AN EXPLICIT DESCRIPTION OF THE RADIATION FIELD IN $3+1$-DIMENSIONS

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Abstract. In previous work with A. Vasy and J. Wunsch, the author established an asymptotic expansion for the radiation field on asymptotically Minkowski spacetimes and showed that the exponents seen in the expansion are given by the poles of a meromorphic family of operators on the spacetime's "boundary at infinity". This note provides an explicit accounting of these poles when the spacetime is $3+1$-dimensional Minkowski space. We conclude by stating the "resonant states" for the first few resonances and then posing a combinatorial problem.

1. Introduction

For a forward solution u of the inhomogeneous wave equation on Minkowski space,

$$\Box u = f \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}),$$

(or, equivalently, a solution u of the homogeneous wave equation with compactly supported initial data), the *Friedlander radiation field* of u encodes the behavior of u near null infinity. With A. Vasy and J. Wunsch [1, 2], the author established an asymptotic expansion of the radiation field on a class of asymptotically Minkowski spacetimes and showed that the exponents of the expansion were given by the poles of a meromorphic family of operators (called P_σ^{-1} in those papers) on the boundary at infinity. The purpose of this note is to identify explicitly these poles in the setting of $(3+1)$-dimensional Minkowski space. In particular, we prove the following theorem:

Theorem 1. *The poles of P_σ^{-1} are simple and located at $-\imath(k+1)$ for $k = 0, 1, 2, \dots$. The rank of the polar part of P_σ^{-1} at $\sigma = -\imath(k+1)$ is $\sum_{j=0}^{k} \dim(E_j) = (k+1)^2$, where E_j is the eigenspace of $\Delta_{\mathbb{S}^2}$ with eigenvalue j.*

The rest of the introduction is devoted to explaining and motivating Theorem 1.

Suppose that u is the solution of the homogeneous wave equation (or, equivalently, a forward solution of the inhomogeneous wave equation) on $3+1$-dimensional Minkowski space:

$$\Box u = \partial_t^2 - \Delta u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^3$$
$$(u, \partial_t u)|_{t=0} = (\phi, \psi) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$$

We now introduce polar coordinates (r, ω) in the spatial variables as well as a "lapse" parameter $s = t - r$ and define an auxiliary function

$$v(r, s, \omega) = r^{-1} u(s + r, r\omega).$$

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Friedlander [3] observed that the function v is smooth in $\rho = r^{-1}$ and so can be extended to $\rho = 0$. The *Friedlander radiation field* of u is then given by

$$\mathcal{R}_+[u](s,\omega) = \lim_{r\to\infty} \partial_s v(r,s,\omega).$$

In Minkowski spaces (and other static spacetimes), the radiation field has a number of desirable properties: Not only is it a unitary translation representation of the wave group, it can also be thought of as generalizing the Radon transform of the initial data. (Indeed, in $\mathbb{R}\times\mathbb{R}^3$, if the initial data are $(0,\psi)$, then the radiation field is just the Radon transform of ψ.)

In previous work [1, 2], the author and collaborators showed that the radiation field admits an asymptotic expansion for suitably nice data. Indeed, on a class of asymptotically Minkowski spacetimes, the radiation field exists and admits an asymptotic expansion in powers of s^{-1}. The exponents seen arise as the poles of a meromorphic family of Fredholm operators, denoted P_σ^{-1} on the "boundary at infinity".

We let M denote the radial compactification of Minkowski space with a defining function ρ for the boundary. In other words, we can consider $\mathbb{R}\times\mathbb{R}^3$ as the interior of a compact manifold with boundary via the coordinate change

$$t = \frac{1}{\rho}\cos\theta, \quad x = \frac{1}{\rho}\omega_j \sin\theta,$$

where $\omega_j \in \mathbb{S}^2$ and $\theta \in \mathbb{S}^1$. We denote by $C_\pm$ (depending on the sign of t) the regions of the boundary sphere $X \cong \mathbb{S}^3$ corresponding to where $|t| \gg |x|$, while we denote by C_0 the region of X where $|t| \ll |x|$. These regions of X naturally inherit conformal families of metrics; on Minkowski space, $C_\pm$ are naturally conformal to $\mathbb{H}^3$ and C_0 is naturally conformal to $2+1$-dimensional de Sitter space.

It is the region where $|t| \sim |x|$ that is of the most interest; we denote these regions by $S_\pm$ depending on the sign of t. By blowing up (in the algebro-geometric sense) the submanifolds $S_\pm$ in M, we obtain a manifold with corners that has two new boundary faces, denoted $\mathcal{I}^\pm$ and corresponding to past and future null infinity. Figure 1 provides a schematic view of this blow-up. The radiation field $\mathcal{R}_+[u]$ is then the rescaled restriction of u to $\mathcal{I}^+$.

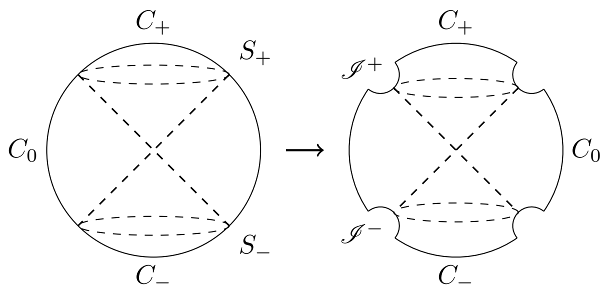

Figure 1. A schematic view of the blow-up. The lapse function s increases along $\mathcal{I}^+$ towards C_+. In the typical Penrose diagram of Minkowski space, $C_\pm$ are collapsed to $i_\pm$ and C_0 is collapsed to i_0.

Conjugating $\Box$ by ρ, multiplying by ρ^{-2}, and then taking the Mellin transform in ρ yields a family of operators P_σ on the boundary sphere $X \cong \mathbb{S}^3$. This has the effect of replacing all factors of $\rho\partial_\rho$ in $\rho^{-3}\Box\rho$ by $\imath\sigma$. Although P_σ is semiclassically hyperbolic (because $\Box$ is hyperbolic), on $C_\pm$ it is classically elliptic (indeed, it can be conjugated to the spectral family for the Laplacian on hyperbolic space). On C_0, P_σ is hyperbolic and can be conjugated to a Klein-Gordon equation on de Sitter space, while at $S_\pm$ it is degenerate.

The Hamilton vector field of the symbol of P_σ is radial at the conormal bundle of $S_\pm$ and so techniques dating back to Melrose [5] and refined by Vasy [6] provide a blueprint for establishing propagation estimates there.

The operator family P_σ is not Fredholm on standard Sobolev spaces, but it is when considered on variable-order Sobolev spaces whose regularity lies below some threshold at S_- (depending on the imaginary part of σ) and is larger than a similar threshold at S_+ (again, depending on the imaginary part of σ). As P_σ is then invertible on these spaces for very large $\operatorname{Im}\sigma$, we may invert to obtain a meromorphic family of Fredholm operators P_σ^{-1}. Because all light rays in Minkowski space escape to infinity, P_σ^{-1} has only finitely many poles in any horizontal strip in $\mathbb{C}$. The main result of both previous papers [1, 2] is that the radiation field for a forward solution has an asymptotic expansion whose exponents are these poles, which are identified as the resonances of the asymptotically hyperbolic operator at C_+.

For $3+1$-dimensional Minkowski space, however, this asymptotically hyperbolic operator is the spectral family of the Laplacian on $\mathbb{H}^3$, *which has no resonances.* In this case, the resonant states associated to the poles of P_σ^{-1} must be supported in S_+ (rather than $\overline{C_+}$). Theorem 1 describes the locations of these poles and the dimension of the corresponding nullspace of P_σ.

The proof of Theorem 1 proceeds in several steps. We first recall from [1] that any resonant state must be supported at the intersection of the light cones and the boundary at infinity because odd-dimensional hyperbolic space has no resonances. This reduces the problem of finding the poles of P_σ^{-1} (and the corresponding resonant states) to understanding when P_σ has nullspace consisting of a distribution of the form

$$\sum_{k=0}^{M} a_k \delta^{(k)}(v) \otimes \phi_\lambda,$$

where ϕ_λ is a spherical harmonic with eigenvalue λ. This immediately implies that the poles of P_σ^{-1} are contained in the negative imaginary integers and reduces the problem of finding the null spaces of an explicit family of matrices. For $\sigma = -\imath(M+1)$, P_σ preserves the family of such distributions and so the problem of finding the null space of P_σ reduces to a linear algebra problem. We write down the matrix representing P_σ and compute its determinant explicitly. This shows that the matrix has one-dimensional null space precisely when $\lambda = k(k+1)$ for $k = 0, 1, \ldots, M$ and hence that $P_{-\imath(M+1)}$ has null space of dimension

$$\sum_{k=0}^{M} \dim(E_k),$$

where E_k is the space of spherical harmonics with eigenvalue $k(k+1)$.

Unfortunately, finding the elements of the null space explicitly is sufficiently complicated that we were unable to solve it here by purely combinatorial means. It is perhaps surprising how difficult it is to find an explicit expression for elements of the nullspace of $P_{-\imath(k+1)}$. However, given the connection of the radiation field with the Radon transform, such an expression would provide an explicit formula for the Radon transform in terms of spherical harmonics. Such formulas exist but are similarly complicated (and have representation-theoretic underpinnings).

In Section 2 we describe some of the geometry of the radial compactification of Minkowski space, and then in Section 3 we define the operator P_σ and recall some of its properties. We also introduce a convenient coordinate system that simplifies the linear algebra in the following section. Section 4 recasts the problem of describing the poles and corresponding resonant states in terms of linear algebra and finishes the proof of Theorem 1. Included in Section 4 is the exact form of the resonant states corresponding to the first few poles of P_σ^{-1}. Finally, in Section 5 we conclude by appealing for a combinatorial expression for the elements of the null space of P_σ in this context.

2. Geometry

The radiation field is the rescaled restriction of a solution u of the wave equation to null infinity. In this section we describe a compactification of Minkowski space on which this statement is a natural one.

We begin by introducing coordinates on Minkowski space given by

$$t = \frac{1}{\rho}\cos\theta$$
$$x = \frac{1}{\rho}\omega_j \sin\theta$$

where $\omega_j \in \mathbb{S}^2$. In terms of these coordinates, the metric on Minkowski space is given by

$$g := -dt^2 + \sum_{j=1}^{3} dx_j^2 = -\cos 2\theta \frac{d\rho^2}{\rho^4} - 4\sin\theta\cos\theta\frac{d\theta}{\rho}\frac{d\rho}{\rho^2} + \cos 2\theta\frac{d\theta^2}{\rho^2} + \sin^2\theta\frac{d\omega^2}{\rho^2}$$

We now replace the coordinate θ by $v = \cos 2\theta$ to obtain

$$g = -v\frac{d\rho^2}{\rho^4} + \frac{dv}{\rho}\frac{d\rho}{\rho^2} + \frac{v}{4(1-v^2)}\frac{dv^2}{\rho^2} + \frac{1-v}{2}\frac{d\omega^2}{\rho^2}$$

The inverse metric (in coordinates (ρ, v, ω)) is then given by

$$g^{-1} \to \begin{pmatrix} -v\rho^4 & 2(1-v^2)\rho^3 & 0 \\ 2(1-v^2)\rho^3 & 4v(1-v^2)\rho^2 & 0 \\ 0 & 0 & \frac{2\rho^2}{1-v}h^{-1} \end{pmatrix},$$

where h is the standard (round) metric on $\mathbb{S}^2$.

This radial compactification of Minkowski space has two distinguished submanifolds where $\rho = 0$ and $v = 0$, which we call $S_\pm$ ($S_\pm$ are distinguished by the sign of t – S_+ is the set in the future where $\rho = 0$ and $v = 0$, while S_- is the corresponding set in the past).

3. The operators

The central object of study is the operator L, given by

$$L = \rho^{-3}\Box_g \rho.$$

Here the conjugation by ρ should be thought of as accounting for the standard decay for solutions of the wave equation, while the prefactor of ρ^{-2} turns a "scattering operator" in the sense of Melrose [5] into a "b-operator" [4].

We record here the precise form of L:

$$L = v(\rho\partial_\rho)^2 + (2+4v)\rho\partial_\rho - 4(1-v^2)\rho\partial_\rho\partial_v - 4v(1-v^2)\partial_v^2 - 4(1-v-3v^2)\partial_v - \frac{2}{1-v}\Delta_\omega + (2+3v)$$

The operator P_σ is the reduced normal operator $\hat{N}(L)(\sigma)$, which effectively replaces $\rho\partial_\rho$ by $\imath\sigma$ and is obtained by conjugating L by the Mellin transform in ρ:

$$P_\sigma = -v\sigma^2 + (2+4v)\imath\sigma - 4\imath\sigma(1-v^2)\partial_v - 4v(1-v^2)\partial_v^2 - 4(1-v-3v^2)\partial_v - \frac{2}{1-v}\Delta_\omega + (2+3v)$$

Although the expression above for P_σ is useful for the global problem of identifying the Fredholm properties of P_σ, for our explicit computation it is more convenient to work with a different coordinate system valid near S_+. For the remainder of this note, we instead take

$$\rho = \frac{1}{t+r}, \quad v = \frac{t-r}{t+r}.$$

In these coordinates, we may write

$$\Box = \partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\Delta_\omega = 4\rho^2\left[-\rho\partial_\rho\partial_v - \partial_v - v\partial_v^2 + \frac{1}{1-v}(\rho\partial_\rho + (1+v)\partial_v) - \frac{1}{(1-v)^2}\Delta_\omega\right]$$

We then have

$$L = \rho^{-3}\Box\rho = 4\left[-\rho\partial_\rho\partial_v - 2\partial_v - v\partial_{v^2} + \frac{1}{1-v}(\rho\partial_\rho + 1 + (1+v)\partial_v) - \frac{1}{(1-v)^2}\Delta_\omega\right]$$

and

$$P_\sigma = 4\left[-(\imath\sigma+2)\partial_v - v\partial_v^2 + \frac{\imath\sigma+1}{1-v} + \frac{1+v}{1-v}\partial_v - \frac{1}{(1-v)^2}\Delta_\omega\right].$$

We may then multiply P_σ by $(1-v)^2/4$ and group the terms with the same degree of homogeneity:

$$\frac{(1-v)^2}{4}P_\sigma = \left[-(\imath\sigma+1)\partial_v - v\partial_v^2\right] + \left[2(\imath\sigma+2)v\partial_v + 2v^2\partial_v^2 + (\imath\sigma+1) - \Delta\omega\right] + \left[-(\imath\sigma+3)v^2\partial_v - v^3\partial_v^2 - (\imath\sigma+1)v\right]$$

4. The poles of P_σ^{-1}

At each pole of P_σ^{-1}, the residue can be identified with an operator whose image is a "resonant state". As P_σ is not self-adjoint, the residue operators do not project onto these states, but we abuse terminology by calling them resonant states anyway. We know from the results of [1] that if f is supported away from $\overline{C_-}$, then $P_\sigma^{-1}f$ is supported in $\overline{C_+}$. Moreover, if $P_\sigma^{-1}f$ is not supported in S_+, then the pole (and corresponding state) can be identified with a resonance of a Laplace-like operator on C_+. In $n+1$-dimensional Minkowski space, this operator is the Laplacian on $\mathbb{H}^n$.

Because $\mathbb{H}^3$ has no resonances, we can conclude that all resonant states of P_σ^{-1} on $3+1$-dimensional Minkowski space must be supported in S_+. The resonant states must therefore be sums of the following form:

$$\sum_{k=0}^{M} \alpha_k \delta^{(k)}(v) \otimes \phi_\lambda(\omega), \tag{1}$$

where ϕ_λ is a spherical harmonic with eigenvalue λ.

In the rest of this section, we prove Theorem 1 in several steps. We first compute the action of P_σ on such a sum, which shows that P_σ has no null space unless $\sigma = -\imath(M+1)$ for $M = 0, 1, \dots$. For such a σ, we then interpret $\frac{(1-v)^2}{4} P_\sigma$ (which has the same null space as P_σ when acting on such distributions) as a family of matrices depending on λ. We compute this determinant and show that the matrix has a 1-dimensional null space exactly when $\lambda = k(k+1)$ for $k = 0, 1, \dots, M$. Appealing to the well-known dimension of the space of spherical harmonics with eigenvalue $k(k+1)$ then completes the proof of Theorem 1.

We now record the action of $\frac{(1-v)^2}{4} P_\sigma$ on distributions of the form above (1). We rely on the following well-known fact:[1]

$$v^r \delta^k(v) = \frac{k!}{(k-r)!} \delta^{(k-r)}(v)$$

We then have the following:

$$\begin{aligned} \frac{(1-v)^2}{4} P_\sigma(\delta^{(k)}(v) \otimes \phi_\lambda(\omega)) = {} & \left[(-\imath\sigma + 1) + (k+2)\right] \delta^{(k+1)} \otimes \phi_\lambda \\ & + \left[-(\imath\sigma + 1)(2k+1) + 2(k+1)^2 + \lambda\right] \delta^{(k)} \otimes \phi_\lambda \\ & + \left[-(\imath\sigma+1)k^2 + k^2(k+1)\right] \delta^{(k-1)} \otimes \phi_\lambda \end{aligned}$$

In particular, for a sum of the form (1) to lie in the null space of P_σ, the leading term must vanish and so $M + 1 - \imath\sigma = 0$, i.e., $\imath\sigma = M+1$. We may then take $\sigma = -\imath(M+1)$, apply P_σ to such a sum, and rearrange the terms to find the following:

$$\begin{aligned} \frac{(1-v)^2}{4} P_{-\imath(M+1)} \left(\sum_{k=0}^{M} \alpha_k \delta^{(k)}(v) \otimes \phi_\lambda(\omega) \right) = \sum_{k=0}^{M} \Big[& (k-1-M)\alpha_{k-1} \\ & + (\lambda + 2k^2 - M(2k+1))\alpha_k \\ & + (k+1)^2(M-k)\alpha_{k+1} \Big] \delta^{(k)}(v) \otimes \phi_\lambda(\omega) \end{aligned} \tag{2}$$

We have now reduced the problem to finding a vector of coefficients α_k so that the sum (2) vanishes. This is equivalent to finding the null space of a matrix $\lambda I - A_M$, where A_M is the tridiagonal $(M+1) \times (M+1)$-matrix with the following entries:

$$\begin{aligned} a_{k,k} &= (M-k)(2k+1) + k \\ a_{k-1,k} &= -k^2(M+1-k) \\ a_{k,k-1} &= M+1-k \end{aligned}$$

Here all indices (k and $k-1$) should be interpreted as taking the values $0, 1, \dots, M$.

[1]This formula is *a priori* valid only for $r \le k$, but if we interpret $(k-r)! = \Gamma(k-r+1)$, then the denominator is infinite for $r > k$ and so the right-hand side is zero there.

The rest of the proof of Theorem 1 then follows from the following proposition:

Proposition 2. *The matrix A_M has simple eigenvalues $k(k+1)$ for $k = 0, 1, \dots, M$. In particular, we have that*

$$\det(\lambda I - A_M) = p_M(\lambda),$$

where

$$p_k(\lambda) = \prod_{j=0}^{k} (\lambda - j(j+1)).$$

Proposition 2 follows immediately by taking $k = M$ in the following lemma:

Lemma 3. *Let d_k be the determinant of the $(k+1) \times (k+1)$-minor of $\lambda I - A_M$ consisting of the first $k+1$ columns and rows of the matrix (i.e., the columns and rows labeled $0, 1, \dots, k$). If $p_k(\lambda)$ is as in Proposition 2, then*

$$d_k = \sum_{\ell=0}^{k+1} c_{k.\ell} \left(\prod_{j=1}^{\ell} (M - k + \ell - j) \right) p_{k-\ell}(\lambda),$$

where

$$c_{k,\ell} = \frac{(-1)^\ell}{\ell!} \left(\frac{(k+1)!}{(k+1-\ell)!} \right)^2,$$

and we interpret $p_{-1}(\lambda) = 1$.

Observe that all terms containing $p_{k-\ell}(\lambda)$ with $\ell \geq 1$ in the expression for d_k in Lemma 3 are multiplied by a factor $(M-k)$ and hence vanish when $k = M$, leaving only the term $c_{M,0} p_M(\lambda) = p_M(\lambda)$.

Proof of Lemma 3. The matrix A_M is tridiagonal, so d_k can be computed recursively:

$$d_k = (\lambda - a_{k,k}) d_{k-1} + a_{k,k-1} a_{k-1,k} d_{k-2}.$$

We therefore proceed by induction, interpreting $d_{-1} = 1$ so that the lemma holds for $k = -1$. In particular, we have $c_{-1,0} = 1$ and $c_{-1,\ell} = 0$ for $\ell \geq 1$.

In computing below, we use the following relationship between the p_k:

$$\lambda p_{k-1-\ell}(\lambda) = p_{k-\ell}(\lambda) + (k-\ell)(k-\ell+1) p_{k-1-\ell}(\lambda)$$

We now compute the two terms in the recursive expression for d_k. We first have the following:

$$\begin{aligned}
&(\lambda - a_{k,k}) d_{k-1} \\
&\quad = (\lambda - (M-k)(2k+1) - k)) \sum_{\ell=0}^{k} c_{k-1,\ell} \left(\prod_{j=1}^{\ell} (M - k + 1 + \ell - j) \right) p_{k-1-\ell}(\lambda) \\
&\quad = \sum_{\ell=0}^{k} c_{k-1,\ell} \left(\prod_{j=0}^{\ell-1} (M - k + \ell - j) \right) p_{k-\ell}(\lambda) \\
&\qquad + \sum_{\ell=1}^{k+1} c_{k-1,\ell-1} \left[(k-\ell+1)(k-\ell+2) - (M-k)(2k+1) - k \right] \cdot \\
&\qquad\qquad \cdot \left(\prod_{j=1}^{\ell-1} (M - k + \ell - j) \right) p_{k-\ell}(\lambda)
\end{aligned}$$

The second term is given by

$$a_{k,k-1}a_{k-1,k}d_{k-2} = -\sum_{\ell=2}^{k+1} c_{k-2,\ell-2}(M+1-k)^2k^2\left(\prod_{j=1}^{\ell-1}(M-k+\ell-j)\right)p_{k-\ell}(\lambda)$$

Adding the two terms and equating coefficients with the desired expression for d_k, we find that

$$\begin{aligned}(M-k)c_{k,\ell} = {} & (M-k+\ell)c_{k-1,\ell} \\ & + \left[(k-\ell+1)(k-\ell+2)-(M-k)(2k+1)-k\right]c_{k-1,\ell-1} \\ & - k^2(M-k+1)c_{k-2,\ell-2}.\end{aligned}$$

We rewrite this equation suggestively:

$$\begin{aligned}(M-k)c_{k,\ell} = {} & (M-k)\left(c_{k-1,\ell}-(2k+1)c_{k-1,\ell-1}-k^2c_{k-2,\ell-2}\right) \\ & + \ell c_{k-1,\ell} + \left((k-\ell+1)(k-\ell+2)-k\right)c_{k-1,\ell-1} - k^2c_{k-2,\ell-2}\end{aligned} \tag{3}$$

To prove the lemma, it therefore suffices to show that $c_{k,\ell}$ is an integer. We prove this fact by induction. We have already seen that $c_{-1,0} = 1$ and $c_{-1,\ell} = 0$ for $\ell \geq 1$. By the induction hypothesis, we assume that

$$c_{k',\ell'} = \frac{(-1)^{\ell'}}{(\ell')!}\left(\frac{(k'+1)!}{(k'+1-\ell')!}\right)^2$$

for all $(k',\ell') < (k,\ell)$, where we define $(a',b') < (a,b)$ if either

- $b' < b$, or
- $b' = b$ and $a' < a$.

We turn first to the second line of equation (3). By the induction hypothesis,

$$\begin{aligned}&\ell c_{k-1,\ell} + \left((k-\ell+1)(k-\ell+2)-k\right)c_{k-1,\ell-1} - k^2c_{k-2,\ell-2} \\ &= \frac{(-1)^\ell}{(\ell-1)!}\left(\frac{k!}{(k-\ell+1)!}\right)^2\left((k-\ell+1)^2-(k-\ell+1)^2-(k-\ell+1)+k-(\ell-1)\right) = 0\end{aligned}$$

Equation (3) and the induction hypothesis then imply that

$$\begin{aligned}c_{k,\ell} &= c_{k-1,\ell} - (2k+1)c_{k-1,\ell-1} - k^2c_{k-2,\ell-2} \\ &= \frac{(-1)^\ell}{\ell!}\left(\frac{k!}{(k-\ell+1)!}\right)^2\left((k-\ell+1)^2+(2k+1)\ell-\ell(\ell-1)\right) \\ &= \frac{(-1)^\ell}{\ell!}\left(\frac{(k+1)!}{(k-\ell+1)!}\right)^2,\end{aligned}$$

finishing the proof of the lemma. □

4.1. **The first few resonant states.** In this section we record the first five sets of eigenvectors of the matrix A_M.

For $M = 0$, we have that $A_0 = (0)$, so its only eigenvalue is 0 with eigenvector (1).

For $M = 1$, we have that

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that the eigenvectors are

$$\mathbf{v}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $M = 2$, we have

$$A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 4 \\ 0 & 1 & 2 \end{pmatrix},$$

with eigenvectors

$$\mathbf{v}_0 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}.$$

For $M = 3$, we record the matrix

$$A_3 = \begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 7 & 8 & 0 \\ 0 & 2 & 7 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

so that the eigenvectors are

$$\mathbf{v}_0 = \begin{pmatrix} -6 \\ 6 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 6 \\ -2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} -6 \\ -6 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_{12} = \begin{pmatrix} 6 \\ 18 \\ 9 \\ 1 \end{pmatrix}.$$

Finally, for $M = 4$, the matrix is

$$A_4 = \begin{pmatrix} 4 & 4 & 0 & 0 & 0 \\ 4 & 10 & 12 & 0 & 0 \\ 0 & 3 & 12 & 18 & 0 \\ 0 & 0 & 2 & 10 & 16 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix},$$

so that the eigenvectors are

$$\mathbf{v}_0 = \begin{pmatrix} 24 \\ -24 \\ 12 \\ -4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -24 \\ 12 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 24 \\ 12 \\ -12 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_{12} = \begin{pmatrix} -24 \\ -48 \\ 0 \\ 8 \\ 1 \end{pmatrix}, \quad \mathbf{v}_{20} = \begin{pmatrix} 24 \\ 96 \\ 72 \\ 16 \\ 1 \end{pmatrix}.$$

5. A combinatorial problem

We conclude this note by posing a combinatorial problem. In principle it is possible to determine the resonant states of P_σ^{-1} by purely combinatorial means. Specifically, this would be achieved by explicitly finding the eigenvectors of the matrix A_M above. The computation above shows that the eigenvalues are $k(k+1)$ for $k = 0, \ldots, M$.

Problem 4. *Find a general expression for the eigenvectors of the matrix A_M.*

The resolution of this problem would provide an explicit formula for the resonant states of P_σ on Minkowski space and thus give an explicit expression for the radiation field in terms of spherical harmonics. Such a formula would then make it feasible to compute the radiation field explicitly for non-trivial examples.

Moreover, given the connection between the radiation field and the Radon transform, such a formula should also recover a formula for the Radon transform in terms of a spherical harmonic decomposition. Existing formulas typically rely on the Funk–Hecke formula and thus involve the Legendre polynomials. We therefore expect that the general expression for the eigenvectors of A_M ought to be expressible in terms of coefficients of Legendre polynomials.

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