}, \omega)$ coordinate system with $\bar{\tau} \in (-\bar{\tau}_0, \bar{\tau}_0)$ for arbitrarily large $\bar{\tau}_0$.

In the coordinate system $(\bar{\tau}, \bar{\rho}, \omega)$, we choose a pair of time-like functions $(\bar{T}_2, \bar{T}_2')$ for $\bar{\tau} \in (-\bar{\tau}_0, \bar{\tau}_0)$ as follows:

$$\bar{T}_2 = \bar{T}, \quad \bar{T}_2' = -\bar{\rho}(2\bar{\tau}_0 - \tau),$$

Here $\bar{T}_2'$ is asymptotically null when approaching null infinity

$$\langle \nabla \bar{T}_2', \nabla \bar{T}_2' \rangle_{\tilde{g}_\infty} = -\bar{\rho}(2\bar{\tau}_0 - \bar{\tau}) \left(2 - (1 - 2M\bar{\rho})\bar{\rho}(2\bar{\tau}_0 - \bar{\tau}) \right) \leq 0.$$

and the vector field $\mathcal{F}$ is

$$\langle \mathcal{F}_{\tilde{g}_\infty}(\bar{T}_2, v), \nabla \bar{T}_2' \rangle_{\tilde{g}_\infty}$$

$$= \left(1 - \frac{1}{2}(\bar{\rho} + 2\bar{\tau}_0 - \bar{\tau})(1 - 2M\bar{\rho})\bar{\rho} \right) \bar{\rho} \, |\partial_{\bar{\rho}} v|^2$$

$$+ \frac{1}{2}(2\bar{\tau}_0 - \bar{\tau}) \left(\left| (1 - 2M\bar{\rho})\bar{\rho}^2 \partial_{\bar{\rho}} v + \partial_{\bar{\tau}} v \right|^2 + |\partial_{\bar{\tau}}|^2 \right)$$

$$+ \frac{1}{2} \left(2\bar{\tau}_0 - \bar{\tau} + \bar{\rho} - (1 - 2M\bar{\rho})(2\bar{\tau}_0 - \bar{\tau})\bar{\rho}^2 \right) \left(|\nabla_\omega v|^2 + v^2 \right).$$

We use the pair $(\bar{T}_2, \bar{T}_2')$ of time-like functions to obtain estimates similar to those in Propositions 3.2 and 3.7:

Proposition 3.10. *Suppose $r_0 > 2M$ is close to $2M$ so that $\bar{\tau}_0 = -(r_0 + 2M \log(r_0 - 2M)) > 0$. If the initial data is in the following space*

$$\bar{\rho}^{\lambda+1} H_b^{N+1} \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right) \times \bar{\rho}^{\lambda+2} H_b^N \left(\left[0, \frac{1}{r_0} \right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho} \, d\omega}{\bar{\rho}} \right)$$

with $N > 2$ and $\lambda > 0$, then $\tilde{u}$ is C^δ up to $\{\bar{\rho} = 0\}$ for all $\bar{\tau} \in (-\infty, \bar{\tau}_0)$, where $\delta = \min\{\lambda, \frac{1}{2}\}$.

Proof. For $\bar{\tau} \leq -\bar{\tau}_0$, i.e., $\bar{b} \leq \bar{b}_0$, the statement is proved in Proposition 3.7. Let Ω be the domain bounded by $\{t = 0\}$, S_1^+, and

$$S = \{\bar{T}_2 = \log \bar{b}_0\} = \{\bar{T}_2 = -\bar{\tau}_0\}, \quad S' = \{\bar{T}_2 = \bar{\tau}_0\}.$$

Let $\Sigma_s = \{\bar{T}_2' = s\}$, which is space-like for $s < 0$ and approaches null infinity as $s \to 0$. We define the quantities

$$M_2^N(\tilde{u}, s) = \left(\sum_{|I| \leq N} \int_{\Sigma_s \cap \Omega} e^{-c\bar{T}_2} \langle \mathcal{F}_{\tilde{g}_\infty}(\bar{T}_2, Z^I \tilde{u}, \nabla \bar{T}_2') \rangle_{\tilde{g}_\infty} \, d\mu_{\bar{T}_2'} \right)^{\frac{1}{2}},$$

$$L_1^N(\tilde{u}, s) = \left(\sum_{|I| \leq N} \int_{S \cap \{\bar{T}_2' < s\}} e^{-c\bar{T}_2} \langle \mathcal{F}_{\tilde{g}_\infty}(\bar{T}_2, Z^I \tilde{u}), \nabla \bar{T}_1 \rangle_{\tilde{g}_\infty} \, d\mu_{\bar{T}_1} \right)^{\frac{1}{2}},$$

$$L_2^N(\tilde{u}, s) = \left(\sum_{|I| \leq N} \int_{S' \cap \{\bar{T}_2' < s\}} e^{-c\bar{T}_2} \langle \mathcal{F}_{\tilde{g}_\infty}(\bar{T}_2, Z^I \tilde{u}), \nabla \bar{T}_2 \rangle_{\tilde{g}_\infty} \right)^{\frac{1}{2}},$$

where $Z^I \in \{\rho \partial_\rho, \partial_\tau, Z_{ij}\}$. We choose c large enough so that

$$\sum_{|I| \leq N} \mathrm{div}_{\tilde{g}_\infty} \left(e^{-c\bar{T}_2} \mathcal{F}_{\tilde{g}_\infty}(\bar{T}_2, Z^I \tilde{u}) \right) \leq 0.$$

Note that this is possible because $\tilde{u}$ solves the equation $\left(\tilde{\Box}_\infty + \gamma_\infty \right) \tilde{u} = 0$. Here c depends on τ_0 and N.

We now choose $s_0 < 0$ small enough so that $\Sigma_{s_0} \cap \{t = 0\} \cap \Omega = \emptyset$. By a proof similar to the one in Proposition 3.2, we have that

$$M_2^N(\tilde{u}, s_0) < C_N,$$

where C_N is bounded by the initial data norm and depends on s_0. Moreover, the bound in the proof of Proposition 3.7 implies that

$$\partial_s \left(L_1^N(\tilde{u}, s) \right)^2 \leq \begin{cases} C(-s)^{2\lambda - 2} & \lambda < \frac{1}{2} \\ C(-s)^{-\frac{1}{2}} & \lambda \geq \frac{1}{2} \end{cases}$$

for $s > s_0$. Here C depends only on the norm of the initial data, $\bar{\tau}_0$, and s_0.

Consider now the domain $\Omega_{s_0}^s$ bounded by Σ_{s_0}, Σ_s, S, and S', for $s > s_0$. By Stokes' theorem,

$$\left(M_2^N(\tilde{u}, s) \right)^2 - \left(M_2^N(\tilde{u}, s_0) \right)^2 + \left(L_2^N(\tilde{u}, s) \right)^2 - \left(L_2^N(\tilde{u}, s_0) \right)^2$$
$$\leq \left(L_1^N(\tilde{u}, s) \right)^2 - \left(L_1^N(\tilde{u}, s_0) \right)^2.$$

Dividing by $s - s_0$ and taking a limit then implies that

$$\partial_s \left(M_2^N(\tilde{u}, s) \right)^2 \leq \partial_s \left(L_1^N(\tilde{u}, s) \right)^2.$$

Using the bound above and integrating then shows that, for $s > s_0$,

$$M_2^N(\tilde{u}, s) \leq C' \left(1 + (-s)^{\lambda - \frac{1}{2}} \right).$$

An application of Sobolev embedding then shows that $\tilde{u}$ is C^δ up to $\{\bar{\rho} = 0\}$ for $\bar{\tau} \in (-\bar{\tau}_0, \bar{\tau}_0)$. $\qquad\square$

Similarly, Corollaries 3.8 and 3.9 may be extended to the entire null infinity.

Corollary 3.11. *Suppose $\lambda = k + \alpha$ for an integer k and $\alpha \in (0, 1]$. If the initial data (ϕ, ψ) is in the space*

$$\bar{\rho}^{\lambda+1} H_b^{N+1} \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho}\, d\omega}{\bar{\rho}} \right) \times \bar{\rho}^{\lambda+2} H_b^N \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho}\, d\omega}{\bar{\rho}} \right)$$

with $r_0 + 2M \log(r_0 - 2M) < 0$ and $N > 2 + k$, then $\tilde{u}$ is $C^{k,\delta}$ up to $\{\bar{\rho} = 0\}$ for $\bar{\tau} < \bar{\tau}_0 = -(2r_0 + 2M \log(r_0 - 2M))$. Here $\delta = \min\{\alpha, \frac{1}{2}\}$.

Proof. For $\bar{\tau} \leq -\bar{\tau}_0$, this reduces to Corollary 3.8. We now define:

$$\tilde{M}_2^N(\tilde{u}, s; \lambda) = \left(\sum_{i \leq k} \left(M_2^{N-1}(\partial_{\bar{\rho}}^i \tilde{u}, s; \lambda) \right)^2 \right)^{\frac{1}{2}},$$

$$\tilde{L}_1^N(\tilde{u}, s; \lambda) = \left(\sum_{i \leq k} \left(L_1^{N-i}(\partial_{\bar{\rho}}^i \tilde{u}, s; \lambda) \right)^2 \right)^{\frac{1}{2}},$$

$$\tilde{L}_2^N(\tilde{u}, s; \lambda) = \left(\sum_{i \leq k} \left(L_2^{N-i}(\partial_{\bar{\rho}}^i \tilde{u}, s; \lambda) \right)^2 \right)^{\frac{1}{2}}.$$

Notice that

$$\left[\tilde{\Box}_\infty, \partial_{\bar{\rho}}^k \right] = \sum_{|I|+i \leq k+1,\, i \leq k} c_{I,i} Z^I \partial_{\bar{\rho}}^i,$$

where $Z \in \{a\partial_a, b\partial_b, Z_{ij}\}$ and $c_{I,i}$ are smooth coefficients. By a proof similar to that of Corollary 3.3, we have:

$$\partial_s \left(\tilde{M}_2^N(\tilde{u}, s; \lambda) \right)^2 + \partial_s \left(\tilde{L}_2^N(\tilde{u}, s; \lambda) \right)^2$$

$$\leq \partial_s \left(\tilde{L}_1^N(\tilde{u}, s; \lambda) \right)^2 \leq \begin{cases} C\,(-s)^{2\alpha-2}, & \alpha < \frac{1}{2} \\ C\,(-s)^{-\frac{1}{2}}, & \alpha \geq \frac{1}{2} \end{cases}.$$

We thus have that $\partial_{\bar{\rho}}^k \tilde{u}$ is C^δ up to $\{\bar{\rho} = 0\}$ for $\bar{\tau} < \bar{\tau}_0$. $\qquad \square$

Corollary 3.12. *If the initial data are "Schwartz", i.e., (ϕ, ψ) lie in*

$$\bar{\rho}^\infty H_b^\infty \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho}\, d\omega}{\bar{\rho}} \right) \times \bar{\rho}^\infty H_b^\infty \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2, \frac{d\bar{\rho}\, d\omega}{\bar{\rho}} \right),$$

with $r_0 + 2M \log(r_0 - 2M) < 0$, then $\tilde{u}$ is smooth up to $\{\bar{\rho} = 0\}$ for $\bar{\tau} < \bar{\tau}_0 = -(2r_0 + 2M \log(r_0 - 2M))$.

Suppose now that the initial data has a classical asymptotic expansion at S_0 (spatial infinity), i.e.,

$$(\phi, \psi) \in \bar{\rho}^{\lambda+1} C^\infty \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2 \right) \times \bar{\rho}^{\lambda+2} C^\infty \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2 \right).$$

with $r_0 + 2M \log(r_0 - 2M) < 0$. Define the "k-th term" w^k as follows:

$$(3.5) \qquad w^0 = \tilde{u} = \bar{\rho}^{-1} u, \quad w^k = \prod_{i=0}^{k-1} \left(\bar{b} \partial_{\bar{b}} - \lambda - i \right)^{n_i} \tilde{u} \quad \text{for} \quad k \geq 1,$$

where n_i are integers that only depend on i such that

$$(Z^I w^k)|_{t=0} \in \sum_{j=1}^{N(k,I)} \bar{\rho}^{\lambda+k} (\log \bar{\rho})^j C^\infty \left(\left[0, \frac{1}{r_0}\right)_{\bar{\rho}} \times \mathbb{S}^2 \right)$$

for $Z \in \{\bar{a}\partial_{\bar{a}}, \bar{b}\partial_{\bar{b}}, Z^{ij}\}$, multi-index I and integers $N(k, I)$. Here the integers n_i and $N(k, I)$ can be calculated according to the lift of $\bar{a}\partial_{\bar{a}}, \bar{b}\partial_{\bar{b}}$:

$$\bar{a}\partial_{\bar{a}} = \bar{\rho}\partial_{\bar{\rho}} - \frac{1}{\bar{\rho}(1 - 2M\bar{\rho})} \partial_t,$$

$$\bar{b}\partial_{\bar{b}} = \bar{\rho}\partial_{\bar{\rho}} - 2M \left(\log \bar{\rho} + \frac{1}{1 - 2M\bar{\rho}} - \log(1 - 2M\bar{\rho}) \right) \partial_t - t\partial_t.$$

Proposition 3.13. *If $\lambda + k = l + \alpha$ for some integer l and $\alpha \in (0, 1]$, then w^k is $C^{l,\delta}$ up to $\{\bar{\rho} = 0\}$ for all $\bar{\tau} < \bar{\tau}_0 = -(2r_0 + 2M \log(r_0 - 2M))$, where $\delta = \min\{\frac{1}{2}, \alpha - \epsilon\}$ with $\epsilon > 0$ arbitrarily small.*

Proof. We apply a similar proof to that of Proposition 3.5 for $\bar{\tau} < -\bar{\tau}_0$, i.e. for $\bar{b} < \bar{\tau}_0^{-1}$, and then extend it to $\bar{\tau} < \bar{\tau}_0$ by a method similar to the method in Proposition 3.10. $\qquad\square$

4. The radiation field

In this section we define the radiation field, show it extends to an energy space, and then show it is unitary (i.e., norm-preserving) on this space.

Consider now smooth initial data (ϕ, ψ) supported in $r \geq 2M + \epsilon$ and $r \leq R_0$, and let u be the solution of $\Box_S u = 0$ with this initial data. By the results of the last section, u restricts to the event horizon and ru restricts to null infinity.

Definition 4.1. *The* forward radiation field of (ϕ, ψ) at the event horizon *is the restriction of $\partial_t u$ to the event horizon. In the coordinates (τ, ρ, ω) of Section 3.1, it is given by*

$$\mathcal{R}_{E_1^+}(\phi, \psi)(\tau, \omega) = \partial_t u(\tau, \rho, \omega)|_{\rho=0}.$$

The forward radiation field of (ϕ, ψ) at null infinity *is the restriction of $\partial_t v$ to null infinity. In the coordinates $(\bar{\tau}, \bar{\rho}, \omega)$ of Section 3.2, it is given by*

$$\mathcal{R}_{S_1^+}(\phi, \psi)(\bar{\tau}, \omega) = \bar{\rho}^{-1} \partial_t u(\bar{\tau}, \bar{\rho}, \omega)|_{\bar{\rho}=0}.$$

Finite speed of propagation implies that $\mathcal{R}_{E_1^+}(\phi,\psi)(\tau,\omega)$ vanishes identically for $\tau \leq 2M + \epsilon + 2M \log \epsilon$ and that $\mathcal{R}_{S_1^+}(\phi,\psi)(\bar\tau,\omega)$ vanishes identically for $\bar\tau \leq -(R_0 + 2M \log(R_0 - 2M))$.

The existence of the static Killing field ∂_t implies that the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{S}^2} \int_{2M}^\infty e(t) r^2 \, dr \, d\omega$$

is conserved, where

$$e(t) = \left(1 - \frac{2M}{r}\right)^{-1} (\partial_t u)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r u)^2 + \frac{1}{r^2} |\nabla_\omega u|^2 \, .$$

For a fixed λ, let us write

$$\begin{aligned}
E(t) = &\frac{1}{2} \int_{\mathbb{S}^2} \int_{-r - 2M \log(r - 2M) \geq t - \lambda} e(t) r^2 \, dr \, d\omega \\
&+ \frac{1}{2} \int_{\mathbb{S}^2} \int_{\lambda - t \leq r + 2M \log(r - 2M) \leq t - \lambda} e(t) r^2 \, dr \, d\omega \\
&+ \frac{1}{2} \int_{\mathbb{S}^2} \int_{r + 2M \log(r - 2M) \geq t - \lambda} e(t) r^2 \, dr \, d\omega \\
= &\mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Let $f_{E_1^+} = \mathcal{R}_{E_1^+}(\phi,\psi)(\tau,\omega)$ and $f_{S_1^+} = \mathcal{R}_{S_1^+}(\phi,\psi)(\bar\tau,\omega)$.

We now compute term I in the (τ, ρ, ω) coordinates of Section 3.1. Recall that near event horizon,

$$\tau = t + r + 2M \log(r - 2M), \quad \rho = r - 2M.$$

So ∂_t lifts to ∂_τ and ∂_r lifts to $\partial_\rho + (\rho + 2M)\rho^{-1}\partial_\tau$. Moreover, t is fixed, so we are free to use τ as the variable of integration (so we think of r as a function of τ) and then $r^2 dr d\omega = (\rho + 2M)\rho d\tau d\omega$. We then have that

$$\begin{aligned}
\mathrm{I} = \frac{1}{2} \int_{\mathbb{S}^2} \int_{\tau \leq \lambda} &\left(2 |\partial_\tau u|^2 + \frac{2\rho \partial_\rho u \partial_\tau u}{(\rho + 2M)} + \frac{|\rho \partial_\rho u|^2}{(\rho + 2M)^2} \right. \\
&\left. + \frac{\rho |\nabla_\omega u|^2}{(\rho + 2M)^3} \right) (\rho + 2M)^2 \, d\tau \, d\omega.
\end{aligned}$$

Because $\mathcal{R}_{E_1^+}(\phi,\psi)$ vanishes for $\tau \leq \tau_0$, the convergence of $\partial_\tau u$ (together with its derivatives) to $\mathcal{R}_{E_1^+}(\phi,\psi)$ is uniform on $(-\infty, \lambda]$, and so we may take a limit as $t \to \infty$. This implies that

$$\lim_{t \to \infty} \mathrm{I} = 4M^2 \int_{\mathbb{S}^2} \int_{-\infty}^\lambda f_{E_1^+}^2 \, d\tau \, d\omega.$$

We then compute term III in the $(\bar\tau, \bar\rho, \omega)$ coordinates of Section 3.2. Near future null infinity,

$$\bar\tau = t - r - 2M \log(r - 2M), \quad \bar\rho = \frac{1}{r}.$$

So ∂_t lifts to $\partial_{\bar\tau}$ and ∂_r lifts to $-\bar\rho^2\partial_{\bar\rho} - (1 - 2M\bar\rho)^{-1}\partial_{\bar\tau}$, and the volume form $r^2\,dr\,d\omega$ becomes $\bar\rho^{-4}d\bar\rho\,d\omega$. We are again free to use $\bar\tau$ as the variable of integration, so that $d\bar\tau = (1 - 2M\bar\rho)^{-1}\,\bar\rho^{-2}d\bar\rho$. We then have that

$$\mathrm{III} = \frac{1}{2}\int_{\mathbb{S}^2}\int_{\bar\tau\leq\lambda}\left(2\left|\bar\rho^{-1}\partial_{\bar\tau}u\right|^2 + 2(1 - 2M\bar\rho)\partial_{\bar\rho}u\partial_{\bar\tau}u + (1 - 2M\bar\rho)^2|\bar\rho\partial_{\bar\rho}u|^2\right.$$
$$\left. + (1 - 2M\bar\rho)|\nabla_\omega u|^2\right)d\bar\tau\,d\omega.$$

We now use the vanishing of $\mathcal{R}_{S_1^+}(\phi,\psi)$ for $\bar\tau\leq\bar\tau_0$ to conclude that the convergence of $\bar\rho^{-1}\partial_{\bar\tau}u$ (and its derivatives) to $\mathcal{R}_{S_1^+}(\phi,\psi)$ is uniform on $(-\infty,\lambda]$. This allows us to take a limit as $t\to\infty$ and conclude that

$$\lim_{t\to\infty}\mathrm{III} = \int_{\mathbb{S}^2}\int_{-\infty}^{\lambda} f_{S_1^+}^2\,d\bar\tau\,d\omega.$$

Term II is positive, so we may now take a limit as $\lambda\to\infty$ to conclude that

$$(4.1)\qquad E(0) = 4M^2\left\|f_{E_1^+}\right\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2 + \left\|f_{S_1^+}\right\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}^2$$
$$+ \lim_{\lambda\to\infty}\lim_{t\to\infty}\frac{1}{2}\int_{\mathbb{S}^2}\int_{\lambda-t\leq r+2M\log(r-2M)\leq t-\lambda} e(t)r^2\,dr\,d\omega.$$

We define the energy space H_E of initial data to be the completion of smooth functions compactly supported in $(2M,\infty)\times\mathbb{S}^2$ with respect to the energy norm

$$\|(\phi,\psi)\|_{H_E}^2 = \frac{1}{2}\int_{\mathbb{S}^2}\int_{2M}^{\infty}\left[\left(1 - \frac{2M}{r}\right)^{-1}\psi^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\phi)^2\right.$$
$$\left. + \frac{1}{r^2}\left|\nabla_\omega\phi\right|^2\right]r^2\,dr\,d\omega.$$

The above computation then shows that the radiation field extends to a bounded operator $H_E\to L^2(\mathbb{R}\times\mathbb{S}^2)\oplus L^2(\mathbb{R}\times\mathbb{S}^2)$.

We now show that $\mathcal{R}_+$ is unitary (but not necessarily surjective), i.e., that the last term in equation (4.1) vanishes. We first show this for compactly supported smooth initial data and then extend by density to the full energy space. Versions of this statement were shown by Dimock [Dim85] and Bachelot [Bac94], though they work in a more abstract scattering framework.

The main tool we use to establish unitarity is the formulation of Price's law proved by Tataru [Tat13], which roughly states that waves decay at the order of $\langle t\rangle^{-3}$ on the Schwarzschild exterior. This formulation is somewhat stronger than the standard formulation of Price's law near the event horizon. In particular, we refer to the formulation of Metcalfe–Tataru–Tohaneanu [MTT12]. We do not need the full strength of Price's law; a decay bound on the energy in all forward cones of smaller aperture than a

light cone should suffice. The local energy decay bounds in the literature, however, do not seem quite strong enough for our purposes.

In the "normalized coordinates" of Tataru [Tat13] and Metcalfe–Tataru–Tohaneanu [MTT12], one has that solutions of the wave equation with smooth, compactly supported data satisfy the following pointwise bounds:

$$|u| \lesssim \frac{1}{\langle t^* \rangle \langle t^* - r^* \rangle^2}, \quad |\partial_t u| \lesssim \frac{1}{\langle t^* \rangle \langle t^* - r^* \rangle^3}.$$

These coordinates correspond to modified Regge–Wheeler coordinates and therefore in our setting,

$$t^* \sim t, \ r^* \sim r + 2M \log(r - 2M), \qquad \text{near null infinity},$$
$$t^* \sim \tau, \ r^* \sim r, \qquad\qquad \text{near the event horizon}.$$

In particular, in our coordinates u is bounded[2] by $\langle t \rangle^{-1} \langle \bar{\tau} \rangle^{-2}$ near null infinity and by $\langle \tau \rangle^{-1} \langle t + 2M \log(r - 2M) \rangle^{-2}$ near the event horizon.

We now use the pointwise bounds to bound the integral II above. Let us implicitly insert cutoff functions and treat it as two integrals - one for large r and another for small r. For large r, inserting the pointwise bounds yields an integral bounded by

$$\int_{r_0}^{R^{-1}(t-\lambda)} \int_{\mathbb{S}^2} \frac{1}{\langle t \rangle^2 \langle \bar{\tau} \rangle^4} r^2 \, dr \, d\omega \lesssim \frac{1}{\langle t \rangle^2 \lambda} + \frac{2}{\langle t \rangle \lambda^2} + \frac{1}{\lambda^3}.$$

For r near $2M$, the pointwise bounds yield an integral bounded by

$$\int_{R^{-1}(\lambda-t)}^{r_0} \int_{\mathbb{S}^2} \frac{1}{\langle \tau \rangle^2 \langle t + 2M \log(r - 2M) \rangle^4} r^2 \, dr \, d\omega$$
$$\lesssim \frac{C_{r_0}}{\langle t + (\lambda - t - 2M) \rangle^6} \lesssim \lambda^{-6}.$$

Here R^{-1} is the inverse function of $R : (2M, \infty) \longrightarrow (-\infty, \infty)$ defined by $R(r) = r + 2M \log(r - 2M)$. By sending $t \to \infty$ and then sending $\lambda \to \infty$, we see that the additional contribution of term II tends to 0. This yields that for smooth, compactly supported data, we have

$$E(0) = 4M^2 \left\| f_{E_1^+} \right\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2 + \left\| f_{S_1^+} \right\|_{L^2(\mathbb{R} \times \mathbb{S}^2)}^2,$$

i.e., the radiation field is unitary on this subpace. A standard density argument then shows that the radiation field is a unitary operator on the space of initial data with finite energy norm. This finishes the proof of Theorem 1.3.

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[2]One should think of the $\langle t \rangle^{-1}$ as governing the decay we factor out when defining the radiation field.

5. Support theorems for the radiation field

In this section we prove two support theorems for smooth compactly supported initial data using methods of Sá Barreto [SB03] and the first author and Sá Barreto [BSB12]. We first show that for such data, the support of the radiation field on the event horizons provides a restriction on the support of the data. We then show a similar theorem for null infinity. In particular, for initial data $(0, \psi)$, $\psi \in C_c^\infty\left((2M, \infty) \times \mathbb{S}^2\right)$, we show that if the radiation field vanishes on E_1^+, then $\psi \equiv 0$. In other words, it is impossible to find initial data in this class so that the radiation field is supported only at null infinity. We also show the corresponding statements for null infinity (S_1^+).

First, let us define the two components of the backward radiation field as follows:

$$(5.1) \qquad \mathcal{R}_{S_1^-}(\phi, \psi)(\tau, \omega) = \lim_{r \to 2M} \partial_t u(\tau + r + 2M \log(r - 2M), r, \omega),$$
$$\mathcal{R}_{E_1^-}(\phi, \psi)(\bar\tau, \omega) = \lim_{r \to \infty} r \partial_t u(\bar\tau - r - 2M \log(r - 2M), r, \omega).$$

5.1. At the event horizon.

The main result of this section is the following theorem:

Theorem 5.1. *Suppose $\phi, \psi \in C_c^\infty\left((2M, \infty) \times \mathbb{S}^2\right)$. If $\mathcal{R}_{E_1^+}(\phi, \psi)$ vanishes for $\tau \leq \tau_0$ and $\mathcal{R}_{E_1^-}(\phi, \psi)$ vanishes for $\tau \geq -\tau_0$, then both ϕ and ψ are supported in $[r_0, \infty) \times \mathbb{S}^2$, where r_0 is given implicitly by*

$$r_0 + 2M \log(r_0 - 2M) = \tau_0.$$

Corollary 5.2. *Suppose $\psi \in C_c^\infty\left((2M, \infty) \times \mathbb{S}^2\right)$. If $\mathcal{R}_{E_1^+}(0, \psi)$ vanishes for $\tau \leq \tau_0$, then ψ is supported in $[r_0, \infty) \times \mathbb{S}^2$. In particular, if $\mathcal{R}_{E_1^+}(0, \psi)$ vanishes identically, then $\psi \equiv 0$.*

Proof of Corollary 5.2. By the time-reversibility of the wave equation, we know that

$$\mathcal{R}_{E_1^+}(0, \psi)(\tau, \omega) = \mathcal{R}_{E_1^-}(0, \psi)(-\tau, \omega),$$

and so we may apply Theorem 5.1. □

We showed in Section 3.1 that solutions with compactly supported smooth initial data are smooth up to the event horizons. In fact, the argument above shows they are jointly smooth at the event horizons. Here μ and ν are Kruskal coordinates, given in terms of $\tau_+ = t + r + 2M \log(r - 2M)$ and $\tau_- = t - r - 2M \log(r - 2M)$ as follows:

$$\mu = e^{\frac{\tau_+}{4M}}, \ \nu = e^{-\frac{\tau_-}{4M}}.$$

Note that near $r = 2M$ (and therefore in the entire exterior domain), r is a smooth function of μ and ν. In these coordinates, the Schwarzschild metric

has the following form:

$$g_S = \frac{16M^2 e^{-r/2M}}{r}\, d\mu\, d\nu + r^2\, d\omega^2.$$

Observe that μ is a defining function for the past event horizon E_1^- while ν is a defining function for the future event horizon E_1^+.

We summarize the joint smoothness in the following lemma:

Lemma 5.3. *If $\phi, \psi \in C_c^\infty\big((2M, \infty) \times \mathbb{S}^2\big)$ and u solves the initial value problem (1.1) with data (ϕ, ψ), then u is smooth as a function of μ and ν.*

Proof of Theorem 5.1. Let $c_0 = e^{\tau_0/4M}$. We start by showing that u must vanish to infinite order at $\nu = 0$ for $\mu \in [0, c_0]$ and at $\mu = 0$ for $\nu \in [0, c_0]$. Indeed, consider the D'Alembertian $\Box_S$, which is given in these coordinates by

$$\Box_S = \frac{e^r}{8M^2 r}\partial_\mu\left(r^2\partial_\nu\right) + \frac{e^r}{8M^2 r}\partial_\nu\left(r^2\partial_\mu\right) + \frac{1}{r^2}\Delta_\omega,$$

where r is regarded as a smooth function of μ and ν. Observe that near $\mu = 0$ or $\nu = 0$, $r - 2M \sim \mu\nu$.

Because the initial data is compactly supported and smooth, there is some constant a_0 so that $u(\mu, \nu, \omega)$ vanishes for $(\mu, \nu) \in [0, a_0]^2$. Moreover, the assumption on the radiation field implies that $\partial_\mu u$ vanishes when $\nu = 0$ and $\mu \in [0, c_0]$, while $\partial_\nu u$ vanishes for $\mu = 0$ and $\nu \in [0, c_0]$. Moreover, because the initial data is smooth and compactly supported, one may write (with a similar expression for $u(\mu, 0, \omega)$

$$u(0, \nu, \omega) = \int_0^\nu \partial_\nu u(0, s, \omega)\, ds,$$

to conclude that in fact u vanishes for $\{\mu = 0, \nu \in [0, c_0]\}$ and $\{\nu = 0, \mu \in [0, c_0]\}$.

We now work exclusively at $\nu = 0$ (as the $\mu = 0$ case is handled in the same way). Because u satisfies $\Box_S u = 0$, we have that for $\nu = 0$ and $\mu \in [0, c_0]$,

$$\frac{2e^r}{8M^2 r}\partial_\nu\partial_\mu u + \frac{e^r}{8M^2 r}\partial_\mu(r^2)\partial_\nu u + \frac{1}{r^2}\Delta_\omega u = 0.$$

Because $u(\mu, 0, \omega) = 0$ for $\mu \in [0, c_0]$, $\frac{1}{r^2}\Delta_\omega u = 0$ here as well. Moreover, we note that $\partial_\mu r^2 \sim \nu$, so the second term vanishes at $\nu = 0$ as well. (Recall that we already know u is smooth as a function of (μ, ν, ω).) In particular, $\partial_\nu\partial_\mu u(\mu, 0, \omega) = 0$. By integrating in μ, we may also conclude that $\partial_\nu u(\mu, 0, \omega) = 0$ for $\mu \in [0, c_0]$.

Differentiating the equation $\frac{8M^2 r}{e^r}\Box_S u = 0$ in ν yields

$$2\partial_\nu^2\partial_\mu u + \partial_\mu(r^2)\partial_\nu^2 u + \partial_\nu\partial_\mu(r^2)\partial_\nu u + \frac{8M^2}{r^2 e^r}\Delta_\omega\partial_\nu u + \partial_\nu\left(\frac{8M^2}{re^r}\right)\Delta_\omega u = 0.$$

Using that $\partial_\nu u$, and u vanish at $(\mu, 0, \omega)$ and that $\partial_\mu(r^2)$ vanishes at $\nu = 0$, we conclude that $\partial_\nu^2 \partial_\mu u(\mu, 0, \omega) = 0$. Integrating again, we also conclude that $\partial_\nu^2 u(\mu, 0, \omega) = 0$.

Continuing inductively, after differentiating the equation k times, we may conclude that $\partial_\nu^{k+1} u$ vanishes at $(\mu, 0, \omega)$ for $\mu \in [0, c_0]$. In particular, u vanishes to infinite order at $\nu = 0$ for $\mu \in [0, c_0]$.

We now decompose u into spherical harmonics. We set w_j to be an eigenbasis of $L^2(\mathbb{S}^2)$ with eigenvalues $-\lambda_j^2$ and write $u(t, r, \omega) = \sum_{j=0}^\infty u_j(t, r) w_j(\omega)$. Note that this decomposition extends to the radiation field as well. The smoothness of the initial data (and thus the solution) implies that the terms in the series are rapidly decreasing in j and so the series converges uniformly. In particular, each u_j vanishes to infinite order at $\nu = 0$ for $\mu \in [0, c_0]$ (with a corresponding statement at the other horizon). In terms of the coordinates μ and ν, each u_j solves the equation

$$(5.2) \qquad \partial_\nu \left(r^2 \partial_\mu u_j \right) + \partial_\mu \left(r^2 \partial_\nu u_j \right) - \frac{8M^2 \lambda_j^2}{re^r} u_j = 0.$$

In particular, we may extend u_j by 0 to $\nu < 0$, $\mu \in [0, c_0]$ and $\mu < 0$, $\nu \in [0, c_0]$. In other words, we may extend u_j to a smooth function for $\mu, \nu \in (-\infty, c_0)$ that vanishes to infinite order at $\mu + \nu = 0$ for $\mu \in [-c_0, c_0]$.

We now use a unique continuation argument for the $1 + 1$-dimensional wave equation to conclude that each u_j must vanish in the region we claim. Indeed, because the extension of u_j vanishes to infinite order at $\mu + \nu = 0$ for $\mu \in [-c_0, c_0]$, finite speed of propagation (with respect to $\mu + \nu$) implies that u vanishes identically for μ and ν in the domain of dependence of $\{\mu + \nu = 0, \mu \in [-c_0, c_0]\}$, i.e., in the region $\{\mu \leq c_0, \nu \leq c_0\}$. In particular, the solution vanishes identically for $\{\mu = \nu, \mu, \nu \leq c_0\}$. Because the hypersurface $\{\mu = \nu\}$ agrees with the hypersurface $\{t = 0\}$ away from $\mu = \nu = 0$, we may conclude that the spherical components of the initial data are supported in $\mu \geq c_0$, i.e., in $r \geq r_0$. □

5.2. **At null infinity.** The main result of this section is the corresponding theorem at null infinity:

Theorem 5.4. *Suppose $\phi, \psi \in C_c^\infty \left((2M, \infty) \times \mathbb{S}^2 \right)$. If $\mathcal{R}_{S_1^+}(\phi, \psi)$ vanishes for $\bar\tau \leq -\bar\tau_0$ and $\mathcal{R}_{S_1^-}(\phi, \psi)$ vanishes for $\bar\tau \geq \bar\tau_0$, then both ϕ and ψ are supported in $(2M, r_0] \times \mathbb{S}^2$, where r_0 is given implicitly by*

$$r_0 + 2M \log(r_0 - 2M) = \bar\tau_0.$$

Corollary 5.5. *Suppose $\psi \in C_c^\infty \left((2M, \infty) \times \mathbb{S}^2 \right)$. If $\mathcal{R}_{S_1^+}(0, \psi)$ vanishes for $\bar\tau \leq -\bar\tau_0$, then ψ is supported in $(2M, r_0] \times \mathbb{S}^2$. In particular, if $\mathcal{R}_{S_1^+}(0, \psi)$ vanishes identically, then $\psi \equiv 0$.*

Proof of Corollary 5.5. Again by the time-reversibility of the wave equation, one has that

$$\mathcal{R}_{S_1^-}(0, \psi)(\bar\tau, \omega) = \mathcal{R}_{S_1^+}(0, \psi)(-\bar\tau, \omega),$$

and we may apply Theorem 5.4.   □

Proof of Theorem 5.4. We showed in Section 3.2 that solutions with compactly supported smooth initial data have expansions at null infinity. If the initial data are in $C_c^\infty\left((2M,\infty)\times\mathbb{S}^2\right)$, then the rescaled solution ru is smooth as a function of $\bar\tau$ up to $\bar\rho=0$. Here $\bar\rho=1/r$ and we use $\bar\tau_+,\bar\tau_-$ to distinguish the coordinates at future and past null infinity if necessary, i.e.,

$$\bar\tau_+ = t - r - 2M\log(r-2M), \quad \bar\tau_- = t + r + 2M\log(r-2M).$$

We start by showing that if the initial data (ϕ,ψ) are smooth and compactly supported, and $\mathcal{R}_{S_1^+}(\phi,\psi)$ vanishes for $\bar\tau_+ \le -\bar\tau_0$, then $v = ru$ vanishes to infinite order there. A similar argument applies near past null infinity as well. First note that by integrating in $\bar\tau_+$, we see that v vanishes at $\bar\rho=0$ for $\bar\tau_+ \le -\bar\tau_0$. We now use that v satisfies the following equation

$$\tilde\Box_\infty v + \gamma_\infty v = 0,$$

where γ_∞ is a smooth function across $\bar\rho=0$ and $\tilde\Box_\infty$ is the D'Alembertian for the conformal metric $\bar\rho^2 g_S$ and is given by

$$\tilde\Box_\infty = 2\partial_{\bar\tau_+}\partial_{\bar\rho} + (1 - 2M\bar\rho)(\bar\rho\partial_{\bar\rho})^2 + (1 - 4M\bar\rho)\bar\rho\partial_{\bar\rho} + \Delta_\omega.$$

Note that because v vanishes at $\bar\rho=0$ for $\bar\tau_+ \le -\bar\tau_0$, $\Delta_\omega v$ vanishes there as well. Thus at $\bar\rho=0$, because v is smooth as a function of $\bar\rho$, we have that

$$2\partial_{\bar\tau_+}\partial_{\bar\rho}v|_{\bar\rho=0} = 0,$$

for $\bar\tau_+ \le -\bar\tau_0$. Integrating in $\bar\tau_+$ shows that $\partial_{\bar\rho}v$ vanishes there as well.

Differentiating the equation in $\bar\rho$ yields

$$2\partial_{\bar\tau_+}\partial_{\bar\rho}^2 v + O(\bar\rho)v = 0,$$

where $O(\bar\rho)$ is the product of $\bar\rho$ and smooth differential operator in $\bar\rho$ and ω. In particular, we have that at $\bar\rho=0$, $\partial_{\bar\tau_+}\partial_{\bar\rho}^2 v = 0$ for $\bar\tau \le -\bar\tau_0$. Integrating again implies that $\partial_{\bar\rho}^2 v = 0$ there.

Proceeding inductively, after differentiating the equation k times, we may conclude that $\partial_{\bar\rho}^{k+1}v = \partial_{\bar\tau_+}\partial_{\bar\rho}^{k+1}v = 0$ for $\bar\rho=0$ and $\bar\tau_+ \le -\bar\tau_0$. In particular, v vanishes to infinite order at $\bar\rho=0$ for $\bar\tau_+ \le -\bar\tau_0$.

A similar argument shows that if $\mathcal{R}_{S_1^-}(\phi,\psi)$ vanishes for $\bar\tau_- \ge \bar\tau_0$, then v vanishes to infinite order there as well.

We now decompose the solution u (and thus also the rescaled solution v) into spherical harmonics. Letting w_j be an eigenbasis of $L^2(\mathbb{S}^2)$ with eigenvalues $-\lambda_j^2$, we write $u(t,r,\omega) = \sum u_j(t,r)w_j(\omega)$ and $v(\bar\tau,\bar\rho,\omega) = \sum v_j(\bar\tau,\bar\rho)w_j(\omega)$. Note that v_j and u_j are related by a change of coordinates and the rescaling, i.e., $v_j = rw_j$. This decomposition extends to the radiation field as well, so each v_j vanishes to infinite order at $\bar\rho=0$ for $\bar\tau_+ \le -\bar\tau_0$ and $\bar\tau_- \ge \bar\tau_0$. In $(\bar\tau_+,\bar\rho)$ coordinates, each function v_j satisfies

$$2\partial_{\bar\tau_+}\partial_{\bar\rho}v_j + (1 - 2M\bar\rho)(\bar\rho\partial_{\bar\rho})^2 v + (1 - 4M\bar\rho)\bar\rho\partial_{\bar\rho}v - \lambda_j^2 v + \gamma_\infty v = 0.$$

Similarly, in $(\bar{\tau}_-, \bar{\rho})$ coordinates, each v_j satisfies

$$-2\partial_{\bar{\tau}_-}\partial_{\bar{\rho}}v + (1 - 2M\bar{\rho})(\bar{\rho}\partial_{\bar{\rho}})^2 v + (1 - 4M\bar{\rho})\bar{\rho}\partial_{\bar{\rho}}v - \lambda_j^2 v + \gamma_\infty v = 0.$$

We may thus extend v_j by 0 to a smooth solution in a neighborhood of $\bar{\rho} = 0$ for $\bar{\tau}_+ \leq -\bar{\tau}_0$. We may perform a similar extension at past null infinity as well.

We now use an argument similar to the one in the proof of Theorem 5.1 to conclude that v (and hence u) must vanish in the region we claim.[3] Suppose instead that the corresponding initial data for some u_j is supported in $r \leq r_1$, corresponding to $\bar{\tau}_1$. Near $\bar{\rho} = 0$ and $\bar{\tau}_+ = -\bar{\tau}_1$, v_j vanishes to infinite order at $\bar{\tau}_+ + \bar{\rho} = -\bar{\tau}_1$ and thus vanishes in a full neighborhood of $\bar{\rho} = 0$, $\bar{\tau}_+ = -\bar{\tau}_1$. A similar argument holds for $\bar{\rho} = 0$ and $\bar{\tau}_- = \bar{\tau}_1$. Let us say that v_j vanishes $\bar{\tau}_+ \leq -\bar{\tau}_1 + \delta$ when $\bar{\rho} \leq \delta$ as well as for $\bar{\tau}_- \geq \bar{\tau}_1 - \delta$ when $\bar{\rho} \leq \delta$. In terms of t and $\bar{\rho}$, this implies that u_j vanishes if

$$r \geq \delta^{-1}$$
$$\tau_1 - \delta - r - 2M\log(r - 2M) \leq t \leq \delta - \tau_1 + r + 2M\log(r - 2M).$$

We now use the hyperbolicity of the $1 + 1$-dimensional operator

$$-\left(\frac{r}{r - 2M}\right)\partial_t^2 + \left(\frac{r - 2M}{r}\right)\partial_r^2 + \frac{2(r - M)}{r^2}\partial_r - \frac{\lambda_j^2}{r^2}$$

with respect to ∂_r to conclude that the initial data for u_j vanishes if $r + 2M\log(r - 2M) \geq \bar{\tau}_1 - \delta$, i.e., for $r \geq \tilde{r}_1$ with $\tilde{r}_1 < r_1$. Figure 5 illustrates the process of improving from vanishing for $r \geq r_1$ to $r \geq \tilde{r}_1$.

Proceeding in this manner, we find that both ϕ and ψ must be supported in $r \leq r_0$. □

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[3]One key difference between the two arguments is that we must proceed incrementally near null infinity because u is not smooth as a function of $\bar{\tau}_+^{-1}$ and $\bar{\tau}_-^{-1}$.
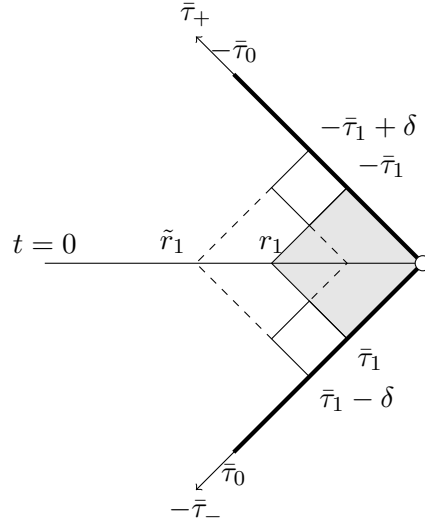
FIGURE 5. Iteratively improving the support of the initial data near null infinity. Knowing that the solution vanishes identically in the shaded region, we find first that it vanishes in the small white region, then use the hyperbolicity of the one-dimensional operator to improve it to the region outlined by the dashed line.

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